# A HIERARCHY OF MODELS FOR MULTILANE VEHICULAR TRAFFIC I: MODELING 

A. Klar<br>Fachbereich Mathematik, Universität Kaiserslautern<br>Kaiserslautern, Germany<br>R. Wegener<br>Institut für Techno- und Wirtschaftsmathematik<br>Kaiserslautern, Germany

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#### Abstract

In the present paper multilane models for vehicular traffic are considered. A microscopic multilane model based on reaction thresholds is developed. Based on this model an Enskog like kinetic model is developed. In particular, care is taken to incorporate the correlations between the vehicles. From the kinetic model a fluid dynamic model is derived. The macroscopic coefficients are deduced from the underlying kinetic model. Numerical simulations are presented for all three levels of description in [10]. Moreover, a comparison of the results is given there.


## 1 Introduction

There are essentially three types of approaches towards the modeling of traffic flow phenomena. The first and most basic one concerns microscopic or follow the leader models, modeling the actual response of single cars to their predecessor, see, e.g., [2, 23, 3]. Macroscopic models based on fluid dynamic equations have been proposed by a large number of authors, see, e.g., [22, 16, 11, 7, 5, 20]. However some of these models have been subject to a considerable controversy, concerning their validity and applicability to traffic flow. Kinetic or Boltzmann-like models [18], [12, 15, 17], $[14,13]$ may present an intermediate step between the above two types of models. On the one side, they can be derived from microscopic considerations. On the other side, fluid dynamic models can be derived from kinetic traffic models as has been shown in a heuristic way, e.g., in $[19,17,15,4]$. A multilane model has been considered,
e.g., in [6]. In [21, 9] a new kinetic model is described and numerically investigated. For a survey on the different types of models, see [8].

In the above mentioned papers multilane effects are usually included in a cumulative way neglecting the exact dynamics of the multilane model. In this paper we are especially concerned with derivation procedures and the links between the different levels of the hierarchy for a full multilane model. Each level is derived from the lower one. The derivation is supported by numerical results. Quantities like the vehicle distributions, the distribution of the leading vehicles and the equilibrium values for mean velocity (fundamental diagram), traffic pressure, etc. are determined numerically on different levels and compared to each other.

The paper is organized in the following way: In Section 2 we describe a microscopic multilane model using so called reaction thresholds. The values of this thresholds are given by investigations of the behaviour of individual drivers. The basic assumption in this model is that the time scales allow an instantaneous treatment of the interactions. Based on this model we derive in Section 3 a new kinetic multilane model. Correlations between the vehicles are taken into account by an ansatz for the leading vehicle distribution. This distribution is also used to determine the probability for lane changing. From the multilane model a cumulative model is developed. Section 4 contains the derivation of fluid dynamic equations. The stationary solution of the homogeneous cumulative kinetic equation is used to determine the coefficients in the macroscopic equations. This leads to a better foundation of the macroscopic model and an explanation of the coefficients in the model. The links between the different models are summarized in Figure 1.

Numerical investigations of the equations derived in this part are presented in [10]. This work is in the following refered to as II.

## 2 The Microscopic Model

In this section we present a microscopic model based on the description of individual cars. The model is based on reaction thresholds. The cars change velocity and lanes instantaneously, once certain reaction thresholds are crossed, i.e. once the distance between a car and its following or leading car is becoming larger or smaller than the threshold distance. As long as no threshold is crossed, the cars move with their respective velocities in free motion. The thresholds usually depend on the velocities of the cars.

We consider a highway with $N$ lanes. For the notations see Figure 2. The car under consideration is denoted by $c$. Leading car and follower on the same line are $c_{+}$and $c_{-}$. On the left and right lanes they are denoted by $c_{l_{+}}, c_{l_{-}}$and $c_{r_{+}}, c_{r_{-}}$, respectively. Velocities are denoted using the same subscripts. Velocities before an interaction


Figure 1: Links between models


Figure 2: Notations
are denoted by $v$ after the interaction by $v^{\prime}$. The maximal velocity is denoted by $w$, i.e. the velocities range between 0 and $w$.

We introduce the following thresholds for lane changing to the left $\left(H_{L}\right)$, lane changing to the right $\left(H_{R}\right)$, braking $\left(H_{B}\right)$, acceleration $\left(H_{A}\right)$ and free driving $\left(H_{F}\right)$ :

$$
\begin{aligned}
H_{L}(v) & =H_{0}+v T_{L} \\
H_{R}(v) & =H_{0}+v T_{R} \\
H_{B}(v) & =H_{0}+v T_{B} \\
H_{A}(v) & =H_{0}+\delta+v T_{A} \\
H_{F} & =H_{0}+\delta+w T_{F} .
\end{aligned}
$$

$T_{L}, T_{R}, T_{B}, T_{A}, T_{F}$ are the reaction times and $H_{0}$ denotes the minimal distance between the vehicles. These constants can be determined on an experimental basis, see, e.g., [8] for typical values. $\delta$ is a constant accounting for the fact that acceleration is done with a certain delay in comparison with braking. Moreover, we introduce the additional lines

$$
\begin{aligned}
H_{L}^{S}(v) & =H_{0}+v T_{L}^{S} \\
H_{R}^{S}(v) & =H_{0}+v T_{R}^{S}
\end{aligned}
$$

denoting the space required on the left and right lane for a changing car. We assume the following ordering of the lines:

$$
T_{F} \geq T_{A}>T_{L}>T_{R}>T_{B}
$$

and

$$
T_{L}^{S}, T_{R}^{S} \geq T_{B}
$$

In other words, braking takes place at a minimal safety distance, acceleration at a larger distance. For a changing car the available space must be at least such that braking is still possible.

In the following the possible interactions, taking place at the thresholds, are collected. We distinguish between two types of interactions. First interactions, where two lanes are involved, are considered.

Interaction 1 (Lane Changing to the left):
If the velocity of the car under consideration is larger than the velocity of the leading car $v>v_{+}$, i.e. the cars are approaching each other and the lane changing to the
left line $H_{L}(v)$ is crossed, then the following takes place: The car will change to the left lane only if there is enough space on the left line, i.e. if

$$
x_{l_{+}}-x>H_{L}^{S}(v), \quad x-x_{l_{-}}>H_{L}^{S}\left(v_{l_{-}}\right) .
$$

Moreover, $c$ and $c_{-}$are accelerated after the lane change, if there is enough space:

$$
v^{\prime}=\left\{\begin{array}{ll}
\tilde{v} & \text { if } x_{l_{+}}-x>H_{F} \\
v & \text { else }
\end{array}, \quad v_{-}^{\prime}= \begin{cases}\tilde{v}_{-} & \text {if } x_{+}-x_{-}>H_{F} \\
v_{-} & \text {else }\end{cases}\right.
$$

where $\tilde{v}, \tilde{v}_{-}$are distributed due to a probability distribution of desired speeds of the drivers with density $f_{D}$, i.e., for example,

$$
\tilde{v}=F_{D}^{-1}(\xi)
$$

with $F_{D}(v)=\int_{0}^{v} f_{D}(\hat{v}) d \hat{v}$. Here $\xi$ is a random variable uniformly distributed on $(0,1)$.

## Interaction 2 (Lane Changing to the right):

If the velocity of the car under consideration is smaller than the velocity of the following car $v_{-}>v$ and the lane changing to the right line $H_{R}\left(v_{-}\right)$is crossed, then the following takes place: The car will change to the right lane only if there is enough space on the right line, i.e. if

$$
x_{r_{+}}-x>H_{R}^{S}(v), \quad x-x_{r_{-}}>H_{R}^{S}\left(v_{r_{-}}\right) .
$$

Moreover, $c$ and $c_{-}$are accelerated after the lane change, if there is enough space:

$$
v^{\prime}=\left\{\begin{array}{ll}
\tilde{v} & \text { if } x_{r_{+}-x>} H_{F} \\
v & \text { else }
\end{array}, \quad v_{-}^{\prime}= \begin{cases}\tilde{v}_{-} & \text {if } x_{+}-x_{-}>H_{F} \\
v_{-} & \text {else }\end{cases}\right.
$$

with $\tilde{v}, \tilde{v}_{-}$as before.
The second type of interactions take place on one lane:
Interaction 3 (Braking):
If the velocity of the car under consideration is larger than the velocity of the leading car $v>v_{+}$and the braking line $H_{B}(v)$ is crossed, then the following takes place: The car brakes in a range of velocities $[\beta v, v]$ below its actual velocity $v$. The new velocity is given by

$$
v^{\prime}=\beta v+\xi(v-\beta v), \beta<1 .
$$

$\xi$ is uniformly distributed in $[0,1]$. Braking is limited by the requirement that an acceleration is possible again, i.e. for all $v, v^{\prime}$ we require

$$
H_{A}\left(v^{\prime}\right)>H_{B}(v) \quad \text { or } \quad \frac{T_{B}}{T_{A}}-\frac{\delta}{T_{A} w}<\beta<1 .
$$

Interaction 4 (Acceleration I, Following):
If the velocity of the car under consideration is smaller than the velocity of the leading car $v<v_{+}$and the acceleration line $H_{A}(v)$ is crossed, then the following takes place: The car accelerates in a range of velocities $[v, \alpha v]$ above its actual velocity $v$. The new velocity is given by

$$
v^{\prime}=v+\xi(\min (w, v \alpha)-v), \alpha>1
$$

Acceleration is limited by the requirement that braking is possible again, i.e. for all $v, v^{\prime}$ we require

$$
H_{B}\left(v^{\prime}\right)<H_{A}(v) \quad \text { or } \quad 1<\alpha<\frac{T_{A}}{T_{B}}+\frac{\delta}{T_{B} w}
$$

Interaction 5 (Acceleration II, Free):
If the velocity of the car under consideration is smaller than the velocity of the leading car $v<v_{+}$and the acceleration line $H_{F}$ is crossed, then the car accelerates. The new velocity $v^{\prime}$ (the desired velocity) is distributed due to the distribution function $f_{D}$, i.e.

$$
v^{\prime}=F_{D}^{-1}(\xi)
$$

with $F_{D}$ as before.

## Remark 1:

The above microscopic interaction rules have to be changed in a suitable way for the first and last lane. In this case lane changing to the right or left, respectively, is not possible.

Remark 2:
The above model describes on the one hand cars in a following behaviour oscillating between braking and acceleration line. On the other hand cars driving freely accelerating to their desired velocities are described. Accidents are avoided by prohibiting a space of size $H_{B}(v)$ in front of each vehicle for the leading car. These facts will be used to set up an analytic model for the leading vehicle distribution, which will be used in the kinetic model in the next section.

## Remark 3:

Obviously, a variety of other features could be included in the model. For example, lane changing to the right is not only caused by a following car but may be performed without any influence of another car. In general, lane changing can also be done in a spontaneous way without interactions with surrounding cars. In particular, this type of lane changing is important for inhomogeneous situations like the simulation of a reduction of lanes. We refer to II, Section 4 for an approach to include spontaneous lane changing into the macroscopic model.

## Remark 4:

Finally we mention that to derive the kinetic equation in the next section we consider for simplicity a microscopic model without the additional acceleration terms in the lane changing interactions. Moreover, lane changing and braking lines are put together. This means we consider a model, where after reaching the braking line, the driver will - according to the above rules - first try to change the lane to the left, second - if this is not possible - the leading car will try to change to the right and third - if lane changing is not possible at all - the driver will brake.

## 3 The Kinetic Model

The second level of our hierarchy of models is given by a kinetic description, using the distribution function in space and velocity instead of a description of individual cars. We assume as in the last section a highway with $N$ lanes. They are numbered by

$$
\alpha=1, \ldots, N
$$

The basic quantities in a kinetic approach are the single car distribution function and the leading vehicle distribution on each lane. The single car distribution function denoted by $f_{\alpha}(x, v)$ describes the number of cars at $x$ with velocity $v$ on lane $\alpha$. The leading vehicle distribution denoted by $f_{\alpha}^{(2)}\left(x, v, h, v_{+}\right)$describes the number of pairs of cars at $x$ with velocity $v$ and leading cars at $x+h$ with velocity $v_{+}$. Here and in the following we do not write explicitely the time dependence.

Integrating $f_{\alpha}^{(2)}$ over $h$ and $v_{+}$we get

$$
\int_{0}^{w} \int_{0}^{\infty} f_{\alpha}^{(2)}\left(x, v, h, v_{+}\right) d h d v_{+}=f_{\alpha}(x, v)
$$

Moreover,

$$
\int_{0}^{w} f_{\alpha}(x, v) d v=\rho_{\alpha}(x)
$$

where $\rho_{\alpha}$ denotes the density on lane $\alpha$. Since the mean space available for each car is $\frac{1}{\rho_{\alpha}}$ we have

$$
\frac{\int_{0}^{w} \int_{0}^{w} \int_{0}^{\infty} h f_{\alpha}^{(2)}\left(x, v, h, v_{+}\right) d h d v d v_{+}}{\int_{0}^{w} \int_{0}^{w} \int_{0}^{\infty} f_{\alpha}^{(2)}\left(x, v, h, v_{+}\right) d h d v d v_{+}}=\frac{1}{\rho_{\alpha}(x)}
$$

or

$$
\begin{equation*}
\int_{0}^{w} \int_{0}^{w} \int_{0}^{\infty} h f_{\alpha}^{(2)}\left(x, v, h, v_{+}\right) d h d v d v_{+}=1 \tag{1}
\end{equation*}
$$

A kinetic equation for the distribution function $f_{\alpha}$ uses the leading vehicle distribution $f_{\alpha}^{(2)}$ to describe the influence of the interactions. To obtain a closed equation
for $f_{\alpha}$ we have to aproximate the leading vehicle distribution $f_{\alpha}^{(2)}$ in a suitable way using $f_{\alpha}$ and a correlation function. A connection between $f_{\alpha}^{(2)}$ and $f_{\alpha}$ is given by the following considerations:

Let $F_{\alpha}(x, v)$ denote the probability distribution in $v$ of cars at $x$, i.e. $f_{\alpha}(x, v)=$ $\rho_{\alpha}(x) F_{\alpha}(x, v)$. Moreover, we denote by $F_{\alpha}^{+}\left(v_{+} ; h, v, x\right)$ the probability distribution in $v_{+}$of the leading cars at distance $h$ for cars at $x$ with velocity $v . Q_{\alpha}(h ; v, x)$ denotes the probability distribution of leading cars in $h$ for a car at $x$ with velocity $v$. Then

$$
\begin{equation*}
f_{\alpha}^{(2)}\left(x, v, h, v_{+}\right)=F_{\alpha}^{+}\left(v_{+} ; h, v, x\right) Q_{\alpha}(h ; v, x) f_{\alpha}(x, v) . \tag{2}
\end{equation*}
$$

We introduce now the following assumptions: The leading vehicles are distributed according to the probability distribution $F_{\alpha}$ at $x+h$ :

$$
F_{\alpha}^{+}\left(v_{+} ; h, v, x\right)=F_{\alpha}\left(x+h, v_{+}\right) .
$$

Moreover, for $Q_{\alpha}$ we take the ansatz

$$
Q_{\alpha}(h ; v, x)=q\left(h ; v, f_{\alpha}(x, \cdot)\right) .
$$

In the next subsection the space homogeneous case for a one lane situation is considered. An explicit expression for $q(h ; v, f)$ is given there. This expression is then taken for the general case considered here.

As a consequence of (1) and (2) the above mentioned explicit expression for $q$ has to fulfill

$$
\int_{0}^{\infty} q\left(h ; v, f_{\alpha}(x, \cdot)\right) d h=1
$$

and

$$
\int_{0}^{w} \int_{0}^{\infty} h q\left(h ; v, f_{\alpha}(x, \cdot)\right) F_{\alpha}(x, v) d h d v=\frac{1}{\rho_{\alpha}(x)} .
$$

Finally this leads to the following approximation of $f_{\alpha}^{(2)}$ :

$$
\begin{equation*}
f_{\alpha}^{(2)}\left(x, v, h, v_{+}\right) \sim q\left(h ; v, f_{\alpha}(x, \cdot)\right) F_{\alpha}\left(x+h, v_{+}\right) f_{\alpha}(x, v) . \tag{3}
\end{equation*}
$$

### 3.1 The Homogeneous Case

We consider first a space homogeneous one lane situation with vehicles having velocities that are distributed according to a given distribution function $f(v)$ with mass

$$
\rho=\int_{0}^{w} f(v) d v
$$

## Leading Vehicle Distribution

In this subsection we define $q(h ; v, f)$, i.e. the probability density in $h$ that for a vehicle with velocity $v$ there is a leading vehicle with headway equal to $h$, if the velocities of the vehicles are distributed according to $f$.
We introduce the notation

$$
<g>=\int_{0}^{w} g(v) F(v) d v
$$

for any function $g=g(v) . F$ is defined as before by $f=\rho F$. Due to the last section $q$ has to fulfill

$$
\begin{equation*}
\int_{0}^{\infty} q(h ; v, f) d h=1 \tag{4}
\end{equation*}
$$

and

$$
\begin{equation*}
<\int_{0}^{\infty} h q(h ; \cdot, f) d h>=\frac{1}{\rho}, \tag{5}
\end{equation*}
$$

i.e. the average distance of the leading car is $\frac{1}{\rho}$.

We assume that one part $(0 \leq \lambda<1)$ of the vehicles has a following behaviour, i.e. is moving between braking line $H_{B}$ and acceleration lines $H_{A}$, and the other part $(1-\lambda)$ behaves independently from each other having a distance at least larger than the braking line $H_{B}$, compare Section 2, Remark 2. Then one obtains, see Part II Section 5.1 for a detailed discussion, the following expression for $q$ :

$$
\begin{align*}
q(h ; v, f)= & (1-\lambda) \tilde{\rho} e^{-\tilde{\rho}\left(h-H_{B}(v)\right)} \chi_{\left[H_{B}(v), \infty\right)}(h)  \tag{6}\\
& +\lambda \frac{1}{H_{A}(v)-H_{B}(v)} \chi_{\left[H_{B}(v), H_{A}(v)\right]}(h)
\end{align*}
$$

where $\tilde{\rho}$ is determined using (5):

$$
\begin{equation*}
\tilde{\rho}=\frac{(1-\lambda) \rho}{1-\rho\left[(1-\lambda)<H_{B}>+\frac{\lambda}{2}\left(<H_{B}>+\left\langle H_{A}>\right)\right]\right.} \tag{7}
\end{equation*}
$$

$\chi$ denotes the characteristic function. $\tilde{\rho}$ is the reduced density, since the available space for a free car is $\left.\left[(1-\lambda)<H_{B}>+\frac{\lambda}{2}\left(<H_{B}\right\rangle+\left\langle H_{A}\right\rangle\right)\right]$.
The above expression takes into account a completely chaotic behaviour of the cars ( $\lambda$ near 0 ), as well a strongly correlated behaviour ( $\lambda$ near 1 ) which leads to a following behaviour. For (6), (7) we need

$$
\rho<\frac{1}{(1-\lambda)<H_{B}>+\frac{\lambda}{2}\left(<H_{B}>+<H_{A}>\right)} .
$$

This is a condition on the type of distribution functions $f$ allowed in the model.
For a comparison of the averaged leading vehicle distribution $\langle q(h ; \cdot, f)\rangle$ and a leading vehicle distribution determined directly from the microscopic multilane model in Section 2 we refer to Part II, Section 2 and in particular to Figure II, 2.1.

## Lane Changing Probabilities

In the following sections we need the probability that a lane change to the left and right is performed. This is again determined from a homogeneous situation. We assume that the velocities of the vehicles on the new lane are distributed according to the distribution function $f$. We consider a car with velocity $v$ and determine the probability $P_{Y}(v, f), Y=L, R$ that lane changing is done, if the respective threshold is crossed, i.e. the probability that there is enough space on the other lane.

For lane changing, the distance after the lane change between the changing car with velocity $v$ and its leading car on the new lane must be at least $H_{Y}^{S}(v), Y=L, R$. Moreover, the distance between the changing car and its follower on the new lane with velocity $v^{\prime}$ must be at least $H_{Y}^{S}\left(v^{\prime}\right), Y=L, R$. The probability $p_{Y}\left(v, v^{\prime}, f\right)$ of a lane change of a car with velocity $v$ having a follower on the new line with velocity $v^{\prime}$ is given by

$$
\begin{equation*}
p_{Y}\left(v, v^{\prime}, f\right)=1-\rho \int_{0}^{H_{Y}^{S}\left(v^{\prime}\right)+H_{Y}^{S}\left(v^{\prime}\right)}[1-\langle Q(h ; \cdot, f)\rangle] d h \tag{8}
\end{equation*}
$$

with the distribution function $Q$ defined by

$$
Q(h ; v, f)=\int_{0}^{h} q\left(h^{\prime} ; v, f\right) d h^{\prime}
$$

The derivation of this formula and the exact assumptions to obtain it are given in II, Section 5.2. Moreover, an explicit expression for $P_{Y}$ using (6) is given there as well. The averaged version yields the desired probability for lane changing for a car with velocity $v$ :

$$
\begin{equation*}
P_{Y}(v, f)=<p_{Y}(v, \cdot, f)> \tag{9}
\end{equation*}
$$

### 3.2 The Kinetic Multilane Model

The kinetic equation for the distibution functions $\left(f_{1}, \cdots, f_{N}\right)$ on the $N$ lanes is obtained from similiar considerations as in the kinetic theory of gases. One determines the kinetic interaction operators, i.e. the gain $(\mathrm{G})$ and loss $(\mathrm{L})$ operators. This is done using the microscopic interactions as a basis combined with the standard procedure to derive kinetic equations. We get

$$
\begin{align*}
\partial_{t} f_{\alpha}+v \partial_{x} f_{\alpha}= & \tilde{C}_{\alpha}^{+}\left(f_{1}^{(2)}, \cdots, f_{N}^{(2)}, f_{1}, \cdots, f_{N}\right)  \tag{10}\\
= & \left(\tilde{G}_{B}^{+}-\tilde{L}_{B}^{+}\right)\left(f_{\alpha-1}, f_{\alpha}^{(2)}, f_{\alpha+1}\right) \\
& +\left(\tilde{G}_{A}^{+}-\tilde{L}_{A}^{+}+\tilde{G}_{F}^{+}-\tilde{L}_{F}^{+}\right)\left(f_{\alpha}^{(2)}\right) \\
& +\left[\tilde{G}_{L}^{+}\left(f_{\alpha-1}^{(2)}, f_{\alpha}\right)-\tilde{L}_{R}^{+}\left(f_{\alpha-1}, f_{\alpha}^{(2)}, f_{\alpha+1}\right)\right]\left(1-\delta_{\alpha, 1}\right) \\
& +\left[\tilde{G}_{R}^{+}\left(f_{\alpha}, f_{\alpha+1}^{(2)}, f_{\alpha+2}\right)-\tilde{L}_{L}^{+}\left(f_{\alpha}^{(2)}, f_{\alpha+1}\right)\right]\left(1-\delta_{\alpha, N}\right) .
\end{align*}
$$

$\delta_{i, j}$ denotes the Kronecker symbol. $f_{\alpha}^{(2)}\left(x, v, h, v_{+}\right)$can be approximated, see (3), by

$$
\begin{equation*}
f_{\alpha}^{(2)}\left(x, v, h, v_{+}\right) \sim q\left(h ; v, f_{\alpha}(x, \cdot)\right) f_{\alpha}(x, v) F_{\alpha}\left(x+h, v_{+}\right) \tag{11}
\end{equation*}
$$

with $\rho_{\alpha}=\int_{0}^{w} f_{\alpha}(x, v) d v, f_{\alpha}=\rho_{\alpha} F_{\alpha}$ and $q(h ; v, f)$ the leading vehicle distribution defined in (6). This approach resembles Enskog's theory of a dense gas, see e.g., [1], rather than a Boltzmann type treatment. The necessity to do such an Enskog type approach is explained in detail in [9]. In particular, it is shown there that a Boltzmann type treatment leads to completely wrong results even for simple inhomogeneous situations.

In the following we use for $X=B, A, F$ the notation

$$
q_{X}(v, f)=q\left(H_{X}(v), v, f\right)
$$

to denote the correlation function. The probability $P_{Y}, Y=L, R$ for a lane change has been defined in (9). Additionally, we use the convention

$$
P_{L}\left(v, f_{N+1}\right)=0=P_{R}\left(v, f_{0}\right) .
$$

The interaction terms appearing in (10) are stated and approximated using (11) in the following:

Interaction 1 (Lane changing to the left):
The car is changing to the left, if the braking line is reached and a lane change is possible (probability $P_{L}$ ).

Gainterm:

$$
\tilde{G}_{L}^{+}\left(f_{\alpha-1}^{(2)}, f_{\alpha}\right)=\int_{v>\hat{v}_{+}} P_{L}\left(v, f_{\alpha}(x)\right)\left|v-\hat{v}_{+}\right| f_{\alpha-1}^{(2)}\left(x, v, H_{B}(v), \hat{v}_{+}\right) d \hat{v}_{+}
$$

writing $f_{\alpha}(x)$ instead of $f_{\alpha}(x, \cdot)$. This is approximated by

$$
=\int_{v>\hat{v}_{+}} G_{L}^{+}\left(f_{\alpha-1}, f_{\alpha}\right) .
$$

Lossterm:
With the same arguments one obtains an approximation of $\tilde{L}_{L}^{+}$:

$$
\tilde{L}_{L}^{+}\left(f_{\alpha}^{(2)}, f_{\alpha+1}\right)=\tilde{G}_{L}^{+}\left(f_{\alpha}^{(2)}, f_{\alpha+1}\right) \sim G_{L}^{+}\left(f_{\alpha}, f_{\alpha+1}\right)=L_{L}^{+}\left(f_{\alpha}, f_{\alpha+1}\right) .
$$

Interaction 2 (Lane change to the right):
A car changes to the right if its follower reaches the braking line and is not able to overtake (change to the left). Moreover, a change to the right must be possible (probability $P_{R}$ ).

## Gainterm:

$$
\begin{aligned}
& \tilde{G}_{R}^{+}\left(f_{\alpha}, f_{\alpha+1}^{(2)}, f_{\alpha+2}\right) \\
= & \int_{\hat{v}_{-}>v} P_{R}\left(v, f_{\alpha}(x)\right)\left[1-P_{L}\left(\hat{v}_{-}, f_{\alpha+2}\left(x-H_{B}\left(\hat{v}_{-}\right)\right)\right)\right]\left|v-\hat{v}_{-}\right| \\
& f_{\alpha+1}^{(2)}\left(x-H_{B}\left(\hat{v}_{-}\right), \hat{v}_{-}, H_{B}\left(\hat{v}_{-}\right), v\right) d \hat{v}_{-}
\end{aligned}
$$

is approximated by

$$
=\begin{aligned}
& G_{R}^{+}\left(f_{\alpha}, f_{\alpha+1}, f_{\alpha+2}\right) \\
& \int_{\hat{v}_{-}>v} P_{R}\left(v, f_{\alpha}(x)\right)\left[1-P_{L}\left(\hat{v}_{-}, f_{\alpha+2}\left(x+H_{B}\left(\hat{v}_{-}\right)\right)\right)\right]\left|v-\hat{v}_{-}\right| \\
& q_{B}\left(\hat{v}_{-}, f_{\alpha+1}\left(x-H_{B}\left(\hat{v}_{-}\right)\right)\right) f_{\alpha+1}\left(x-H_{B}\left(\hat{v}_{-}\right), \hat{v}_{-}\right) F_{\alpha+1}(x, v) d \hat{v}_{-} .
\end{aligned}
$$

Lossterm:

$$
\begin{aligned}
& \tilde{L}_{R}^{+}\left(f_{\alpha-1}, f_{\alpha}^{(2)}, f_{\alpha+1}\right)=\tilde{G}_{R}^{+}\left(f_{\alpha-1}, f_{\alpha}^{(2)}, f_{\alpha+1}\right) \\
\sim \quad & G_{R}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right)=L_{R}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right)
\end{aligned}
$$

## Interaction 3 (Braking):

A car brakes, if it reaches the braking line and the driver is not able to change to the left and if the leading car cannot change to the right.

## Gainterm:

We define $P_{B}$, the probability for braking, as

$$
\begin{aligned}
& P_{B}\left(v, v_{+}, f_{\alpha-1}\left(x+H_{B}(v)\right), f_{\alpha+1}(x)\right) \\
= & {\left[1-P_{L}\left(v, f_{\alpha+1}(x)\right)\right]\left[1-P_{R}\left(v_{+}, f_{\alpha-1}\left(x+H_{B}(v)\right)\right)\right] }
\end{aligned}
$$

One obtains an approximation of

$$
\begin{aligned}
& \tilde{G}_{B}^{+}\left(f_{\alpha-1}, f_{\alpha}^{(2)}, f_{\alpha+1}\right) \\
= & \iint_{\hat{v}>\hat{v}_{+}} P_{B}\left(\hat{v}, \hat{v}_{+}, f_{\alpha-1}\left(x+H_{B}(v)\right), f_{\alpha+1}(x)\right)\left|\hat{v}-\hat{v}_{+}\right| \\
& \sigma_{B}(v, \hat{v}) f_{\alpha}^{(2)}\left(x, \hat{v}, H_{B}(\hat{v}), \hat{v}_{+}\right) d \hat{v} d \hat{v}_{+}
\end{aligned}
$$

by

$$
\begin{aligned}
& G_{B}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right) \\
= & \iint_{\hat{v}>\hat{v}_{+}} P_{B}\left(\hat{v}, \hat{v}_{+}, f_{\alpha-1}\left(x+H_{B}(v)\right), f_{\alpha+1}(x)\right)\left|\hat{v}-\hat{v}_{+}\right| \\
& \sigma_{B}(v, \hat{v}) q_{B}\left(\hat{v}, f_{\alpha}(x)\right) f_{\alpha}(x, \hat{v}) F_{\alpha}\left(x+H_{B}(\hat{v}), \hat{v}_{+}\right) d \hat{v} d \hat{v}_{+}
\end{aligned}
$$

with

$$
\sigma_{B}(v, \hat{v})=\frac{1}{\hat{v}(1-\beta)} \chi_{[\beta \hat{v}, \hat{v}]}(v) .
$$

## Lossterm:

$$
\begin{aligned}
& \tilde{L}_{B}^{+}\left(f_{\alpha-1}, f_{\alpha}^{(2)}, f_{\alpha+1}\right) \\
= & \int_{v>\hat{v}_{+}} P_{B}\left(v, \hat{v}_{+}, f_{\alpha-1}\left(x+H_{B}(v)\right), f_{\alpha+1}(x)\right)\left|v-\hat{v}_{+}\right| f_{\alpha}^{(2)}\left(x, v, H_{B}(v), \hat{v}_{+}\right) d \hat{v}_{+}
\end{aligned}
$$

is approximated by

$$
\begin{aligned}
& L_{B}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right) \\
& \int_{v>\hat{v}_{+}} P_{B}\left(v, \hat{v}_{+}, f_{\alpha-1}\left(x+H_{B}(v)\right), f_{\alpha+1}(x)\right)\left|v-\hat{v}_{+}\right| \\
& q_{B}\left(v, f_{\alpha}(x)\right) f_{\alpha}(x, v) F_{\alpha}\left(x+H_{B}(v), \hat{v}_{+}\right) d \hat{v}_{+} .
\end{aligned}
$$

Interaction 4 (Acceleration):
The car accelerates, if the acceleration line is reached.
Gainterm:

$$
\tilde{G}_{A}^{+}\left(f_{\alpha}^{(2)}\right)=\iint_{\hat{v}<\hat{v}_{+}}\left|\hat{v}-\hat{v}_{+}\right| \sigma_{A}(v, \hat{v}) f_{\alpha}^{(2)}\left(x, \hat{v}, H_{A}(\hat{v}), \hat{v}_{+}\right) d \hat{v} d \hat{v}_{+}
$$

is approximated by

$$
G_{A}^{+}\left(f_{\alpha}\right)=\iint_{\hat{v}<\hat{v}_{+}}\left|\hat{v}-\hat{v}_{+}\right| \sigma_{A}(v, \hat{v}) q_{A}\left(\hat{v}, f_{\alpha}(x)\right) f_{\alpha}(x, \hat{v}) F_{\alpha}\left(x+H_{A}(\hat{v}), \hat{v}_{+}\right) d \hat{v} d \hat{v}_{+}
$$

with

$$
\sigma_{A}(v, \hat{v})=\frac{1}{\min (w, \alpha \hat{v})-\hat{v}} \chi_{[\hat{v}, \min (w, \alpha \hat{v})]}(v)
$$

Lossterm:

$$
\tilde{L}_{A}^{+}\left(f_{\alpha}^{(2)}\right)=\int_{v<\hat{v}_{+}}\left|v-\hat{v}_{+}\right| f_{\alpha}^{(2)}\left(x, v, H_{A}(v), \hat{v}_{+}\right) d \hat{v}_{+}
$$

is approximated by

$$
L_{A}^{+}\left(f_{\alpha}\right)=\int_{v<\hat{v}_{+}}\left|v-\hat{v}_{+}\right| q_{A}\left(v, f_{\alpha}(x)\right) f_{\alpha}(x, v) F_{\alpha}\left(x+H_{A}(v), \hat{v}_{+}\right) d \hat{v}_{+}
$$

Interaction 5 (Free acceleration):
Using $q_{F}, H_{F}$ and

$$
\sigma_{F}(v, \hat{v})=f_{D}(v)
$$

instead of $q_{A}, H_{A}$ and $\sigma_{A}$ one defines $G_{F}^{+}$and $L_{F}^{+}$in the same way as $G_{A}^{+}$and $L_{A}^{+}$, respectively.

Using the above approximations the kinetic equation reads for $\alpha=1, \cdots, N$ :

$$
\begin{align*}
\partial_{t} f_{\alpha}+v \partial_{x} f_{\alpha}= & C_{\alpha}^{+}\left(f_{1}, \cdots, f_{N}\right)  \tag{12}\\
= & \left(G_{B}^{+}-L_{B}^{+}\right)\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right) \\
& +\left(G_{A}^{+}-L_{A}^{+}+G_{F}^{+}-L_{F}^{+}\right)\left(f_{\alpha}\right) \\
& +\left[G_{L}^{+}\left(f_{\alpha-1}, f_{\alpha}\right)-L_{R}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right)\right]\left(1-\delta_{\alpha, 1}\right) \\
& +\left[G_{R}^{+}\left(f_{\alpha}, f_{\alpha+1}, f_{\alpha+2}\right)-L_{L}^{+}\left(f_{\alpha}, f_{\alpha+1}\right)\right]\left(1-\delta_{\alpha, N}\right)
\end{align*}
$$

### 3.3 A Cumulative Kinetic Model

We derive in this subsection a cumulative model from the multilane model above. The homogeneous version of this model will in the following section be used to derive macroscopic coefficients. The basic assumption underlying the following derivation is that traffic is homogenized over all lanes.

In the model derived in this section the lane changing interactions influence the dynamics only by reducing the number of braking interactions in the cumulative model. This is similar to the standard kinetic models used in traffic flow, see, e.g., $[14,19,15,21]$. However here the dynamics in the cumulative model is derived from the multilane model. This is done by introducing a probability for braking in the equations according to the lane changing rules derived above.

The cumulative model is obtained from the multilane one by assuming that the distribution function $f_{\alpha}$ is the same on all lanes and by summing the equations over all lanes $1, \cdots, N$. We consider the cumulative distribution functions

$$
f=f_{1}=\cdots=f_{N}=\frac{1}{N} \sum_{\alpha=1}^{N} f_{\alpha}, \quad F=F_{1}=\cdots=F_{N}=\frac{1}{N} \sum_{\alpha=1}^{N} F_{\alpha}
$$

This means that $N f(x, v)$ is the total distribution function on the highway and $f=\rho F$, where $\rho$ is the average density per lane.

The probability $P_{B}$ for braking on lane $\alpha$ is due to the last section

$$
\begin{aligned}
& P_{B}\left(v, v_{+}, f_{\alpha-1}\left(x+H_{B}(v)\right), f_{\alpha+1}(x)\right) \\
= & {\left[1-P_{L}\left(v, f_{\alpha+1}(x)\right)\right]\left[1-P_{R}\left(v_{+}, f_{\alpha-1}\left(x+H_{B}(v)\right)\right)\right] }
\end{aligned}
$$

with the conventions

$$
P_{L}\left(v, f_{N+1}\right)=0=P_{R}\left(v, f_{0}\right)
$$

Averaging over all lanes gives the cumulative braking probability which we denote by $P_{B}^{C}$ as

$$
\begin{aligned}
& P_{B}^{C}\left(v, v_{+}, f\left(x+H_{B}(v)\right), f(x)\right) \\
= & \frac{1}{N}\left(1-P_{L}(v, f(x))\right)+\frac{1}{N}\left[1-P_{R}\left(v_{+}, f\left(x+H_{B}(v)\right)\right)\right] \\
+ & \frac{N-2}{N}\left[1-P_{L}(v, f(x))\right]\left[1-P_{R}\left(v_{+}, f\left(x+H_{B}(v)\right)\right)\right]
\end{aligned}
$$

These considerations yield the following equation for the cumulative model:

$$
\partial_{t} f+v \partial_{x} f=C_{C}^{+}(f)
$$

with

$$
\begin{equation*}
C_{C}^{+}(f)=G_{B}^{+}(f)-L_{B}^{+}(f)+G_{A}^{+}(f)-L_{A}^{+}(f)+G_{F}^{+}(f)-L_{F}^{+}(f) \tag{13}
\end{equation*}
$$

where with a slight abuse of notation we have defined

$$
\begin{aligned}
G_{B}^{+}(f)= & \iint_{\hat{v}>\hat{v}_{+}} P_{B}^{C}\left(\hat{v}, \hat{v}_{+}, f\left(x+H_{B}(\hat{v})\right), f(x)\right)\left|\hat{v}-\hat{v}_{+}\right| \sigma_{B}(v, \hat{v}) \\
& q_{B}(\hat{v}, f(x)) f(x, \hat{v}) F\left(x+H_{B}(\hat{v}), \hat{v}_{+}\right) d \hat{v} d \hat{v}_{+}
\end{aligned}
$$

and

$$
\begin{aligned}
L_{B}^{+}(f)= & \int_{v>\hat{v}_{+}} P_{B}^{C}\left(v, \hat{v}_{+}, f\left(x+H_{B}(v)\right), f(x)\right)\left|v-\hat{v}_{+}\right| \\
& q_{B}(v, f(x)) f(x, v) F\left(x+H_{B}(v), \hat{v}_{+}\right) d \hat{v}_{+}
\end{aligned}
$$

$G_{A}^{+}, L_{A}^{+}, G_{F}^{+}, L_{F}^{+}$are defined as before.

## Remark:

This model should be compared with the one developed in [9] or with the standard model in [19]. In the present case the terms $\sigma_{A}, \sigma_{B}$ defined by the microscopic interactions rules are not depending on the local density $\rho$ but only on the behaviour of a single driver, i.e. no macroscopic effects are included in these terms.

## 4 The Fluid Dynamic Model

The third and last level of the hierarchy is given by a fluid dynamic description on the basis of the moments density and velocity. To derive a macroscopic equation we proceed similarly to the gas kinetic case:

### 4.1 Balance Equations

We start the derivation of fluid dynamic equations by multiplying the inhomogeneous kinetic equation (12) with the property $\phi(v)$ and integrating it with respect to $v$. One obtains the following set of balance equations:

$$
\partial_{t} \int_{0}^{w} \phi f_{\alpha} d v+\partial_{x} \int_{0}^{w} v \phi f_{\alpha} d v=\int_{0}^{w} \phi(v) C_{\alpha}^{+}(f)(x, v, t) d v .
$$

We define the density $n_{\alpha}^{\phi}$ of the property $\phi$ as

$$
n_{\alpha}^{\phi}=\int_{0}^{w} \phi f_{\alpha} d v .
$$

The important point in deriving fluid dynamic equations from kinetic Enskog equations is to identify clearly the flux and the source terms in the equation. The flux of $\phi$ due to the kinetic advection part is as usual

$$
q_{\alpha}^{\phi}=\int_{0}^{w} v \phi f_{\alpha} d v .
$$

However, there is a second contribution to the flux coming from the Enskog collison term due to the finite size of the interaction thresholds. To obtain this flux we separate the Enskog interaction term into a local interaction term and a deviation from the local term:

$$
C_{\alpha}^{+}=C_{\alpha}-\left(C_{\alpha}-C_{\alpha}^{+}\right),
$$

where the local term $C_{\alpha}$ is defined by

$$
\begin{aligned}
C_{\alpha}\left(f_{1}, \cdots, f_{N}\right) & =\left(G_{B}-L_{B}\right)\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right) \\
& +\left(G_{A}-L_{A}+G_{F}-L_{F}\right)\left(f_{\alpha}\right) \\
& +\left[G_{L}\left(f_{\alpha-1}, f_{\alpha}\right)-L_{R}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right)\right]\left(1-\delta_{\alpha, 1}\right) \\
& +\left[G_{R}\left(f_{\alpha}, f_{\alpha+1},\left(f_{\alpha+2}\right)-L_{L}\left(f_{\alpha}, f_{\alpha+1}\right)\right]\left(1-\delta_{\alpha, N}\right) .\right.
\end{aligned}
$$

The gain and loss terms $G_{B}, L_{B}$, etc. are defined as $G_{B}^{+}, L_{B}^{+}$, etc. without a shift in the $x$-dependence, i.e., all functions appearing depend only on $x$. We mention that we do not proceed here exactly as in Enskogs theory of a dense gas. The fact that the velocities are only positive requires a slightly different treatment.

Rewriting the balance equations we get

$$
\partial_{t} n_{\alpha}^{\phi}+\partial_{x} q_{\alpha}^{\phi}+E_{\alpha}^{\phi}=S_{\alpha}^{\phi}
$$

with Enskog flux term

$$
E_{\alpha}^{\phi}=\int_{0}^{w} \phi(v)\left[C_{\alpha}(f)(x, v, t)-C_{\alpha}^{+}(f)(x, v, t)\right] d v
$$

and source term

$$
\begin{equation*}
S_{\alpha}^{\phi}=\int_{0}^{w} \phi(v) C_{\alpha}(f)(x, v, t) d v \tag{14}
\end{equation*}
$$

Using $\phi(v)=1$ and $\phi(v)=v$ one obtains equations for the traffic flow density

$$
n_{\alpha}^{1}=\rho_{\alpha}=\int_{0}^{w} f_{\alpha} d v
$$

and the traffic flux

$$
n_{\alpha}^{v}=\rho_{\alpha} u_{\alpha}=\int_{0}^{w} v f_{\alpha} d v
$$

$u_{\alpha}=n_{\alpha}^{v} / n_{\alpha}^{1}$ denotes the mean velocity. For $\phi(v)=1$ we get the continuity equations

$$
\partial_{t} \rho_{\alpha}+\partial_{x}\left(\rho_{\alpha} u_{\alpha}\right)+E_{\alpha}^{1}=S_{\alpha}^{1}
$$

Moreover, for $\phi(v)=v$ the acceleration equations

$$
\begin{equation*}
\partial_{t}\left(\rho_{\alpha} u_{\alpha}\right)+\partial_{x}\left(p_{\alpha}+\rho_{\alpha} u_{\alpha}^{2}\right)+E_{\alpha}^{v}=S_{\alpha}^{v} \tag{15}
\end{equation*}
$$

are obtained with the 'traffic pressure'

$$
p_{\alpha}=\int_{0}^{w}\left(v-u_{\alpha}\right)^{2} f_{\alpha} d v
$$

To obtain closed equations for $\rho_{\alpha}$ and $u_{\alpha}$ one has to specify the dependence of $p_{\alpha}, S_{\alpha}^{1}, S_{\alpha}^{v}$ and $E_{\alpha}^{1}, E_{\alpha}^{v}$ on $\rho_{\alpha}$ and $u_{\alpha}$.

### 4.2 Closure Relations

There are a variety of possible closure relations, which could be borrowed from gas dynamics. We restrict here to the derivation of nonviscous fluid dynamic equations. As usual, to find closure relations for the balance equations one has to use the stationary solutions of the cumulative kinetic equation (16). All parameters of the fluid dynamic equation can be identified from these solutions. For the following compare the derivation of macroscopic equations in the case of a multicomponent gas with chemical reactions.

On each lane we consider the homogeneous cumulative equation for $f(v)$ derived from equation (13) given in Section 3.3:

$$
\begin{equation*}
\partial_{t} f=C_{C}(f) \tag{16}
\end{equation*}
$$

with $C_{C}(f)$ defined by

$$
C_{C}(f)=G_{B}(f)-L_{B}(f)+G_{A}(f)-L_{A}(f)+G_{F}(f)-L_{F}(f)
$$

$G_{B}, L_{B}, G_{A}, L_{A}, G_{F}, L_{F}$ have been defined above. For this kinetic equation $\rho=$ $\int_{0}^{w} f(v) d v$ is constant in time. As before the definition $f(v)=\rho F(v)$ is used.

For the following arguments the homogeneous equation (16) must have a one parameter family of stationary equilibrium distributions $f^{e}(\rho, v)$ depending only on the density, i.e. for $\rho$ fixed we have

$$
f(v) \rightarrow f^{e}(\rho, v) \text { for } t \rightarrow \infty
$$

no matter what the inital distribution of the homogenous equation is. See Part II for numerical experiments on this issue.

The fundamental diagram, i.e. the equilibrium mean velocity, is given by

$$
\begin{equation*}
u^{e}(\rho)=\frac{1}{\rho} \int_{0}^{w} v f^{e}(\rho, v) d v . \tag{17}
\end{equation*}
$$

Equation (15) is now closed by the following procedure:
We approximate the traffic pressure $p_{\alpha}$ in (15) by its equilibrium value:

$$
p_{\alpha}=\int_{0}^{w}\left(v-u_{\alpha}\right)^{2} f_{\alpha} d v \sim \int_{0}^{w}\left(v-u^{e}(\rho)\right)^{2} f^{e}(\rho, v) d v=p^{e}\left(\rho_{\alpha}\right) .
$$

The Enskog terms $E_{\alpha}^{\phi}$ are approximated in the following way: We linearize the expressions for $E_{\alpha}^{\phi}$ in $H$ and substitute the stationary distributions $f^{e}\left(\rho_{\alpha}, v\right)$ for $f_{\alpha}$. This yields a contribution from each of the terms appearing in the definition of $E_{\alpha}^{\phi}$. $E_{\alpha}^{\phi}$ is given by

$$
E_{\alpha}^{\phi}=\sum_{X=L, R, B, A, F} E_{X}^{\phi}
$$

with

$$
E_{X}^{\phi}=\int_{0}^{w} \phi(v)\left[G_{X}-G_{X}^{+}\right] d v-\int_{0}^{w} \phi(v)\left[L_{X}-L_{X}^{+}\right] d v
$$

for $X=L, R, B, A, F$. In the following the terms $E_{L}^{\phi}, E_{R}^{\phi}$ due to lane changing are neglected. This means we neglect the nonlocal Enskog corrections due to the lane changing terms and keep the corrections due to acceleration and braking. This is justified by the numerical analysis of the corresponding interaction frequencies, see Part II. A further simplification arises, since the remaining terms with $\phi=1$ are zero. We are left with $E_{\alpha}^{v}$ given by

$$
E_{\alpha}^{v}=E_{B}^{v}+E_{A}^{v}+E_{F}^{v}
$$

The procedure is shown in detail for the term

$$
\int_{0}^{w} v\left[G_{B}-G_{B}^{+}\right] d v
$$

The results for the other terms are stated without derivation .

$$
\begin{aligned}
& \int_{0}^{w} v\left[G_{B}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right)-G_{B}^{+}\left(f_{\alpha-1}, f_{\alpha}, f_{\alpha+1}\right)\right] d v \\
= & \int_{0}^{w} v \iint_{\hat{v}>\hat{v}_{+}}\left|\hat{v}-\hat{v}_{+}\right| \sigma_{B}(v, \hat{v}) q_{B}\left(\hat{v}, f_{\alpha}(x)\right) \\
& {\left[P_{B}\left(\hat{v}, \hat{v}_{+} f_{\alpha-1}(x), f_{\alpha+1}(x)\right) f_{\alpha}(x, \hat{v}) F_{\alpha}\left(x, \hat{v}_{+}\right)\right.} \\
& \left.-P_{B}\left(\hat{v}, \hat{v}_{+} f_{\alpha-1}\left(x+H_{B}(\hat{v})\right), f_{\alpha+1}(x)\right) f_{\alpha}(x, \hat{v}) F_{\alpha}\left(x+H_{B}(\hat{v}), \hat{v}_{+}\right)\right] d \hat{v} d \hat{v}_{+} d v
\end{aligned}
$$

The probability for braking is treated by using $P_{B}^{C}\left(\hat{v}, \hat{v}_{+}, f_{\alpha}(x), f_{\alpha}(x)\right)$ instead of $P_{B}\left(\hat{v}, \hat{v}_{+}, f_{\alpha-1}\left(x+H_{B}(\hat{v})\right), f_{\alpha+1}(x)\right)$. Moreover, we write $P_{B}^{C}\left(\hat{v}, \hat{v}_{+}, f\right)$ for $P_{B}^{C}\left(\hat{v}, \hat{v}_{+}, f, f\right)$.
Using

$$
F_{\alpha}\left(x+H_{B}(v), \hat{v}_{+}\right) \sim F_{\alpha}\left(x, \hat{v}_{+}\right)+H_{B}(v) \partial_{x} F_{\alpha}\left(x, \hat{v}_{+}\right),
$$

we get for $\int_{0}^{w} v\left[G_{B}-G_{B}^{+}\right] d v$ :

$$
\begin{aligned}
- & \int_{0}^{w} v \iint_{\hat{v}>\hat{v}_{+}} P_{B}^{C}\left(\hat{v}, \hat{v}_{+} f_{\alpha}(x)\right)\left|\hat{v}-\hat{v}_{+}\right| \sigma_{B}(v, \hat{v}) H_{B}(\hat{v}) q_{B}\left(\hat{v}, f_{\alpha}(x)\right) \\
& f_{\alpha}(x, \hat{v}) \partial_{x} F_{\alpha}\left(x, \hat{v}_{+}\right) d \hat{v} d \hat{v}_{+} d v .
\end{aligned}
$$

Moreover, we introduce $f^{e}\left(\rho_{\alpha}(x), v\right)$ instead of $f_{\alpha}(x, v)$ and use

$$
f^{e}\left(\rho_{\alpha}, \hat{v}_{+}\right)=\rho_{\alpha} F^{e}\left(\rho_{\alpha}, \hat{v}_{+}\right)
$$

This yields the following final approximation for $\int_{0}^{w} v\left[G_{B}-G_{B}^{+}\right] d v$ :

$$
\begin{array}{r}
-\int_{0}^{w} v \iint_{\hat{v}>\hat{v}_{+}} P_{B}^{C}\left(\hat{v}^{2}, \hat{v}_{+}, f^{e}\left(\rho_{\alpha}(x)\right)\right)\left|\hat{v}-\hat{v}_{+}\right| \sigma_{B}(v, \hat{v}) H_{B}(\hat{v}) q_{B}\left(\hat{v}, f^{e}\left(\rho_{\alpha}(x)\right)\right) \\
f^{e}\left(\rho_{\alpha}(x), \hat{v}\right) \partial_{\rho} F^{e}\left(\rho_{\alpha}(x), \hat{v}_{+}\right) d \hat{v} d \hat{v}_{+} d v \partial_{x} \rho_{\alpha} .
\end{array}
$$

The other terms are treated similarly. With $a_{B}^{e}, a_{A}^{e}$ and $a_{F}^{e}$ defined by

$$
\begin{aligned}
a_{B}^{e}(\rho)= & -\iint_{\hat{v}>\hat{v}_{+}} P_{B}^{C}\left(\hat{v}, \hat{v}_{+}, f^{e}(\rho)\left|\hat{v}-\hat{v}_{+}\right| H_{B}(\hat{v}) q_{B}\left(\hat{v}, f^{e}(\rho)\right) f^{e}(\rho, \hat{v}) \partial_{\rho} F^{e}\left(\rho, \hat{v}_{+}\right)\right. \\
& {\left[\int_{0}^{w} v \sigma_{B}(v, \hat{v}) d v-\hat{v}\right] d \hat{v}_{+} d \hat{v} } \\
a_{A}^{e}(\rho)= & -\iint_{\hat{v}<\hat{v}_{+}}\left|\hat{v}-\hat{v}_{+}\right| H_{A}(\hat{v}) q_{A}\left(\hat{v}, f^{e}(\rho)\right) f^{e}(\rho, \hat{v}) \partial_{\rho} F^{e}\left(\rho, \hat{v}_{+}\right) \\
& {\left[\int_{0}^{w} v \sigma_{A}(v, \hat{v}) d v-\hat{v}\right] d \hat{v}_{+} d \hat{v} } \\
a_{F}^{e}(\rho)=- & -\iint_{\hat{v}<\hat{v}_{+}}\left|\hat{v}-\hat{v}_{+}\right| H_{F} q_{F}\left(\hat{v}, f^{e}(\rho)\right) f^{e}(\rho, \hat{v}) \partial_{\rho} F^{e}\left(\rho, \hat{v}_{+}\right) \\
& {\left[\int_{0}^{w} v \sigma_{F}(v, \hat{v}) d v-\hat{v}\right] d \hat{v}_{+} d \hat{v} }
\end{aligned}
$$

we obtain alltogether

$$
E_{\alpha}^{v} \sim a^{e}\left(\rho_{\alpha}\right) \partial_{x} \rho_{\alpha}
$$

with the definition

$$
a^{e}(\rho)=a_{B}^{e}(\rho)+a_{A}^{e}(\rho)+a_{F}^{e}(\rho) .
$$

We introduce the integrated Enskog coefficent

$$
A^{e}(\rho)=\int_{0}^{\rho} a^{e}(\rho) d \rho
$$

This gives finally

$$
E_{\alpha}^{v} \sim \partial_{x} A^{e}\left(\rho_{\alpha}\right) .
$$

The source term $S_{\alpha}^{\phi}$ defined in (14) is treated as for example in kinetic semiconductor theory, by using a relaxation time approximation for $C_{\alpha}$ :

$$
\begin{align*}
C_{\alpha} \sim & \left(\frac{1}{T_{\alpha-1}^{L}} f_{\alpha-1}-\frac{1}{T_{\alpha}^{R}} f_{\alpha}\right)\left(1-\delta_{\alpha, 1}\right)  \tag{18}\\
& +\left(\frac{1}{T_{\alpha+1}^{R}} f_{\alpha+1}-\frac{1}{T_{\alpha}^{L}} f_{\alpha}\right)\left(1-\delta_{\alpha, N}\right) \\
& +\frac{1}{T^{e}\left(\rho_{\alpha}\right)}\left[f^{e}\left(\rho_{\alpha}\right)-f_{\alpha}\right],
\end{align*}
$$

where

$$
\begin{aligned}
\frac{1}{T_{\alpha}^{L}} & =P_{L}^{e}\left(\rho_{\alpha+1}\right) \nu_{B}^{e}\left(\rho_{\alpha}\right) \\
\frac{1}{T_{\alpha}^{R}} & =P_{R}^{e}\left(\rho_{\alpha-1}\right)\left(1-P_{L}^{e}\left(\rho_{\alpha+1}\right)\right) \nu_{B}^{e}\left(\rho_{\alpha}\right) \\
\frac{1}{T^{e}\left(\rho_{\alpha}\right)} & =\left(\nu_{B}^{e}+\nu_{A}^{e}+\nu_{F}^{e}\right)\left(\rho_{\alpha}\right)
\end{aligned}
$$

$\nu_{B}^{e}$ is defined by

$$
\nu_{B}^{e}(\rho)=\frac{1}{\rho} \iint_{v>\hat{v}_{+}} P_{B}^{C}\left(v, \hat{v}_{+}, f^{e}(\rho)\right)\left|v-\hat{v}_{+}\right| q_{B}\left(v, f^{e}(\rho)\right) f^{e}(\rho, v) F^{e}\left(\rho, \hat{v}_{+}\right) d \hat{v}_{+} d v
$$

Moreover,

$$
\nu_{X}^{e}(\rho)=\frac{1}{\rho} \iint_{v<\hat{v}_{+}}\left|v-\hat{v}_{+}\right| q_{X}\left(v, f^{e}(\rho)\right) f^{e}(\rho, v) F^{e}\left(\rho, \hat{v}_{+}\right) d \hat{v}_{+} d v
$$

for $X=A, F$. The lane changing probabilities $P_{Y}^{e}, Y=L, R$ are given by averaging $P_{Y}(v, f), Y=L, R$ with the equilibrium distribution $f^{e}$, i.e. $P_{Y}^{e}(\rho)$ is defined by

$$
P_{Y}^{e}(\rho)=\int_{0}^{w} P_{Y}\left(v, f^{e}(\rho)\right) F^{e}(\rho, v) d v
$$

Multiplication of (18) with $\phi(v)$ and integration over $v$ then gives an approximation for the source terms $S_{\alpha}^{\phi}$.

Using the above approximations, one obtains fluid dynamic equations in the form

$$
\begin{align*}
\partial_{t} \rho_{\alpha}+\partial_{x}\left(\rho_{\alpha} u_{\alpha}\right)= & \left(\frac{1}{T_{\alpha-1}^{L}} \rho_{\alpha-1}-\frac{1}{T_{\alpha}^{R}} \rho_{\alpha}\right)\left(1-\delta_{\alpha, 1}\right)  \tag{19}\\
& +\left(\frac{1}{T_{\alpha+1}^{R}} \rho_{\alpha+1}-\frac{1}{T_{\alpha}^{L}} \rho_{\alpha}\right)\left(1-\delta_{\alpha, N}\right)
\end{align*}
$$

and

$$
\begin{align*}
& \partial_{t}\left(\rho_{\alpha} u_{\alpha}\right)+\partial_{x}\left(p^{e}\left(\rho_{\alpha}\right)+\rho_{\alpha} u_{\alpha}^{2}\right)+\partial_{x}\left(A^{e}\left(\rho_{\alpha}\right)\right)  \tag{20}\\
= & \left(\frac{1}{T_{\alpha-1}^{L}} \rho_{\alpha-1} u_{\alpha-1}-\frac{1}{T_{\alpha}^{R}} \rho_{\alpha} u_{\alpha}\right)\left(1-\delta_{\alpha, 1}\right) \\
& +\left(\frac{1}{T_{\alpha+1}^{R}} \rho_{\alpha+1} u_{\alpha+1}-\frac{1}{T_{\alpha}^{L}} \rho_{\alpha} u_{\alpha}\right)\left(1-\delta_{\alpha, N}\right) \\
& +\frac{1}{T^{e}\left(\rho_{\alpha}\right)} \rho_{\alpha}\left[u^{e}\left(\rho_{\alpha}\right)-u_{\alpha}\right]
\end{align*}
$$

for $\alpha=1, \cdots, N$. Terms on the right hand side describe sources and sinks due to lane changing. All equilibrium quantities are determined from the stationary solution of the homogeneous kinetic equations (16).

## Remark:

The cumulative fluid dynamic equation derived from the cumulative kinetic model in Section 3.3 is

$$
\begin{aligned}
\partial_{t} \rho+\partial_{x}(\rho u) & =0 \\
\partial_{t}(\rho u)+\partial_{x}\left(p^{e}(\rho)+\rho u^{2}\right)+\partial_{x}\left(A^{e}(\rho)\right) & =\frac{1}{T^{e}(\rho)} \rho\left[u^{e}(\rho)-u\right]
\end{aligned}
$$

$\rho$ denotes the density per lane, i.e. $N \rho$ is the total density on the highway. This is obtained from the model above by adding all lanes and assuming that density und velocity are equal on all lanes.

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# A HIERARCHY OF MODELS FOR MULTILANE VEHICULAR TRAFFIC II: NUMERICAL AND STOCHASTIC INVESTIGATIONS 

A. Klar<br>Fachbereich Mathematik, Universität Kaiserslautern<br>Kaiserslautern, Germany<br>R. Wegener<br>Institut für Techno- und Wirtschaftsmathematik<br>Kaiserslautern, Germany

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#### Abstract

In this paper the work presented in [6] is continued. The present paper contains detailed numerical investigations of the models developed there. A numerical method to treat the kinetic equations obtained in [6] are presented and results of the simulations are shown. Moreover, the stochastic correlation model used in [6] is described and investigated in more detail.


## 1 Introduction

In this part we present numerical methods and results for the equations of vehicular traffic, which have been obtained in [6]. We refer to [6] as Part I. The microscopic, the kinetic and the macroscopic model are considered and detailed numerical results are given.

The paper is organized in the following way: In Section 2 we describe results obtained with the microscopic model described in I, Section 2 . We evaluate explicitely the velocity distribution functions, the fundamental diagram, and the leading vehicle distribution from the microscopic model. Section 3 contains the description of the method to simulate the homogeneous kinetic model and describes the way, the coefficients of the macroscopic model are determined numerically. To obtain these coefficients one uses the stationary distributions of the homogenous cumulative kinetic equation as described in I. The results of microscopic and kinetic simulations
are compared. Section 4 describes inhomogeneous situations. The macroscopic equations with the above mentioned coefficients are solved and results for a highway with a reduction of lanes are shown. In Section 5 a stochastic model is included, which is used in I, section 3 to obtain several quantities needed to set up the kinetic model, like the correlation function and the lane changing probabilities. This stochastic model is defined and investigated in more detail.

The physical units in the following numerical computations are fixed by setting the maximal velocity $w$ equal to 1 and the bumper to bumper distance $H_{0}$ equal to 1 . Thus, the maximal density per lane is $\rho_{m}=\frac{1}{H_{0}}=1$ and the unit time $t_{0}$ is given by $t_{0}=\frac{H_{0}}{w}=1$.

## 2 Simulation of the Microscopic Model

The microscopic model defined in I, Section 2 is considered for an equilibrium situation. We consider a periodic highway with length $L$. The highway has $N$ lanes and a total density $N \rho$, where $\rho$ denotes the average density per lane. $\rho$ is given by $\rho=\frac{M}{L}$ where $N M$ denotes the total number of cars on the highway.

The simulation is based on an event oriented scheme, i.e. the exact trajectory of any single vehicle is calculated from one event (interaction) to the next. During the interaction the velocities are changed according to the rules set up in I, Section 2.

In the actual computation the length of the highway under consideration is chosen as $L=500$. The number of lanes is choosen as $N=3$. For the reaction times the following values have been used: $T_{B}=5, T_{A}=10, T_{F}=20, T_{L} \sim T_{B} \sim T_{R}, T_{L}^{S}=$ $T_{B}=T_{R}^{S}$. Moreover, we choose $\alpha=2, \beta=0.5, \delta=0.1$ and $f_{D}=\frac{1}{0.05} \chi_{[0.95,1]}$. We refer to [4] for experimental data.

The number of vehicles is then defined by the desired value of $\rho$. Starting with an uniform distribution in space, a random distribution in velocity is chosen such that the distance between vehicle $i$ and its leading car is at least the braking distance $H_{B}\left(v_{i}\right)$, where $v_{i}$ is the velocity of vehicle $i, i=1, \cdots, N M$. The evolution is computed until a stationary state is reached. We use a large number of iterations and time averaging at the end of these iterations.

In Figure (1) the time development of the mean velocity $\frac{1}{N M} \sum_{i=1}^{N M} v_{i}$, computed with the microscopic model for situations with densities $\rho=0.1,0.2,0.4,0.6$, is shown. One observes a fast tendency towards equilibrium with fluctuations around the equilibrium state. The fluctuations depend on the number of vehicles. We mention that other quantities like higher order moments need a longer time to reach the final equilibrium state. The numerical simulations support the assumption that the equilibrium state is determined by one parameter, the density $\rho$.


Figure 1: Approach to the stationary state of the mean velocity for $\rho=$ 0.1, 0.2, 0.4.0.6 (averaging for $t>4 \cdot 10^{4}$ )

In Figure (2) we plot the velocity distribution functions for different values of $\rho$. The plot shows the distribution of the velocities after the final stationary state has been reached. The kinetic distribution functions for the same values of $\rho$ are plotted in Figure (4).

The mean velocities for the whole range of values of $\rho$, associated to these distribution functions, i.e. the fundamental diagram, is shown in Figure (3). It is plotted together with the kinetic fundamental diagram $u^{e}(\rho)$ defined in I, Section 4.2, (17), obtained from the stationary distributions of the homogeneous kinetic equation.

In Figure (9) we show the distribution of the distances of the leading vehicles obtained from the stationary state of the microscopic model for $\rho=0.4$. It is compared with the velocity averaged kinetic distribution of the leading vehicles. This distribution is obtained by using assumption (2), see also (6) in I, Section 3.1, combined with the stationary solution of the homogeneous kinetic equation.

If, e.g., the fundamental diagram is compared with measured data, as reported in [7], one observes good qualitative agreement.

Moreover, one observes, as will be discussed in more detail in the next section, good agreement of the microscopic with the kinetic results. This gives a numerical justification for the derivation procedure leading to the kinetic equation presented in Part I.


Figure 2: Distribution function for $\rho=0.1,0.2,0.4 .0 .6$

## 3 Simulation of the Homogeneous Kinetic Model and Macroscopic Coefficients

To obtain the coefficients for the fluid dynamic equations we have to compute the stationary distributions of the homogeneous cumulative kinetic equation I, (16). We treat the kinetic equation by a discretization scheme, that is described in the following:

A simple standard discretization of the equation in velocity-space needs a large number of discretization points in order to describe correctly the influence of the singularities appearing at $v=0$ and $v=w=1$. Therefore, we divide the velocity space into a certain number of cells and calculate the transition rates between the cells given by the kinetic equation. One uses either a fixed or an adaptive grid in velocity space. To get accurate solutions the use of an adaptive grid, concentrating the grid points around the peaks of the distribution function, gives a big advantage in computation time. Here we describe for simplicity the procedure with a fixed discretization. We introduce gridpoints

$$
v_{i}=\frac{i}{K-1}, \quad i=0, \ldots, K-1
$$

in $[0,1]$. Integrating the distribution function $f$ over each cell $M_{i}$ with $M_{0}=$

$$
\left[0, \frac{1}{2(K-1)}\right], M_{i}=\left[\frac{i-\frac{1}{2}}{K-1}, \frac{i+\frac{1}{2}}{K-1}\right], i=1, \cdots K-2 \text { and } M_{K-1}=\left[1-\frac{1}{2(K-1)}, 1\right], \text { gives }
$$

$$
h_{i}=\int_{M_{i}} f(v) d v
$$

The discretized kinetic equation is then given by integrating the kinetic equation I, (16) with respect to $v$ over each cell $M_{i}$. This gives

$$
\partial_{t} h_{i}=\int_{M_{i}} C_{C}(f)(v) d v
$$

Using the integration rule

$$
\int_{0}^{w} \psi(v) f(v) d v \sim \sum_{i=0}^{K-1} \psi\left(v_{i}\right) h_{i}
$$

one obtains a discrete velocity model:

$$
\partial_{t} h_{i}=\sum_{j, k=0}^{K-1} S_{i j k} h_{j} h_{k}
$$

The transition rates $S_{i j k}$ are determined by an explicit integration of the collision kernels over the cells $M_{i}$. The most important fact about this type of discretization is the conservation of density (number of vehicles). One shows

$$
\sum_{i=0}^{K-1} S_{i j k}=0
$$

This gives the assertion due to $\rho=\sum_{j=0}^{K-1} h_{i}$.
In the following simulations of the kinetic model the same parameters for the reaction times as in the microscopic model are used.

Figure (3) shows plots of the kinetic fundamental diagrams $u^{e}(\rho)$. This means we plot the mean values of the stationary distributions of the homogeneous kinetic equation $I$, (16) for the whole range of values of $\rho$. For the numerical simulation the parameter $\lambda$ in the definition I , (6) of the leading vehicle distribution $q$ has been choosen equal to 0 and $1-\epsilon$, where $\epsilon$ is small. ( $\lambda$ did denote the part of the vehicles having a following behaviour, see also Section 5.1.). We remark that the results are not sensitive to the exact choice of $\epsilon$. Moreover, the microscopic fundamental diagram is plotted. The plot of the mean velocity obtained from the kinetic model with $\lambda=1-\epsilon$ agrees very well with the microscopic fundamental diagram. This shows that in our microscopic simulation essentially all cars have a following behaviour. In particular, for densities larger than $\rho=0.15$ the agreement is very good. This corresponds to the fact that for higher densities all cars are trapped between braking and acceleration line. This behaviour is generally not true


Figure 3: Kinetic $(\lambda=0,1-\epsilon)$ and microscopic fundamental diagrams
in practice for small values of $\rho$. Independent cars appear as well. Another choice of $\lambda$ could be more appropriate in these situations. We mention, that measured data for the fundamental diagram are reported, e.g., in [7]. As mentioned above the qualitative agreement with these data is good.

In order to obtain a good agreement of microscopic and kinetic results and for reasons of simplicity, we have choosen in the following $\lambda=1-\epsilon$. Figure (4) shows the stationary distributions $f^{e}(\rho)$ of the homogeneous kinetic equation for different values of $\rho$. This may be compared to the microscopic distribution functions shown in Figure (2). One observes a good aggreement for most of the values of $\rho$. However, for $\rho$ very small or very large the form of the microscopic and kinetic distribution functions deviates. This is, for example, due to the additional accelerations in the microscopic model.

In the following all quantities defined in I, Section 4.2 are determined. Figure (5) shows a plot of the traffic pressure $p_{e}(\rho)$. The Enskog coefficient $A^{e}(\rho)$ is shown in Figure (6). We mention that the values of the Enskog coefficient are much larger than those of $p_{e}(\rho)$. Figure (7) shows a plot of the lane changing probabilities $P_{Y}^{e}(\rho)=P_{R}^{e}(\rho)=P_{L}^{e}(\rho)$. Equality is due to the fact that we have choosen $T_{L}^{S}=T_{R}^{S}$. Figure (7) shows the lane changing probabilities due to interactions. In general, also spontaneous lane changing has to be taken into account, which is not caused by another car. This type of lane changing has for simplicity not been considered in the model up to now, see also Remark 3 in I, Section 2. However, in particular, for inhomogeneous situations like the one treated in the next section, it is important to include this kind of lane changing. Figure (8) shows a plot of the interaction frequencies $\nu_{B}^{e}(\rho)$ and $\nu_{A}^{e}(\rho)$. We remark that the results for the pressure, the lane changing probabilities and the collision frequencies have been slightly smoothed.

Finally, we plot in Figure (9) the leading vehicle distributions obtained in two ways.


Figure 4: Kinetic distribution functions for $\rho=0.1,0.2,0.4,0.6$


Figure 5: Traffic pressure $p_{e}(\rho)$


Figure 6: Enskog coefficient $A^{e}(\rho)$


Figure 7: Lane changing probabilities $P_{Y}^{e}(\rho)$ due to interactions


Figure 8: Interaction frequencies $\nu_{B}^{e}(\rho), \nu_{A}^{e}(\rho)$

We fix $\rho=0.4$ and do not distinguish between vehicles with different velocities. First the microscopic model is used directly to obtain the distribution of the distances of the leading vehicle. This is compared to a distribution obtained by using the special form for $q$ assumed in (2) or $\mathrm{I},(6) . q(h ; v, f)$ is computed using the distribution functions $f=f^{e}(\rho)$ obtained from the stationary distribution of the cumulative homogeneous kinetic equation. We plot again the velocity averaged version, i.e. we plot $<q\left(h ; \cdot, f^{e}(\rho)\right)>$. One observes, that the leading vehicle distribution are nearly coincident for this value of $\rho$. For very small or very large values of $\rho$ a slightly larger deviation is observed. This justifies the ad hoc choice of the leading vehicle distribution $q$ for the kinetic model in I, Section 3.1., (6).

## 4 Inhomogeneous Simulations

In the following series of figures an inhomogeneous traffic flow situation is shown. We refer, e.g., to [2], for other simulations of macroscopic multilane models. We consider a highway with a reduction of the number of lanes from 3 to 2 after two thirds of the highway under consideration. The length $L$ of the highway is equal to 1000. The lane drop is at the point $x=600$.

The example is calculated with the multilane fluid dynamic equations stated in (19), (20) in I, Section 4.2. The coefficients are determined from the kinetic model and have been computed in the last section. Additionally, we increase the lane changing rates $\frac{1}{T_{\alpha}^{L}}$ and $\frac{1}{T_{\alpha}^{R}}$ defined in I, Section 4.2 by adding a fixed quantity independent of $\rho$ in order to account for spontaneous lane changing not due to interactions with other cars. This has been neglected in the model so far. Looking at the values of $\nu_{B}^{e}(\rho)$ in Figure (8), we have chosen the value 0.005 as the additional lane changing


Figure 9: Leading vehicle distributions for $\rho=0.4$
rate due to spontaneous lane changing. Here, obviously other models can be taken. For example, models, where the above lane changing rate is assumed to depend on $\rho$, may be used. We refer to [2] for different models and to [8] for experimental investigations. In front of the lane drop an additional strong increase of the lane changing frequency to the right is included on the left lane in order to obtain an empty highway on this lane just in front of the lane drop. (The size of this lane changing area is $\Delta x=50$, the increase in lane changing frequency is $10 \frac{w}{\Delta x}$.)

We start with an empty highway and prescribe the incoming values at 0 . The number of ingoing vehicles is equal on all lanes. Moreover, a constant flux of incoming vehicles is used. The solutions of the macroscopic equation is shown in Figures (10) and (11) for all lanes and different times. The density $\rho$ on the three lanes is plotted in Figure (10), the flux $q=\rho u$ in (11). Starting with an empty highway one observes in Figure (10) free flow of the vehicles until the stretch is completely filled with vehicles. The overall density is small compared to the maximal density, such that there is no influence of the lane drop. When the stretch finally is filled with vehicles, the density rises at the bottleneck. In particular, on the lane in the middle, to which the cars are changing from the right lane, one observes an increase in density. Later one observes the formation of a traffic jam, which is finally running backwards on all lanes. These results are, at least qualitatively, similiar to those that are observed in real traffic flow situations. A detailed comparison with measured data is left to future work. For a numerical simulation of a cumulative kinetic model and a comparison of the results with those of the associated macroscopic model, we refer to [5].


Figure 10: Time development of the density computed by the macroscopic multilane equations: Lane drop from 3 to 2 lanes at $x=600$


Figure 11: Time development of the flux computed by the macroscopic multilane equations: Lane drop from 3 to 2 lanes at $x=600$

## 5 Stochastic correlation model

In this section the space homogeneous situation is treated in more detail using stochastic arguments. We introduce a basic stochastic model. A one lane highway with a large number of vehicles is considered. The distances between the vehicles are represented by probability variables $D_{1}, D_{2}, \cdots$. They are assumed to be independent. The location of the vehicles is given by the probability variables $X_{1}, X_{2}, \cdots$ defined by $X_{n+1}=X_{n}+D_{n}$ with $X_{1}$ given. The probability variables representing the velocities of the vehicles are denoted by $V_{1}, V_{2}, \cdots$. The velocities are distributed according to a given distribution function $f$ with $\int_{0}^{w} f d v=\rho$. The stochastic process $(V, X)$ can be viewed as a Markov renewal process, see [1].

In Part I, Section 3.1, the leading vehicle distribution, i.e. the distribution of the distances between the vehicles and their leading vehicles and the lane changing probabilities are used to obtain the kinetic model. The distribution of the distances $D_{1}, D_{2}, \cdots$ is discussed in the next subsection. An approximation for the lane changing probabilities is given in the second subsection.

### 5.1 Leading Vehicle Distribution

In this section the definition of the leading vehicle distribution is discussed. Each vehicle drives with an individual velocity $v$.

Looking at the microscopic model in I, Section 2 one observes that the braking line $H_{B}(v)$ represents the minimal distance between the vehicles. One part of the vehicles is assumed to be independent or freely driving. More exactly, they are assumed to have exponentially distributed leading vehicles, i.e. the density of the leading vehicle distribution for a vehicle with velocity $v$ is

$$
q(h ; v, f)=\tilde{\lambda} e^{-\tilde{\lambda}\left(h-H_{B}(v)\right)} \chi_{\left[H_{B}(v), \infty\right)}(h)
$$

The parameter $\tilde{\lambda}$ is determined by the requirement that the mean space between the cars is equal to $\frac{1}{\rho}$ given by

$$
\begin{equation*}
<\int_{0}^{\infty} h q(h ; v, f) d h>=\frac{1}{\rho} \tag{1}
\end{equation*}
$$

Looking again at the microscopic model one observes that most of the cars are trapped between braking and acceleration line. They are in a following behaviour oscillating between the braking and acceleration line. We assume therefore that only a part of the vehicles $((1-\lambda), \lambda<1)$ has exponentially distributed leading vehicles and the other part $(\lambda)$ has a following behaviour. For this part we assume that the
headway is uniformly distributed between braking and acceleration line $H_{B}$ and $H_{A}$. One obtains

$$
\begin{align*}
q(h ; v, f) & =(1-\lambda) \tilde{\rho} e^{-\tilde{\rho}\left(h-H_{B}(v)\right)} \chi_{\left[H_{B}(v), \infty\right)}(h)  \tag{2}\\
& +\lambda \frac{1}{H_{A}(v)-H_{B}(v)} \chi_{\left[H_{B}(v), H_{A}(v)\right.}(h)
\end{align*}
$$

with the reduced density $\tilde{\rho}$ determined in such a way that (1) is fulfilled:

$$
\begin{equation*}
\tilde{\rho}=\frac{(1-\lambda) \rho}{1-\rho\left[(1-\lambda)<H_{B}>+\frac{\lambda}{2}\left(<H_{B}>+<H_{A}>\right)\right]} \tag{3}
\end{equation*}
$$

Here $(1-\lambda)<H_{B}>+\frac{\lambda}{2}\left(<H_{B}>+<H_{A}>\right)$ is the average space required per vehicle with exponentially distributed leading vehicles, if the rest is assumed to be distributed between $H_{B}$ and $H_{A}$.

### 5.2 Lane Changing Probabilities

Determination of the probability of a gap is difficult for the general situation considered above. Moreover, this is not really the situation one has in mind. Instead one considers a homogeneous situation not depending on the special starting point $X_{1}$. Therefore, one determines an asymptotic distribution at infinity of the above process or, equivalently, one uses a so called stationary renewal process, see $[3,1]$.

The probability variables $D_{i}, i=1,2, \cdots$ are distributed according to the density $q\left(h ; V_{i}, f\right)$. The velocity variables $V_{i}, i=1,2, \cdots$ are distributed according to $f$ and independent of the $D_{i}$. We mention that the expectation of $D_{i}$ is given by $E\left(D_{i}\right)=\mu=\frac{1}{\rho}$ according to the definition of $q$. In particular $E\left(D_{i}\right)$ is independent of $i$.

One looks at a fixed spatial point $x$ and determines the distribution of the distance $B_{x}$ between the point $x$ and the next car behind $x$ and the distance $F_{x}$ between $x$ and the next car in front of $x$ :

$$
\begin{array}{r}
B_{x}=x-X_{N_{x}} \\
F_{x}=X_{N_{x}+1}-x
\end{array}
$$

if $X_{N_{x}} \leq x<X_{N_{x}+1}$. In the language of renewal processes $B_{x}$ and $F_{x}$ are the so called current and excess life. Obviously, $P\left(F_{x} \geq h_{1}, B_{x} \geq h_{2}\right)$ gives the probability for a gap of length $h_{1}$ in front of $x$ and of length $h_{2}$ behind $x$. The asymptotic value of this probability as $x$ tends to infinity is obtained using the renewal theorem, see e.g. $[3,1]$. Rewriting the results obtained in [1] for the present context, one obtains:

$$
P\left(F_{x} \geq h_{1}, B_{x} \geq h_{2}\right)=\frac{1}{\mu} \int_{h_{1}+h_{2}}^{\infty}[1-<Q(h ; \cdot, f)>] d h
$$

with the distribution function $Q$ defined by

$$
\begin{equation*}
Q(h ; v, f)=\int_{0}^{h} q\left(h^{\prime} ; v, f\right) d h^{\prime} . \tag{4}
\end{equation*}
$$

This leads to

$$
\begin{equation*}
P\left(F_{x} \geq h_{1}, B_{x} \geq h_{2}\right)=<\rho \int_{h_{1}+h_{2}}^{\infty} \int_{h}^{\infty} q\left(h^{\prime}, \cdot, f\right) d h^{\prime} d h>. \tag{5}
\end{equation*}
$$

These considerations are now used to determine the lane changing probabilities $p_{Y}\left(v, v^{\prime}, f\right)$ :

Due to the considerations in I, Section 3.1 the driver changes to the new lane, if the distance after the lane change between the changing car with velocity $v$ and its leading car on the new lane is at least $H_{Y}^{S}(v), Y=L, R$. Moreover, the distance between the changing car and its follower on the new lane with velocity $v^{\prime}$ must be at least $H_{Y}^{S}\left(v^{\prime}\right), Y=L, R$.

Setting $h_{1}=H_{Y}^{S}(v)$ and $h_{2}=H_{Y}^{S}\left(v^{\prime}\right)$ in the above formula leads therefore to the desired lane changing probability $p_{Y}\left(v, v^{\prime}, f\right)$ used in I, Section 3.1:

$$
\begin{equation*}
p_{Y}\left(v, v^{\prime}, f\right)=<\rho \int_{H_{Y}^{S}(v)+H_{Y}^{S}\left(v^{\prime}\right)}^{\infty} \int_{h}^{\infty} q\left(h^{\prime}, \cdot, f\right) d h^{\prime} d h>. \tag{6}
\end{equation*}
$$

Using the expression (2) for $q$ a more explicit expression for $p_{Y}\left(v, v^{\prime}, f\right)$ is given by averaging the function $\tilde{p}_{Y}\left(v, v^{\prime}, \tilde{v}, f\right)$ given below with respect to $\tilde{v}$ :

$$
p_{Y}\left(v, v^{\prime}, f\right)=<\tilde{p}_{Y}\left(v, v^{\prime}, \cdot, f\right)>.
$$

$\tilde{p}_{Y}\left(v, v^{\prime}, f\right)$ is given by

$$
p_{Y}\left(v, v^{\prime}, \tilde{v}, f\right)=\rho R\left(H_{Y}^{S}(v)+H_{Y}^{S}\left(v^{\prime}\right), \tilde{v}, f\right)
$$

with

$$
R(h, v, f)=(1-\lambda) R_{0}(h, v, f)+\lambda R_{1}(h, v, f),
$$

where

$$
R_{0}(h, v, f)= \begin{cases}\frac{1}{\hat{\rho}}+H_{B}(v)-h & \text { if } h<H_{B}(v) \\ \frac{1}{\hat{\rho}} e^{\tilde{\rho}( }\left(-H_{B}(v)\right) & \text { if } H_{A}(v)>h>H_{B}(v) \\ \frac{1}{\hat{\rho}} e^{\tilde{\rho}\left(h-H_{B}(v)\right)} & \text { if } H_{A}(v)<h\end{cases}
$$

and

$$
R_{1}(h, v, f)=\left\{\begin{array}{ll}
H_{B}(v)-h+\frac{H_{A}(v)-H_{B}(v)}{2} & \text { if } h<H_{B}(v) \\
\frac{\left(H_{A}(v)-h\right)^{2}}{2\left(H_{A}(v)-H_{B}(v)\right)} & \text { if } H_{A}(v)>h>H_{B}(v) . \\
0 & \text { if } H_{A}(v)<h
\end{array} .\right.
$$

## 6 Conclusions

We have thus obtained a consistent new hierarchy of models ranging from a microscopic follow the leader model to a macroscopic fluid dynamic multilane model. In particular, a derivation procedure for kinetic and macroscopic traffic flow models is given. The basic features of this hierarchy are:

- The models are based on reaction thresholds with values derived from experimental data.
- The kinetic model uses a leading vehicle distribution derived from the behaviour on the microscopic level. This takes into account the strongly correlated behaviour of the vehicles.
- An Enskog like kinetic multilane model and a new cumulative model is derived. The cumulative model is derived from the multilane one.
- Macroscopic multilane models are derived by determining the coefficients from the stationary solution of the cumulative, homogeneous kinetic equation.
- The derivation of the macroscopic and kinetic equations is supported by numerical analysis. Numerical computations are presented on all levels and a comparison of the results on different levels is given.
- Further work is required for numerical simulations of the inhomogeneous kinetic multilane equations and a comparison of kinetic and macroscopic multilane results.


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