Equivalence of Turn-Regularity and Complete Extensions

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Abstract: The aim of the two-dimensional compaction problem is to minimize the total edge length or the area of an orthogonal grid drawing. The coordinates of the vertices and the length of the edges can be altered while all angles and the shape of the drawing have to be preserved. The problem has been shown to be *NP*-hard.
Two commonly used compaction methods are the turn-regularity approach by (Bridgeman et al., 2000) and the approach by (Klau and Mutzel, 1999) considering complete extensions. We formally prove that these approaches are equivalent, i. e. a face of an orthogonal representation is turn-regular if and only if there exists a unique complete extension for the segments bounding this face.

1 INTRODUCTION

The compaction problem has been one of the challenging tasks in graph drawing for many years, as orthogonal drawings suffered from insufficient compaction algorithms, and as compaction plays an important role for various applications.

To give a practical example: In VLSI design the vertices of a graph represent electrical components, such as transistors, contacts, or logic gates, while the edges represent wires connecting these components (Lengauer, 1990). Minimizing the area, width, height or total edge length of such an orthogonal drawing representing a chip layout, with a certain distance between all the electrical components though, is essential for this use case.

Schematic drawings are almost always subject to size limitations. Thus, apart from VLSI design, compaction is important in many other contexts in information visualization. The types of drawings and domains range from UML diagrams in the area of software engineering, via entity-relationship diagrams for database management, through to subway maps (Batini et al., 1984), (Batini et al., 1986), (Tamassia, 1987), (Tamassia et al., 1988), (Di Battista et al., 1995), (Eiglsperger, 2003), (Eiglsperger et al., 2003).

1.1 Topology-Shape-Metrics Scheme

Orthogonal grid drawings are usually generated in three phases, according to the topology-shape-metrics scheme (Batini et al., 1986):

In the first phase, the graph is planarized, i. e. a plane embedding is computed while the number of crossings is minimized. For non-planar graphs, edge crossings are replaced by artificial vertices.

In the second phase, the orthogonal shape of the drawing is determined. This means that bends along the edges and the angles between the edges around each vertex are determined while the number of bends is minimized.

In the third phase, the drawing is compacted. Here, the coordinates of the vertices are mapped to a grid and the length of each edge is determined while the shape of the drawing is preserved. The goal of the compaction phase is to minimize the total edge length of the drawing. Patrignani (2001) has proven this problem to be *NP*-hard; Bannister and Eppstein (2012) have given inapproximability results for non-planar drawings.

1.2 State-of-the-Art

Since the orthogonal compaction problem is *NP*-hard, for a long time one-dimensional compaction heuristics were applied. These one-dimensional heuristics transform the two-dimensional compaction problem into two one-dimensional problems, and solve them by applying minimum-cost flow techniques (Tamassia, 1987), (Eiglsperger et al., 2001). In one dimension – either the x- or y-dimension – the orthogonal drawing is considered to be fixed, while in the other dimension the coordinates of the vertices can be altered. This, in general, does not lead to optimal results.

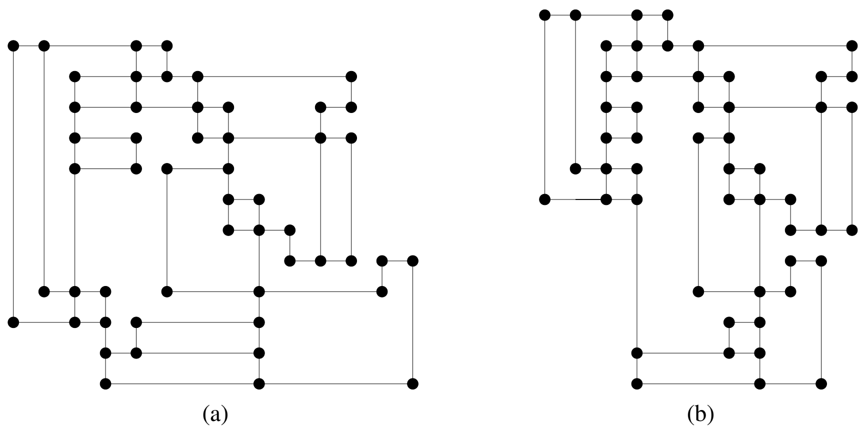


Figure 1: Drawings for the same graph and the same orthogonal shape – with different total edge length.

The complete-extension approach by (Klau and Mutzel, 1999) was one of the first approaches not splitting up the compaction problem into two one-dimensional problems but solving it as a whole, by formulating it as an integer linear program (ILP).

Eiglsperger and Kaufmann (2002) presented a linear-time heuristic building up on the basic idea of (Klau and Mutzel, 1999). Recent research on the compaction problem was published by (Jünger et al., 2018), who allow to change the orthogonal shape of an edge and, under this condition, present a polynomial-time algorithm. An experimental comparison of compaction methods can be found in (Klau et al., 2001). For an overview of compaction heuristics see e. g. (Eiglsperger et al., 2001).

Another challenge when assigning coordinates to the vertices is to prevent collisions. First heuristics required all faces to be rectangular (Tamassia, 1987). If all faces are rectangular already, the compaction problem can be solved to optimality in polynomial time (Di Battista et al., 1999). Otherwise, Tamassia (1987) inserts artificial edges until all faces are rectangular. If these artificial edges, however, are randomly oriented horizontally or vertically, an optimal drawing is no longer guaranteed. Contrariwise, iterating over all possible ways of inserting artificial edges would take exponential time.

Figure 1 illustrates how a bad local decision – when considering only one dimension of the drawing, or when inserting a "bad" artificial edge – can affect the drawing as a whole. The drawing in Figure 1(a) has a total edge length of 128 units, the drawing in Figure 1(b) – for the same graph and the same orthogonal embedding – has a total edge length of 111 units.

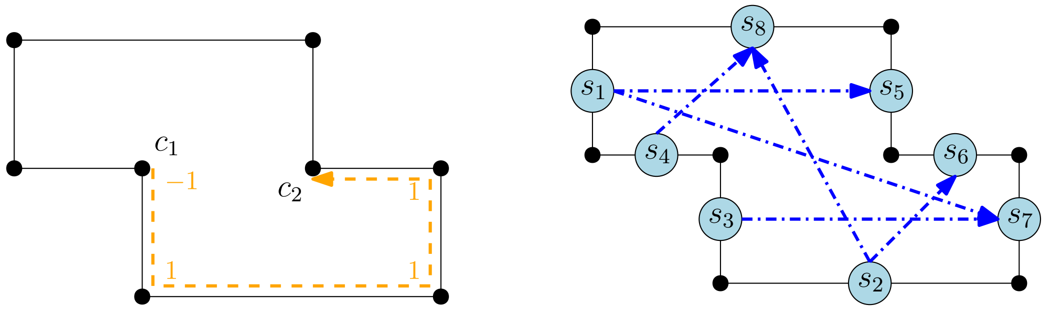
With the turn-regularity approach, Bridgeman et al. (2000) presented a more sophisticated way to prevent collisions. They first determine all vertices which could potentially collide, vertices with so-called *kitty corners*. Only between these pairs of vertices artificial edges are inserted in order to separate these vertices either horizontally or vertically. Thereby, the turn-regularity approach practically requires much less artificial edges than rectangular approaches (Esser, 2014). This becomes important if the problem is solved as an ILP. Then, less artificial edges mean less constraints. This avoids inserting needless place-holders to the drawing.

Nowadays, the turn-regularity approach by (Bridgeman et al., 2000) and the complete-extension approach by (Klau and Mutzel, 1999) are *the* methods for solving the compaction problem. We will prove the equivalence of both approaches, more precisely: A face of an orthogonal representation is turn-regular (as defined in the first approach) if and only if the segments bounding this face are separated or can uniquely be separated (as defined in the second approach). This is, to our best knowledge, the first formal proof of equivalence of both approaches.

This paper is organized as follows: Section 2 summarizes the main ideas of the turn-regularity approach and the complete-extension approach, in Section 3 we prove their equivalence, before we summarize our results in Section 4.

2 COMPACTION METHODS

In this section we describe both the turn-regularity approach and the complete-extension approach. We present basic definitions and theorems from (Bridgeman et al., 2000) and (Klau and Mutzel, 1999), which are required for further conclusions. For more details on graph drawing in general, orthogonal drawings, and compaction, see e. g. (Di Battista et al., 1999), (Kaufmann and Wagner, 2001), or (Tamassia, 2013).



(a) Kitty corners.　　(b) Incomplete shape description.

Figure 2: Basic idea of the turn-regularity approach and the complete-extension approach.

Let $G = (V, E)$ be an undirected 4-graph, consisting of a set of n vertices V, and a set of m edges E. A graph is a 4-*graph* if it is planar, i. e. it admits a drawing in the plane without any edge crossings, and if all vertices have at most four incident edges. Such a drawing of G in the plane induces a *planar embedding* and especially specifies the faces in the drawing – multiple internal faces and one external face.

Let H be an orthogonal representation of G. An *orthogonal representation* is an extension of a planar embedding which contains additional information about the orthogonal shape of the drawing, i. e. information about bends along the edges and about the angles between consecutive edges – 90°, 180°, 270°, or 360° angles. Convex (90°) and reflex (270°) corners are especially important for the turn-regularity approach. Defining the bends and angles in H implicitly specifies which edges are horizontal and which are vertical.

Hereinafter, w.l.o.g. H is assumed to be *simple*, i. e. free of bends, and connected. Existing bends can beforehand be replaced by artificial vertices. If H is not connected, each connected component can be processed separately.

Let Γ be a planar orthogonal grid drawing of H. In addition to H, Γ contains information about the coordinates of each vertex on the grid and about the length of each edge.

Given H, the *2-dimensional compaction problem* is to find a planar orthogonal grid drawing Γ of H with minimum total edge length. We will focus on this formulation of the compaction problem. Variations of this problem, where the area of Γ or the length of the longest edge is minimized, can be solved with nearly the same approaches.

2.1 Turn-Regularity Approach

The idea of the turn-regularity approach is to determine all pairs of vertices which could potentially collide. Unlike for original compaction heuristics (Tamassia, 1987), the faces are not required to be rectangular. The definitions and lemmata within this subsection have been adopted from (Bridgeman et al., 2000).

Definition 1 (Turn)**.** Let f be a face in H. To every corner c in f a **turn** is assigned:

$$\text{turn}(c) := \begin{cases} 1, & \text{if } c \text{ is a convex } 90^\circ \text{ corner,} \\ 0, & \text{if } c \text{ is a flat } 180^\circ \text{ corner,} \\ -1, & \text{if } c \text{ is a reflex } 270^\circ \text{ corner.} \end{cases}$$

Corners enclosing 360° angles are treated as a pair of two reflex corners. Bridgeman et al. (2000) have shown that it is sufficient to replace each 360° vertex by two artificial 270° vertices connected by an artificial edge, to subsequently compact the drawing, and to finally substitute the artificial vertices by the original one again.

Thus, in the following we can assume G to be biconnected, i. e. if an arbitrary vertex was removed from G, G would still remain connected.

Every reflex corner either is a north-east, south-east, south-west or north-west corner. If it is clear which face is considered, we will also speak of north-east, south-east, south-west, and north-west vertices which have a respective corner in this face.

Based on turn(c), Bridgeman et al. (2000) defined the rotation:

Definition 2 (Rotation)**.** Let f be a face in H. The **rotation** of an ordered pair of corners (c_i, c_j) in f is defined as

$$\text{rot}(c_i, c_j) := \sum_{c \in P} \text{turn}(c),$$

where P is a path along the boundary of f from c_i (included) to c_j (excluded) in counter-clockwise direction.

For simplifying notation, if it is clear which face is considered, we also write rot(v_i, v_j) for two vertices v_i, v_j with corresponding corners c_i, c_j.

(a) One-dimensional rectangular approach.

(b) Original turn-regularity approach with randomly oriented artificial edges.

(c) Turn-regularity approach in combination with branch-and-cut methods.

Figure 3: Graph from the "Rome" dataset compacted by different heuristics with regard to a minimum total edge length.

Lemma 3. *Let f be a face in H.*

(i) For all corners c_i in f it holds:

$$rot(c_i, c_i) = \begin{cases} 4, & \text{if } f \text{ is an internal face,} \\ -4, & \text{if } f \text{ is the external face.} \end{cases}$$

(ii) For all corners c_i, c_j in f the following equivalence holds:

$$rot(c_i, c_j) = 2$$
$$\Leftrightarrow\ rot(c_j, c_i) = \begin{cases} 2, & \text{if } f \text{ is an int. face,} \\ -6, & \text{if } f \text{ is the ext. face.} \end{cases}$$

Definition 4 (Kitty Corners, Turn-regular)**.** Let f be a face in H. Two reflex corners c_i, c_j in f are named **kitty corners** if $\mathrm{rot}(c_i, c_j) = 2$ or $\mathrm{rot}(c_j, c_i) = 2$. A face is **turn-regular** if it contains no kitty corners.

Figure 2(a) shows two kitty corners c_1, c_2 within an internal face. The rotation $\mathrm{rot}(c_1, c_2)$ (orange, dashed) sums up to 2.

The kitty corners in H can be determined in a runtime of $O(n)$ (Bridgeman et al., 2000). If all faces are turn-regular, an optimal drawing can be computed in polynomial time by applying minimum-cost flow techniques as in (Tamassia, 1987).

Otherwise, all non-turn-regular faces are made turn-regular. Thereby, the turn-regularity approach defines a heuristic:

1. Determine all non-turn-regular faces.
2. Insert an artificial edge between each pair of kitty corners.
3. Apply minimum-cost flow techniques to determine the final length of each edge.
4. Remove the artificial edges.

2.2 Complete-Extension Approach

The complete-extension approach mainly considers segments, not single edges. A horizontal *subsegment* is a set of connected horizontal edges. Note that each edge is a subsegment itself. A horizontal *segment* is a maximally connected horizontal subsegment. This means that there is no other connected horizontal edge which could be added to this set. Vertical subsegments and segments are defined analogously.

For a subsegment s, $l(s)$ denotes the vertical segment containing the leftmost vertex of s; $r(s)$ is the vertical segment with the rightmost vertex of s, $b(s)$ and $t(s)$ are the horizontal segments with the bottommost and topmost vertex of s. Note that two different subsegments or edges e_1, e_2 can have the same left, right, top or bottom segment, e. g. $l(e_1) = l(e_2)$.

The idea of the complete-extension approach by (Klau and Mutzel, 1999) is to transform the compaction problem into a combinatorial problem. For this purpose so-called shape descriptions are used. The definitions within this subsection have been adopted from (Klau and Mutzel, 1999).

Definition 5 (Shape Description, Constraint Graph)**.** A **shape description** of the simple orthogonal representation H is a tuple $\sigma = \langle D_h, D_v \rangle$ of two directed so-called **constraint graphs** $D_h = (S_v, A_h)$ and $D_v = (S_h, A_v)$ with

$$\begin{aligned} A_h &:= \{(l(e), r(e)) \mid e \text{ horizontal edge in } G\}, \\ A_v &:= \{(b(e), t(e)) \mid e \text{ vertical edge in } G\}, \end{aligned}$$

and two sets of corresponding vertices S_v, S_h.

The arcs in $A_h \cup A_v$ determine the relative position of every pair of segments. However, this information is generally not sufficient to produce an orthogonal embedding. The shape description might need to be extended.

Definition 6 (Separated Segments)**.** A pair of segments $(s_i,s_j) \in S \times S$, where $S := S_h \cup S_v$, is called **separated** if the shape description contains one of the four following paths:

1. $r(s_i) \xrightarrow{\star} l(s_j)$
2. $r(s_j) \xrightarrow{\star} l(s_i)$
3. $t(s_j) \xrightarrow{\star} b(s_i)$
4. $t(s_i) \xrightarrow{\star} b(s_j)$

A shape description is **complete** if all pairs of segments are separated.

The notation $a \longrightarrow b$ denotes an immediate path from vertex a to vertex b, while on the path $a \xrightarrow{\star} b$ intermediate vertices are allowed.

For two segments $(s_i,s_j) \in S$ there can exist multiple paths Definition 6, e. g. for two vertical segments in a rectangular face there exist two paths along the top way and along the bottom way.

Klau (2001) has shown that it is sufficient to consider only segments within the same face. If any two segments that share a common face are separated, the shape description is complete.

Definition 7 (Complete Extension)**.** A **complete extension** of a shape description $\sigma = \langle (S_v, A_h), (S_h, A_v) \rangle$ is a tuple $\tau = \langle (S_v, B_h), (S_h, B_v) \rangle$ with the following properties:

(P1) $A_h \subseteq B_h$ and $A_v \subseteq B_v$.

(P2) B_h and B_v are acyclic.

(P3) Every non-adjacent pair of segments in G is separated.

Figure 2(b) shows a shape description (blue, dash-dotted). The shape description is not complete, as there is no directed path from segment vertex s_3 to s_5 and from s_4 to s_6. If one of the arcs (s_3, s_5) or (s_4, s_6) is added, the other pair becomes separated as well. The shape description then is complete.

From every complete extension an orthogonal drawing can be constructed which respects all constraints of this extension. Thus, the task of compacting an orthogonal grid drawing is equivalent to finding a complete extension that minimizes the total edge length. Klau and Mutzel (1999) have formulated this task as an ILP which can be solved optimally. If a shape description is complete or uniquely completable, the compaction problem can be solved optimally in polynomial time (Klau and Mutzel, 1999).

Figure 3 shows a graph from the "Rome" dataset, which has been introduced by (Di Battista et al., 1997). The dataset consists of about 11,000 real-world graphs and is widely used for benchmarking graph drawing algorithms.

The drawing in Figure 3(a) (total edge length: 3,276 units) has been compacted by the original approach by (Tamassia, 1987), which reduces the problem to two one-dimensional problems and inserts artificial edges until all faces are rectangular.

For Figure 3(b) (total edge length: 3,154 units) the turn-regularity approach by (Bridgeman et al., 2000) has been applied. The original turn-regularity approach first determines all vertices which could collide, inserts artificial edges between these kitty corners, but then randomly orients these artificial edges – either horizontally or vertically. Thus, the approach generally does not lead to optimal results.

Figure 3(c) (total edge length: 2,882 units) shows an optimal drawing which has been created by an "extended" turn-regularity method. If we do not orient the artificial edges randomly, but if we apply branch-and-cut methods to iterate over all feasible combinations, this leads to an optimal solution. The complete-extension approach by (Klau and Mutzel, 1999) makes use of the same branch-and-cut techniques. It iterates over all possible combinations to extend incomplete shape descriptions, and – due to the equivalence, which we prove in Section 3 – the complete-extension approach will return an optimal drawing with the same total edge length. If there exist multiple optimal solutions, the drawings resulting from both approaches can differ due to local decisions, but will both have minimum total edge length.

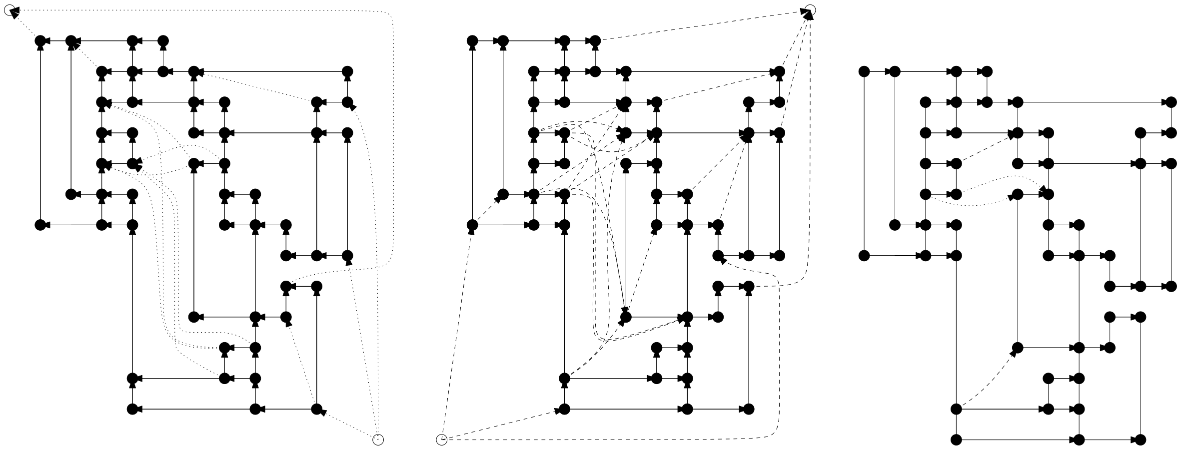
3 EQUIVALENCE

In this section we prove the equivalence of the turn-regularity approach and the complete-extension approach. Note that both approaches consider different components of a drawing. While turn-regularity is a definition based on the shape of faces, completeness is a definition based on segments, more precisely: on the vertices dual to the segments. However, a segment is a set of edges, so that a relation between segments and faces can be established. A segment s is said to *bound* a face f if one of the edges in s is on the boundary of f.

When introducing the concept of turn-regularity, Bridgeman et al. (2000) constructed auxiliary graphs G_l, G_r, H_x, and H_y (note that H_x, H_y are graphs, not orthogonal representations).

G_l is a variation of the original graph G in which all edges are oriented leftward or upward, in G_r all edges are oriented rightward or upward.

Bridgeman et al. (2000) augment G_l and G_r by so-called saturating edges. These saturating edges are constructed based on a previously defined switch property of the edges in E. Effectively, the saturating edges – or the "saturator" – simply form an acyclic directed graph with a source vertex s and a target



(a) Drawing of G_l (solid lines) and its saturator (dotted lines).

(b) Drawing of G_r (solid lines) and its saturator (dashed lines).

(c) Drawing of H_x.

Figure 4: Drawings of the auxiliary graphs G_l, G_r, and H_x.

vertex t. The saturator of G_l consists of

- additional source and target vertices s and t in the external face,
- an arc from s to every external south-east vertex,
- an arc from every external north-west vertex to t,
- an arc from every internal north-west vertex to its opposite convex vertex,
- an arc from the opposite convex vertex to any internal south-east vertex, and
- the subset of all affected vertices from V.

The saturator of G_r is defined analogously, with outgoing arcs from north-east vertices and incoming arcs towards south-west vertices.

Bridgeman et al. (2000) further introduced maximal vertical or horizontal unconstrained chains. A chain of segments in a face f is said to be *unconstrained* if both its end-vertices have a reflex corner.

Based on G_l, G_r, and their saturators, Bridgeman et al. (2000) constructed two more auxiliary graphs H_x, H_y:

H_x describes the x-, i. e the left-to-right relation between the segments in H. H_x contains all original edges from E and exactly those saturating edges from both G_l and G_r which are incident to end-vertices of a maximal unconstrained vertical chain. The original vertical edges are kept without orientation, the original horizontal edges are all oriented from left to right. The saturating edges in H_x are all oriented so that they point from left to right segments (i. e. the saturating edges from G_l are reversed). The source vertex s, the sink vertex t, and all their incident edges are omitted.

H_y, which denotes the bottom-to-top relation between segments, is constructed in a similar way. In H_y, the saturating edges are incident to end-vertices of maximal horizontal unconstrained chains, the vertical edges are all directed upwards, the horizontal ones are unoriented.

Figure 4 shows these auxiliary graphs G_l, G_r and, H_x for the drawing from Figure 1(b). H_y has not been illustrated, as it is – apart from the orientation of the edges – identical with the original graph. There do not exist horizontal unconstrained chains, so that H_y does not contain any saturating edges.

H_x in this example is not uniquely determined, because in the non-turn-regular face there exist various possibilities to add saturating edges. For every maximal unconstrained vertical chain two incident saturating edges have been chosen.

Bridgeman et al. (2000) have proven various characteristics of these auxiliary graphs, in particular:

Lemma 8. *H_x and H_y are uniquely determined if and only if H is turn-regular.*

Proof. See (Bridgeman et al., 2000, p. 71). □

We will use this statement when proving the forward direction of the theorem on equivalence.

Bridgeman et al. (2000) later used Lemma 8 to show that in a turn-regular orthogonal representation there exist so-called orthogonal relations between every two vertices (Bridgeman et al., 2000, Theorem 5). This is where the link to the complete-extension approach is established. A complete extension of a shape description also means that there is some unique relation between every two segments.

Theorem 9. *A face f in H is turn-regular if and only if the segments bounding f are separated or can uniquely be separated.*

Proof. Forward direction: Let f be turn-regular. We can apply Lemma 8 and conclude that H_x and H_y are uniquely determined. Thus, it needs to be proved that H_x and H_y induce a complete extension.

The horizontal arcs in H_x are all directed in right direction, the vertical arcs in H_y in top direction. Thus, from the vertical segments we can deduce a set of segment vertices S_v. From the horizontal arcs in H_x we can deduce a set of horizontal left-to-right arcs A_h connecting these segment vertices. Then, every two segment vertices in S_v are connected by a sequence of arcs from A_h if there is a corresponding path in H_x. Thus, the vertical segments and the horizontal arcs in H_x induce a constraint graph $D_h = (S_v, A_h)$. In the same way, a second constraint graph $D_v = (S_h, A_v)$ can be deduced from the horizontal segments and the vertical arcs in H_y.

Let B_h contain all arcs from A_h and all saturating edges from H_x. Let B_v contain all arcs from A_v and all saturating edges from H_y.

It remains to show that $\tau = \langle (S_v, B_h), (S_h, B_v) \rangle$ is a complete extension of $\sigma = \langle D_h, D_v \rangle$. We will prove that the three properties from Definition 7 are fulfilled. As A_h and A_v have been augmented by additional arcs, for (P1) it obviously holds $A_h \subseteq B_h$ and $A_v \subseteq B_v$.

Regarding (P2), B_h and B_v are acyclic by construction. A_h only contains left-to-right arcs. The arcs from $B_h \setminus A_h$ do not close any cycles in D_h as they establish new left-to-right relations between previously unconnected vertices, i. e. they retain the left-to-right-order of D_h. The same argument applies to D_v and the arcs from $B_v \setminus A_v$.

For (P3) we can argue that the saturating edges in H_x and H_y were added at the ends of maximal unconstrained chains – which exactly bound non-separated segments. The end-vertices of a maximal unconstrained chain both correspond to reflex corners.

This means that, without the saturating edges, the unconstrained chain is non-separated from any other segment. From the other segment vertices in $S_h \cup S_v$ there either exist only incoming or only outgoing arcs to the segment vertices of this unconstrained chain. Without the saturating edges, a path coming in to the unconstrained chain and then going out to another segment, as the paths in Definition 6, could not be established. The saturating edges, however, allow exactly these paths, because they are incident with the end-vertices of unconstrained chains and facilitate an additional way towards or away from these end-vertices. As the saturating edges were added at the ends of all maximal unconstrained chains, all segments are separated.

Summarized, we can deduce a complete extension from H_x and H_y. As H_x and H_y are uniquely determined, this complete extension is unique. Thus, for any turn-regular face f the segments bounding f are already separated or can uniquely be separated.

Backward direction: One possible way to prove the backward direction is to give a constructive proof, to transform an arbitrary complete extension into auxiliary representations H_x, H_y, and to finally apply Lemma 8 again. Another way is to translate the language of complete extensions into the language of turn-regularity, and to argue by the rotation. We will do the latter.

Let f be non-turn-regular, i. e. in f there exists at least one pair of kitty corners (c_1, c_2) with $\mathrm{rot}(c_1, c_2) = 2$. Denote the corresponding vertices by u_1, u_2. Let s_1^h, s_1^v be the horizontal and vertical segment incident to u_1, and let s_2^h, s_2^v analogously be the incident segments to u_2. Figure 5 illustrates this setting.

Let $\mathrm{first}_p(s)$ and $\mathrm{last}_p(s)$ be the first and last vertex, respectively, of a segment s on a path p.

Consider the vertical segments s_1^v, s_2^v and the path $p = u_1 \xrightarrow{\star} \mathrm{last}_p(s_1^v) \xrightarrow{\star} \mathrm{first}_p(s_2^h) \xrightarrow{\star} u_2$. In Figure 5, p is the path along the bottom way.

If f is an internal face, it holds:

$$\begin{aligned} \mathrm{rot}(u_1, u_2) &= 2 \\ &= -1 + \underbrace{\mathrm{rot}(\mathrm{last}_p(s_1^v), u_2)}_{=3} \end{aligned} \qquad (1)$$

As the rotation along the subpath $\mathrm{last}_p(s_1^v) \xrightarrow{\star} u_2$ is 3, on this path there must exist at least three convex corners and at least one other vertical segment $s^\star$. In the constraint graph D_h, the arc between s_1^v and $s^\star$ and the arc between s_2^v, $s^\star$ must both be either incoming or outgoing due to the rotation.

In Figure 5 both arcs to $s^\star$ are incoming, drawn as dash-dotted blue arcs.

For the sake of completeness note that if there are additional vertical segments on the subpath $\mathrm{last}_p(s_1^v) \xrightarrow{\star} u_2$, there will not be immediate arcs between s_1^v and $s^\star$ or between s_2^v and $s^\star$, but a longer sequence of incoming or outgoing arcs.

Summarized, as both arcs are either incoming or outgoing, path p from s_1^v to s_2^v allows none of the four connections from Definition 6.

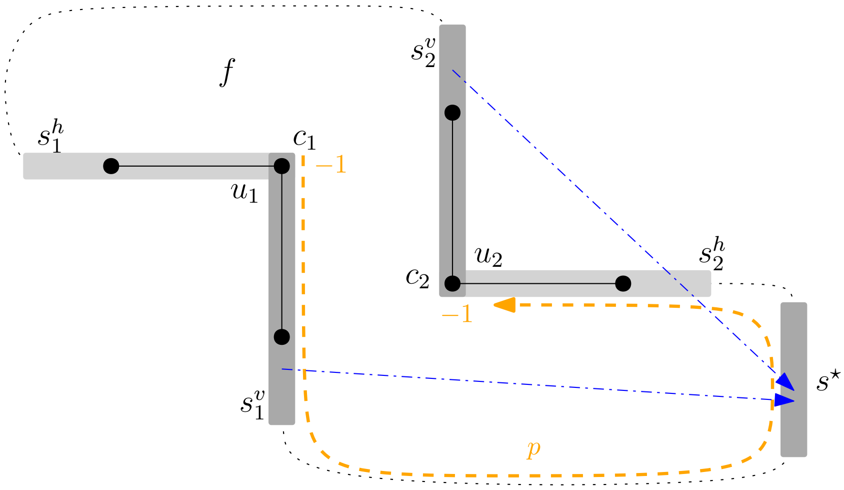


Figure 5: Illustration of the proof of equivalence.

Following from Lemma 3, it also holds:

$$\begin{aligned}\mathrm{rot}(u_2,u_1) &= 2 \\ &= -1+\underbrace{\mathrm{rot}(\mathrm{last}_p(s_2^v),u_1)}_{=3} \qquad (2)\end{aligned}$$

Thereby, the same conclusion as above can be shown for the other path $q = u_2 \xrightarrow{\star} \mathrm{last}_q(s_2^v) \xrightarrow{\star} \mathrm{first}_p(s_1^h) \xrightarrow{\star} u_1$ from s_2^v to s_1^v. In Figure 5, q is the path along the upper way.

Thus, s_1^v and s_2^v are not separated.

If f is the external face, the following two equations (or vice versa) apply and the same conclusion can be deduced:

$$\mathrm{rot}(u_1,u_2) = 2 = -1+\mathrm{rot}(\mathrm{last}_p(s_1^v),u_2) \qquad (3)$$

$$\mathrm{rot}(u_2,u_1) = -6 = -1+\mathrm{rot}(\mathrm{last}_p(s_2^v),u_1) \qquad (4)$$

When considering the horizontal segments s_1^h, s_2^h it can be argued in the same way that these segments are not separated. Thus, neither the vertical segments s_1^v, s_2^v nor the horizontal segments s_1^h, s_2^h are separated.

It remains to show that there is no unique way to complete the shape description. This is because it holds $u_1 = r(s_1^h) = t(s_1^v)$ and $u_2 = l(s_2^h) = b(s_2^v)$. This means that there are two possible ways to complete shape description, not only one. As soon as – either in the horizontal or the vertical constraint graph – a path completing the shape description is chosen, the respective segments in the other constraint graph will be separated as well.

Thus, the shape description is not complete and a complete extension cannot uniquely be chosen. □

The forward direction of the proof induces an algorithm for transferring the turn-regularity formulation into the complete-extension formulation. For turn-regular orthogonal representations the auxiliary graphs G_l, G_r, their unique saturators, and H_x, H_y can be constructed in $O(n)$ time (Bridgeman et al., 2000, Proof of Theorem 8), (Di Battista and Liotta, 1998). Replacing each segment by a segment vertex and constructing the constraint graphs can also be done in linear time. This means that the turn-regularity formulation can be converted into the complete-extension formulation – when not having to choose one of many saturators – in $O(n)$ time.

The compaction problem becomes *NP*-hard if the orthogonal representation is not turn-regular and one has to choose from a possibly exponential number of saturators or, in the complete-extension formulation, from a possibly exponential number of complete extensions.

4 CONCLUSION

We have shown that the turn-regularity approach by (Bridgeman et al., 2000) and the complete-extension approach by (Klau and Mutzel, 1999) are equivalent: A orthogonal representation is turn-regular if and only if there exists a unique complete extension. In the first approach, one must decide how to align new artificial edges in non-turn-regular faces. In the second approach, one must decide how to extend incomplete shape descriptions. Our theorem means that both decisions are equivalent.

Klau and Mutzel (1999) have formulated the compaction problem as an ILP. The equivalence of both approaches suggests that there also exists an ILP formulation for the turn-regularity approach. In (Esser, 2014) a restricted ILP formulation under certain conditions has been presented. We plan to present a general ILP formulation.

Moreover, we want to discuss the meaning of the equivalence from a practical perspective. Which of both approaches – in practice – suits best to which use case? Which approach is more efficient for which visualization problem?

We further intend to apply compaction techniques to the area of document processing. Here, a common issue is to correctly extract tables from documents. These tables could be interpreted as orthogonal drawings.

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