Linear Model for Optical Measurement

Ding Luo

Vision and Fusion Laboratory
Institute for Anthropomatics
Karlsruhe Institute of Technology (KIT), Germany
ding.luo@kit.edu

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Abstract: A linear measurement model is used to describe the measurement system where the measurements are linear combinations of the target signal. Due to its simplicity, it can be applied to various measurement systems. In this article, a comprehensive review of linear measurement model with a focus on optical systems is conducted by considering three different situations. Firstly, the assumption of signal sparsity is made, which turns the model into a compressive sensing problem. In spite of the various potentials demonstrated by the compressive sensing approach, it has been shown that compressive sensing is not fully ready for real-time applications yet due to its computational cost. Secondly, prior information of the target signal is taken into consideration to transform the linear measurement model into a linear manifold learning problem. With classical methods like principal component analysis (PCA), it has been demonstrated with two examples that such approach could simplify the measurement and the recovery process. Last but not least, the postprocessing step for the retrieval of information within the signal is further reduced through holistic design of the measurement system, granting systems with optical computation to make measurement faster and more robust against noise.

1 Introduction

Various real-world signals can be viewed as an n-dimensional vector $\mathbf{x} \in \mathbb{R}^n$, such as sound, image, etc. In a linear measurement model, each measurement of the target signal is a linear combination of all values in the vector $\mathbf{x}$. The complete measurements of the signal can be written as an m-dimensional vector $\mathbf{y} = A\mathbf{x} \in \mathbb{R}^m$ with an $m \times n$ measurement matrix A. Such formulation is very convenient as it covers many practical situations. For example, under the traditional Nyquist-Shannon sampling frame, the dimension of the signal $\mathbf{x}$ can be considered as tending to infinity, which represents a continuous function. Each row of the measurement matrix A is constructed as a bandpass filter which represents the sampling step.

The ultimate goal of the linear measurement model, like any other measurement systems, is to retrieve the signal $\mathbf{x}$ and the information it is carrying. Formulation of the linear measurement model as a linear system naturally leads to a classical problem of linear algebra: conditions for solving the equation $\mathbf{y} = A\mathbf{x}$. In this context, this problem is equivalent to determining which kind of measurements are needed in order to recover the signal.

According to the classical theory of linear algebra, if there are at least as many measurements as unknowns ($m \geq n$) and A has full rank, the problem is determined or overdetermined. Then the equation $\mathbf{y} = A\mathbf{x}$ can be solved uniquely (e.g. by Gaussian elimination). If there are fewer measurements than unknowns ($m < n$), the problem is underdetermined even with A having full rank. The knowledge of $\mathbf{y} = A\mathbf{x}$ restricts $\mathbf{x}$ to an affine subspace of $\mathbb{R}^n$, but does not determine $\mathbf{x}$ completely. Nevertheless, if A has full rank and $\mathbf{x}$ is believed to be "small", the least square approach yields $\mathbf{x}^{\#} = A^T(AA^T)^{-1}\mathbf{y}$, which is the solution of the ℓ_2-minimization problem:

$$\begin{aligned} \underset{\mathbf{z}}{\text{minimize}} \quad & \|\mathbf{z}\|_2 \\ \text{subject to} \quad & A\mathbf{z} = \mathbf{y} \end{aligned}$$

However, the assumption that the signal vector $\mathbf{x}$ is "small" does not apply to most of the signals in practice. Therefore, in order to solve an underdetermined linear system, many researchers have proposed various methods to solve this problem in unconventional ways, such as compressive sensing. This problem has profound practical value as the number of measurements could potentially be much smaller than the length of the signal to be measured.

The following content of the report is divided into several sections. In the second section, a theoretical review of compressive sensing is conducted with the help of an exemplary algorithm. The third section considers a more practical setting where prior information can be utilized to further simplify the recovery of the signal. In the fourth section, possibility is proposed that the information behind the signal can sometimes be retrieved directly without the recovery of the signal, which leads to a measurement system with optical computation capability.

2 Compressive Sensing

Compressive sensing (also known as compressed sensing, compressive sampling, or sparse sampling) consists of reconstructing an s-sparse vector $\mathbf{x} \in \mathbb{C}^N$ from $\mathbf{y} = A\mathbf{x}$, where $A \in \mathbb{C}^{m \times N}$ is the so-called measurement matrix representing an underdetermined linear system ($m < N$). Suppose vector $\mathbf{x}$ is the unique s-sparse solution of $A\mathbf{z} = \mathbf{y}$ with $\mathbf{y} = A\mathbf{x}$, the vector $\mathbf{x}$ can be reconstructed as the solution of:

$$\begin{aligned} \underset{\mathbf{z} \in \mathbb{C}^N}{\text{minimize}} \quad & \|\mathbf{z}\|_0 \\ \text{subject to} \quad & A\mathbf{z} = \mathbf{y} \end{aligned} \tag{2.1}$$

Unlike the assumption of the signal vector $\mathbf{x}$ as "small" mentioned in Sec. 1, the sparsity assumption is much more useful as it applies to most of the real-world signals. Many signals are naturally sparse, such as the signal from a heart beat monitor in the hospital where only values inside the peak are non-zero. For signals which are not sparse, there often exists a certain basis, with which the signal can be transformed into a sparse one. In the worst case, most of the signals can be well approximated by a sparse signal. This is the fundamental reason why compressive sensing has received increasing attention in recent years.

The problem of ℓ_0-minimization described by Eq. (2.1) can be split into two aspects. Firstly, it has to be guaranteed that the recovered signal $\mathbf{x}$ is the unique solution to the problem. Secondly, algorithms have to be developed to perform such recovery.

In order to fulfill the first condition, the measurement matrix has to be constructed according to the sparsity of the signal. This involves both the size of the signal, which is directly linked to the number of measurements, and the property of the matrix. Generally speaking, the measurement matrix has to be *incoherent*, in order to guarantee that the reconstruction is unique [FR13]. The Restricted Isometry Property (RIP) proposed by Emmanuel J. Candes [Can08] is generally considered as a canonical measure of the coherence of a measurement matrix. Deliberate construction of a matrix with RIP is a very hard problem but fortunately with statistical tools it has been proven that random matrices tend to have RIP with high probability [FR13].

The progress of algorithm development for ℓ_0-minimization problem has not been easy as it has been proven that the problem is NP-hard [FR13]. The breakthrough is made by Emmanuel J. Candes *et al.* with their discovery of theoretically guaranteed equivalence between ℓ^0-minimization and ℓ^1-minimization under certain conditions [CRT06]. This leads to a branch of optimization methods based on ℓ^1-minimization, often referred to as Basis Pursuit. As a convex optimization problem, it can be solved fairly fast with linear programming methods. Another major branch of methods are greedy methods, based on iterative construction/modification/thresholding of the support of $\mathbf{x}$, such as Orthogonal Matching Pursuit [TG07] and Iterative Hard Thresholding [BD09].

As a fairly new paradigm, compressive sensing has shown great potential. Firstly, the linear sampling scheme serves as an alternative to the traditional Nyquist-Shannon sampling theory, which allows for much reduced number of measurements under many situations. Secondly, the optimization nature of compressive sensing grants itself intrinsic tolerance against noise, which is very important in practical situations. Such features have helped to realize the application of compressive sensing in various fields, especially in optical imaging/measurement led by the development of the single-pixel camera [DDT$^+$08].

In spite of its popularity, there are various factors preventing compressive sensing from being applied in real-time measurement systems. To begin with, the computational cost of the reconstruction algorithms is still relatively high, in terms of both time and space. Since most algorithms are based on iterative processes in an optimization loop and each step often consists of considerable amount of computational task, the reconstruction process is very time consuming even for state-of-the-art algorithms, which are already thousands of times faster than primary algorithms invented at the beginning of compressive sensing. For example, TVAL3 is an algorithm developed by Chengbo Li with the specific target of image reconstruction [Li09], which can be applied in implementations like single pixel camera. Although it is among the fastest and best performing algorithms for compressive sensing, reconstruction of a single image with 64×64 pixels using 30% measurements still takes 1.9 second on a consumer laptop. To increase the size of the image to 640×640 pixels, the measurement matrix would be 122880×409600 large and takes a memory space of 375GB with double precision. Additionally, empirical experiences have shown that the efficiency of the algorithms relies heavily on the tuning of the input parameters. Therefore, it is in the author's opinion that compressive sensing is still not ready for real-time measurement areas.

3 Linear Manifold Learning

Conventional compressive sensing theories rely only on the sparsity of the signal, whereas in reality more a priori information could be linked to the target signal to facilitate reconstruction. To further utilize such information, model-based compressive sensing is proposed by R.G. Baraniuk *et al.* [BCDH10], where dependencies between values and locations in the signal are taken into consideration to allow for even less measurements with robust recovery.

To push it even further, in the extreme case that all possible forms of the signal are known a priori, the linear measurement model can be transformed into a linear manifold learning problem, denoted by $Y = AX$, where $X \in \mathbb{R}^{n \times p}$ contains p possible n-dimensional signals and $Y \in \mathbb{R}^{m \times p}$ contains p corresponding m-dimensional measurements, which are linked through the linear measurement matrix $A \in \mathbb{R}^{m \times n}$. Since X is considered known a priori, various linear methods can be used to construct the measurement matrix A to yield best measurements for recovery, such as principal components analysis, metric multidimensional scaling, etc.

3.1 Chromatic Confocal Signal

In a chromatic confocal system, an objective lens with controlled chromatic aberration is used to separate different spectral components of the illumination light on to different focal planes. The reflected light from the sample, after the confocal filtering, will be concentrated around the spectral component that is in focus, generating a quasi-Gaussian peak in the spectrum, which is traditionally measured with a spectrometer.

In a predefined optical system, such quasi-Gaussian peaks can be considered as known a priori. All possible peak functions can be collected to form the matrix X. Principal component analysis can be performed on matrix X yielding a series of principal components, the largest of which are used to construct the measurement matrix A. The pseudo-inverse matrix of A can be applied on the measurement Y for the reconstruction of the signals: $X_R = A^T Y = A^T AX$.

As shown by the simulation results in Fig. 3.1, it is possible to recover the shape of the peak function with only three linear measurements. Due to the fact that the calculation of the pseudo-inverse of A is rank deficient, $A^T A$ is not an identity matrix, thus the reconstruction is not exact. Nevertheless, the peak position is well aligned with the original signal, and therefore preserving the information within. By increasing the number of linear measurements, the recovery accuracy will be much improved.

With application of the linear measurement model on a chromatic confocal setup, the spectrometer with thousands of pixels could potentially be replaced by a limited set of filter combinations, allowing much less data flow and thus increased measurement speed. Compared with conventional compressive sensing, the reconstruction process is dramatically simplified as it involves only matrix multiplication.

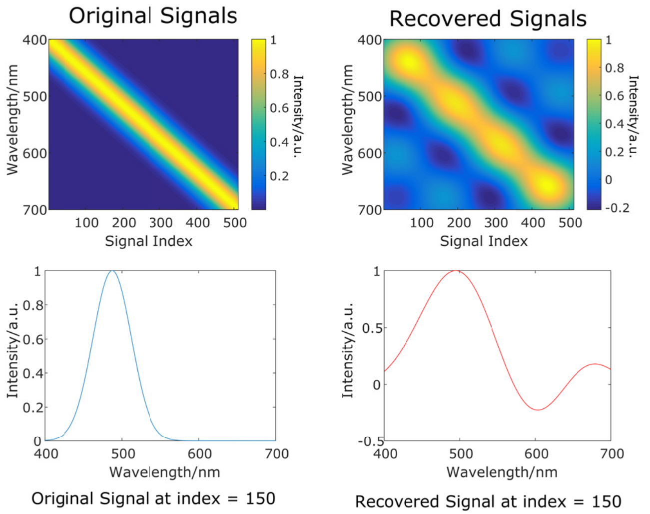

Figure 3.1: Simulation results of Gaussian signals.

3.2 Shape from Focus

Shape from focus is yet another field where linear measurement model with prior information could potentially be applied to enhance the efficiency of the technique. Depending on the size of the aperture, focal length of the objective, focal plane position and size of the imaging pixel, an imaging system has a certain depth of focus so that objects out of focus get blurred. The measure of how well an object is in focus can serve as a datum for measuring the distance of this object to the imaging system. When the measurement is taken across the complete field of the imaging system, the three dimensional profile of the object surface could be retrieved.

Conventional shape from focus methods generally involve several steps. Firstly, an image sequence is captured while the focal plane of the imaging system is shifted by either mechanical scanning of camera/sample or motorized focus shifting within the objective lens. Secondly, focus measure values for each pixel in every image are computed to construct a 3D data cube with 2D transverse spatial coordinates corresponding to camera pixels and 1D axial shift coordinate. Various algorithms have been developed for this purpose [PPG13]. Lastly, the 3D profile of the sample is reconstructed from these focus measure values in the data cube. This can be conducted through either simple approaches like taking axial location with maximum focus value as the target position [PPG13], or with more sophisticated approaches by encoupling optimization algorithms such as Total Variation regularization [Mah13].

Regardless of the focus measure algorithms, the focus measure values of one specific pixel at different axial locations typically form a quasi-Gaussian shape, similar to that of the (chromatic) confocal signal. Therefore in principal, it is possible to apply the same linear measurement model to the process of shape from focus. Instead of taking a complete sequence of axially shifted images, only a limited number of images are required, which are linear combinations of the original image sequence. The complete focus measure signal can be retrieved from the focus measure of these synthetic images provided that the focus measure operation is linear.

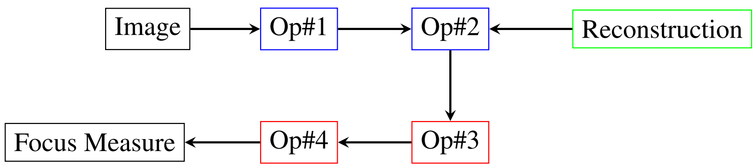

Figure 3.2: Exemplary processes of shape from focus with linear measurement model. Op#1 and Op#2 are linear operations on the image while Op#3 and Op#4 are non-linear.

However, modern focus measure algorithms often consist of multiple steps of operations on the image, many of which are non-linear operations. This prevents direct recovery of the original focus measure signal from the synthetic images. Nevertheless, for an algorithm with multiple steps of operations, as long as there exists one linear operation before all non-linear steps, a reconstruction step can be inserted to recover not the focus measure value directly, but rather the result of the last linear operation. An example is illustrated by the diagram in Fig. 3.2.

Fortunately, many efficient focus measure algorithms indeed have at least one linear operation before all non-linear operations. Take the Modified Laplacian (LAPM) for example [PPG13]. A one dimensional Laplacian filter $M = \begin{bmatrix} -1 & 2 & -1 \end{bmatrix}$ is firstly constructed. Then the image is filtered in both x and y directions respectively. The absolute value of the two filtered images are summed as the final focus measure value. Apparently the 1D filtering operation as a convolution is linear while taking the absolute value is non-linear. Therefore the reconstruction step must be inserted before taking the absolute value.

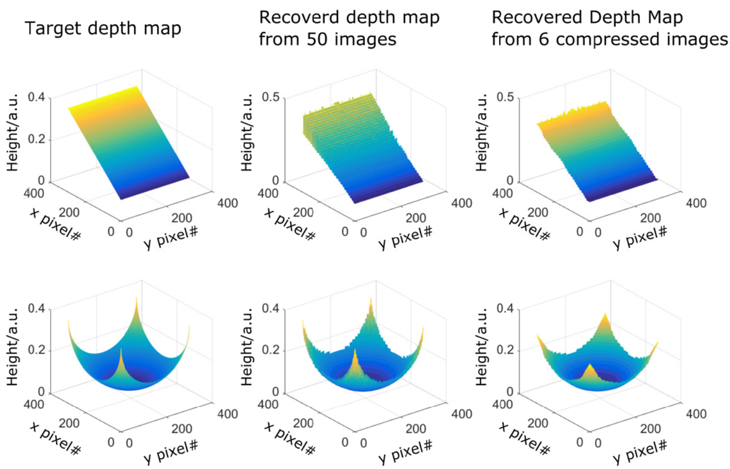

Figure 3.3: Simulation result of compressed shape from focus.

Simulation with this method is performed with synthetic images for two surface profiles. One is a linear ramp while the other one is part of a sphere. Both surface profiles are overlaid with the same texture map for the synthesizing of an axial image sequence with 50 images. With the conventional shape from focus method, LAPM is applied to all 50 images for focus measure, and axial position with the maximum value is taken as the axial position for each pixel. Meanwhile, based on the prior information provided by LAPM and the imaging system, a measurement matrix is constructed to linearly combine the image sequence to form six compressed images. LAPM is applied to these six images with reconstruction before the absolute value operation. The result of both approaches are illustrated in Fig. 3.3. It is clearly demonstrated that such compressed measurement yields comparable result to conventional methods, but largely reduces the number of images to be taken as well as the following image processing tasks.

In practice, the realization of the linear measurement can be achieved in many different ways. One possible implementation is to take a chromatic objective coupled with spectrally filtered illumination/detection. Another approach could be to change the focus position within each exposure with varying speed derived from the constructed measurement matrix, possibly through liquid lenses.

Fundamentally, the conventional compressive sensing is closely related to such linear measurement model with prior knowledge. Both can be compared in a classical linear algebra picture. Conventional compressive sensing considers the linear projection from *a collection of hyperplanes* (derived from *sparsity of the signal*) to a lower dimensional manifold and the recovery of the point in such *hyperplanes* through its counter part in such lower dimensional manifold. As a manifold learning problem, linear measurement model with prior knowledge considers the linear projection from *a particular subspace* (derived from *prior knowledge of the signal*) to a lower dimensional manifold and the recovery of the point in such *subspace* through its counter part in such lower dimensional manifold.

4 Optical Computation

It has been shown that with the linear measurement model, the process of conducting a measurement is much simplified. In compressive sensing, this is realized under the assumption of the signal sparsity while in a linear manifold learning problem, this is achieved with the help of prior knowledge. Nevertheless, signal reconstruction is a necessary step in both approaches.

Fig. 4.1 lists three different information embedding and recovery schemes. In the first case, the signal to be measured and the information to be retrieved are equivalent, for example an image taken for the pure target of recording the scene. In most situations, the information and the signal are not exactly the same thing, like illustrated in the second and third cases. Both Sec. 2 and Sec. 3 are dealing with the second case where the signal is reconstructed from the measurement and then a postprocessing step is implemented to retrieve the target information from the signal. This is often not the optimum case since the process between measurement and signal and the process between signal and information are treated separately. In this section, a different approach is proposed to treat the complete process holistically to further improve the efficiency of the measurement. A major assumption made beforehand is that in many practical measurement scenarios, the information embedded in the signal residing in a high dimensional manifold is often of lower dimensions by itself.

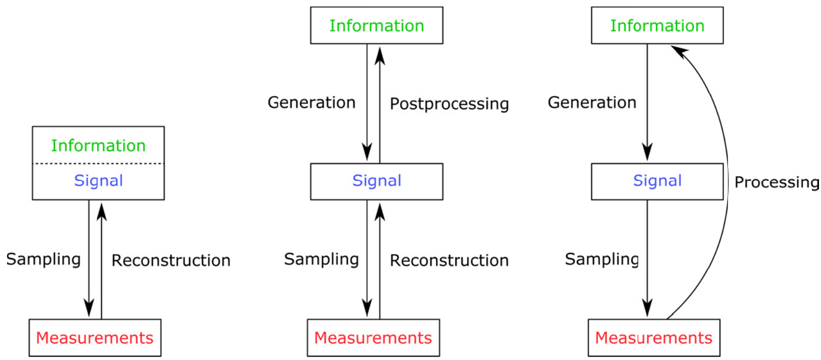

Figure 4.1: Different information embedding and recovery schemes.

Take the quasi-Gaussian signal for example. In the scenarios discussed in previous sections, a one dimensional target information, the axial position of the sample, is embedded in this peak function with a non-linear and implicit manner. After the peak function (signal) is reconstructed from the measurement, a further step is required to retrieve the information, such as through the location of the peak/centroid of the signal.

In fact, this postprocessing step can be effectively reduced/minimized by coupling it into the sampling process. To retrieve the centroid of the signal, a "centroid" preserving measurement matrix can be constructed for the measurement. Construction of such a matrix can be performed using Bernstein Polynomials.

For example, Fig. 4.2 illustrates the measurement results of a Gaussian signal with measurement matrices constructed from Bernstein Polynomials of different degrees. Regardless of the degrees of the polynomials, it can be proven mathematically that the measurement always preserves the centroid of the original signal. One idea to implement such a measurement is to build a chromatic probe using one linear interference filter as the first degree Bernstein polynomial. In this case the height of the sample can be retrieved with minimum processing. Schematic of such optical system is shown in Fig. 4.3

The fundamental difference of this approach with respect to the approaches proposed in previous sections is that the computation/processing part is largely shifted to the optical system through holistic design of the measurement setup.

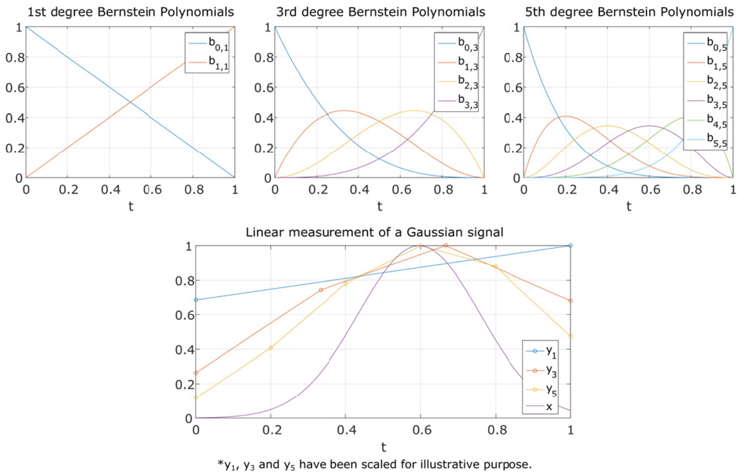

Figure 4.2: Linear measurement of a Gaussian signal with different degrees of Bernstein Polynomials.

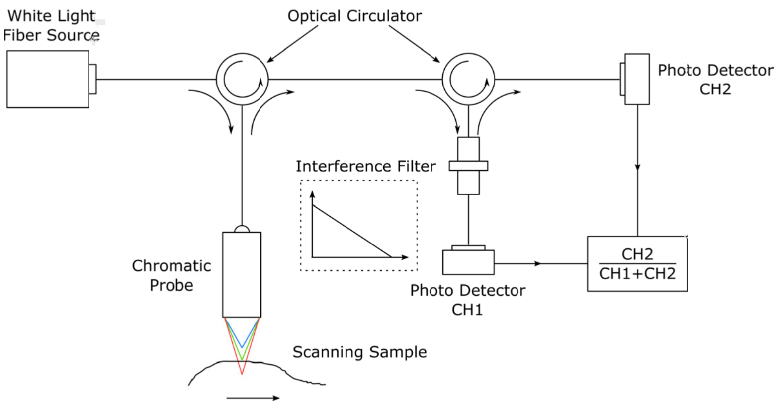

Figure 4.3: Schematic of a two-channel chromatic confocal measurement setup using first degree Bernstein Polynomials to retrieve centroid location directly.

In the proposed chromatic system, the measurement matrix is achieved by optical filters. This not only alleviates the computational burden in many systems, but more importantly, reduces the effect of the optoelectronic noises which propagates in the processing chain in conventional optical measurement systems.

5 Conclusion

The discussion presented in this paper is centered around the linear measurement model $\mathbf{y} = A\mathbf{x}$, where the measurements are linear combinations of the target signal. By making the assumption that the target signal is sparse, this model turns into a compressive sensing problem. In spite of the potential it has demonstrated, the computational cost has prevented compressive sensing from being applied in real-time measurement systems at the moment. With prior information of target signal taken into consideration, the linear measurement model is transformed into a linear manifold learning problem, where classical methods like PCA can be utilized to construct the measurement matrix which allows direct recovery of the signal from the measurement. Examples of chromatic confocal measurement and shape from focus are presented to demonstrate the potential of such approach in real world optical measurement systems. Last but not least, a holistic approach is proposed in order to retrieve the information with minimum postprocessing, allowing optical computation in measurement systems, which is both faster and more robust against noise.

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