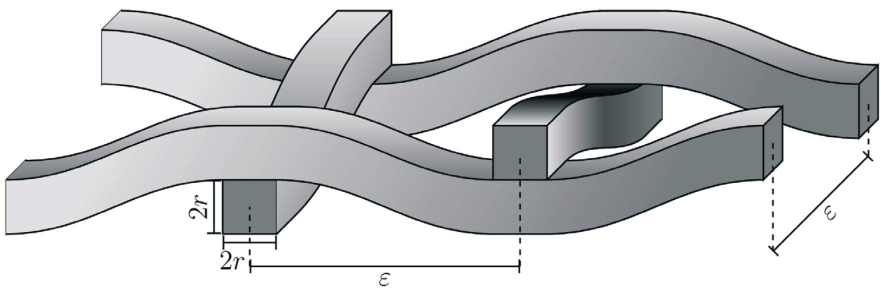
Stephan Wackerle

Homogenization and Dimension Reduction of a Textile Shell and Minimization of Buckling with Microstructure-Optimization



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TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

PhD-Thesis

Homogenization and Dimension Reduction of a Textile Shell and Minimization of Buckling with Microstructure-Optimization

Stephan Wackerle

Vom Fachbereich Mathematik der Technischen Universität Kaiserslautern zur Verleihung des akademischen Grades Doktor der Naturwissenschaften (Doctor rerum naturalium, Dr. rer. nat.) genehmigte Dissertation

Gutachter:

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Technische Universität Kaiserslautern

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Università di Napoli "Federico II"

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Abstract

Fachbereich Mathematik

Homogenization and Dimension Reduction of a Textile Shell and Minimization of Buckling with Microstructure-Optimization

Stephan Wackerle

For technical textiles it is important to understand the influence of weaving pattern, fiber properties and contact between the fibers on the macroscopic properties. In particular, if the macroscopic behavior of the textile should be optimized the exact relation between microstructure and effective properties are crucial.

The thesis begins with the simultaneous homogenization and dimension reduction of textile structures. The rigorous derivation of a homogenized elasticity problem is shown for two different energy regimes, linear and von-Kármán-type. The homogenizations for both energy regimes begin with the decomposition of displacements and general Korn-type estimates of the displacement fields. These estimations yield together with an textile-adapted unfolding operator the limiting problems.

In particular, the linear elasticity problem is augmented with a Signorini-type contact condition, which further defines the limit problem. Here we present two different orders of contact and distinguish a linear elastic plate or a Leray-Lions-type problem in the limit. For the nonlinear elasticity problem with glued fibers, the energy scaling is specifically chosen such that in the limit the von-Kármán plate arises. Due to the nonlinearity of the problem the limit is derived with means of Γ-convergence. Eventually, we prove that for isotropic homogeneous fibers and the given symmetries of the textile the homogenized plate is orthotropic.

The end of the thesis is dedicated to the investigation and optimization of buckling of textiles. The critical strain for buckling of an orthotropic plate is derived for both compression and tension and given in terms of the elastic properties. For a plate under uniaxial compression an optimization scheme for the delay and shape-modification of buckling is presented and illustrated with implemented examples.

TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

Zusammenfassung

Fachbereich Mathematik

Homogenization and Dimension Reduction of a Textile Shell and Minimization of Buckling with Microstructure-Optimization

Stephan WACKERLE

Für technische Textilien ist es essentiell den Einfluss von Webmuster, Fasereigenschaften und Kontakt zwischen den Fasern auf die makroskopischen Eigenschaften zu verstehen. Insbesondere, wenn das makroskopisches Verhalten des Textils optimiert werden soll, ist die genaue Beziehung zwischen Mikrostruktur und effektiven Eigenschaften entscheidend.

Die Arbeit beginnt mit der simultanen Homogenisierung und Dimensionsreduktion von textilen Strukturen. Die Homogenisierung ist mathematisch hergeleitet für zwei verschiedene Energieregime, linear und von-Kármán. Die Homogenisierungen für beide Energieregime beginnen mit der Zerlegung von Verschiebungen (Decomposition of dispalcements) und allgemeinen Korn-Abschätzungen der Verschiebungsfelder. Zusammen mit einem Entfaltungsoperator (Unfolding operator) für die textile Struktur ergeben diese Abschätzungen die Grenzprobleme.

Insbesondere wird das linear elastische Problem durch eine Signorini-Kontaktbedingung ergänzt, die das Limitproblem weiter spezifiziert. Wir stellen hier zwei verschiedene Ordnungen für den Kontakt zwischen Fasern vor und unterscheiden im Limes zwischen einer linear elastischen Platte und einem Leray-Lions-Problem. Für das von-Kármán Elastizitätsproblem mit geklebten Fasern ist die Skalierung der Energie gezielt so gewählt, dass im Grenzfall die von-Kármán-Platte entsteht. Aufgrund der Nichtlinearität des Problems wird der Grenzwert mittels der Γ-Konvergenz hergeleitet. Abschließend beweisen wir, dass für isotrope homogene Fasern und den gegebenen Symmetrien des Textils die homogenisierte Platte orthotrop ist.

Das Abschluss der Arbeit ist der Untersuchung und Optimierung des Knickens von Textilien gewidmet. Die kritische Dehnung, die das Knicken einer orthotropen Platte induziert, wird in Bezug auf die elastischen Eigenschaften sowohl für Kompression, als auch für Zug angegeben. Für eine Platte unter uniaxialer Kompression wird ein Optimierungsschema für die Verzögerung und Formänderung der Knickung vorgestellt und anhand von implementierten Beispielen illustriert.

Contents

Notation

Throughout the thesis the following notation is used:

Notation	Explanation
ε	a small parameter; size of the periodicity cell
κ	small parameter, independent of ε
r	a small parameter, radius of the beams, generally $r \leq \kappa\varepsilon$
$\omega_r = (-r, r)^2$	the beam's cross-section
$(1, q)$	index for the q-th $\mathbf{e}_1$-directed beam, $q \in \{1, \dots, N_\varepsilon\}$
$(2, p)$	index for the p-th $\mathbf{e}_2$-directed beam, $p \in \{0, \dots, N_\varepsilon\}$
$P_r, \mathcal{P}_\varepsilon$	the reference beam domain, curved beam domain
$\mathcal{S}_\varepsilon$ or Ω^*_ε	beam structure
$\Omega = (0, L)^2$	the limit plate domain
$N_\varepsilon = \frac{L}{2\varepsilon} \in \mathbb{N}$	number of periodicity cells in each direction
$\mathbf{C}_{pq}$	mutual contact area of beam q and p
$\mathcal{K}_\varepsilon$	set of contact-mid-points, $\mathcal{K} = \{0, \dots, 2N_\varepsilon\}^2$
λ, μ	Lamé constants
$\mathcal{C}(\Omega)$	space of continuous functions over Ω
$H^1(\Omega)$ or $H^2(\Omega)$	Sobolev-space of functions over Ω
∇u	gradient of a function u
$e(u) = \frac{\nabla u + (\nabla u)^T}{2}$	symmetric gradient of u
$a \wedge b$	the cross-product of two vectors $a, b \in \mathbb{R}^3$
I_d	identity mapping
$\mathbf{I}_3$	unit matrix

Chapter 1

Introduction

The topic of this thesis originated from industrial projects at the Fraunhofer ITWM about the buckling of belt-shaped textiles. While the buckling under compression is a commonly known problem [34, 44], it may arise also under tension, see [9, 42]. In many industrial applications such buckling effects are unwanted, since the shape of an originally flat object is disturbed. This leads to deformations, which can not be handled in a subsequent treatment, or lead to different load distributions within the specimen or the environment. The idea is to modify the textile's micro-structure in such a way that buckling is reduced or delayed. Additionally, it is possible to adjust the buckling shape, which means that once the critical regime is reached, the buckling deformation of the specimen is as uncritical as possible. However, for the production it is necessary to keep the periodic structure for an easy fabrication and to satisfy the requirements for its designated applications. Furthermore, the periodic structure helps to exclude defects of the final product, be it optical or mechanical. In recent investigations textiles are very present, since they are in a way customizable, such that they can be used in many situations ranging from clothes to structural parts in vehicles.

To model the buckling behavior one typically uses the von-Kármán plate. The model is at the verge between linear and a fully non-linear elasticity. As shown in *A hierarchy of plate models derived from nonlinear elasticity by* Γ*-convergence* (see [21]) there are different limit models varying from a fully nonlinear membrane model to the standard linear plate model depending on the inherent energy, cf. Table 1.1. Note, that in [21] the powers are shifted by one, since an energy normalized w.r.t the thickness is considered.

In the following analysis we only consider a linear energy, which corresponds to $\|e(u)\|^2_{L^2(\Omega_h)} < h^5$, and the energy regime for von-Kármán, i.e., $\|e(u)\|_{L^2(\Omega_h)} = h^5$, (see also [20, 36, 46]). For simplicity, we assume indifferently $\|e(u)\|^2_{L^2(\Omega_h)} \leq h^5$, yet start with a linear or a nonlinear strain tensor respectively.

For textile structures it is crucial to resolve the contact between the fibers. This is presented in Chapter 3, where the linear elastic energy is augmented by a non-rigid contact condition between the fibers. The contact, i.e., the order of possible sliding at the contact areas, plays a crucial role for the determination of the limit although the overall energy is assumed to be still in the linear regime. Indeed, strong contact leads to the typical homogenized plate, while for weak contact the limit problem is not even macroscopically a plate anymore. In

$\|e(u)\|^2_{L^2(\Omega_h)} = h^k$	limit model
$k > 5$	Linear plate
$k = 5$	von-Kármán plate
$5 > k > 3$	Linearized isometry constraint
$k = 3$	Bending theory
$k < 3$	Membrane theory

Figure 1.1: Different asymptotic models w.r.t $\|e(u)\|_{L^2(\Omega_h)}$, a measure for the elastic energy, in terms of the plate's thickness h occupying the domain $\Omega_h = (0, L)^2 \times (-\frac{h}{2}, \frac{h}{2})$.

this thesis the contact is restricted to two different orders, which result in either linear cell problems or in a type of *Leray-Lions*-operator (e.g. see [19, 32]).

The general strategy in this whole thesis is to obtain a homogenized model of the textile with an additional dimension reduction from 3D to 2D. The different regimes of linear elasticity with contact and of the von-Kármán plate are investigated separately, due to a different analysis. For the buckling optimization the von-Kármán is important, which is addressed in the last part of this thesis.

Specifically, for the first homogenization and dimension reduction the textile is resolved on the fiber scale with a canvas weaving pattern. These fibers with radius r and periodicity ε are modeled as beams, which are periodically in contact with each other. The contact is modeled via a gap-function g_ε. In the linear energy regime, i.e., of order $\|e(u)\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{5/2}$, the decompositions of displacements (see [25]) are adapted for the beam structure. This decomposition allows to split the displacements into elementary (displacements of the middle-line and rotation of the cross-section) and residual displacements. In particular, the decomposition for beams yields middle-line displacements and rotations of different order (w.r.t. ε and r). On the basis of the beam displacements global fields are introduced. These are defined on the whole plate and give rise to Korn-type inequalities and estimates for the textile displacements. These estimates are necessary to obtain compactness results for the asymptotic analysis. The limit displacements are derived with the help of an unfolding-rescaling operator, i.e., an unfolding operator with an incorporated dimension reduction (see e.g. [16, 28]). Depending on the contact conditions between the fibers we identify two different limit problems. The cases of cubic contact $g_\varepsilon \sim \varepsilon^3$ and $g_\varepsilon \sim \varepsilon^{3+\alpha}$ with $\alpha > 0$ are further investigated, which lead to a *Leray-Lions problem* or a standard homogenized linear plate, respectively.

For the second homogenization concerning the von-Kármán plate the same textile structure with glued fibers, i.e., $g_\varepsilon \equiv 0$, is considered. To utilize here the decomposition of plate displacements, we introduce an extension to the plate domain without holes. For this type of extension the assumption of glued fibers is crucial. The decomposition for plates yields again elementary and residual plate displacements. We again provide the Korn's-type estimates for the displacements. The asymptotic behavior of displacements and the strain tensor is investigated and the results are similar to [4, 5, 16, 25]. In the derivation of the limit problem a Γ-convergence argument yields the limiting energy of von-Kármán-type. The corresponding cell problems remain as in chapter 3 with contact condition $g_\varepsilon = 0$.

The last part of the thesis is dedicated to the investigation and optimization of buckling for textile plates, to show the effectiveness of the derived homogenized models. There are two cases discussed: tension and compression induced buckling. Note, that the tension generates a cross-compression leading to buckling in the lateral direction. For the compression dominated case, direct or induced, a macroscopic optimization scheme is developed, which is designed to improve buckling-shape and delay its appearance. The optimization accounts for constraints coming from textile-industry and is implemented in MATLAB. Numerical examples underline the effectiveness of the macroscopic optimization. Additionally, the influence of the Pareto optimization front is discussed, which arises naturally due to the two considered objectives. By a second step, the macroscopic results are transfered to the micro-structure with a given design space. This is shown for an academic example.

The thesis is concluded by a summary.

Part I

Homogenization for Textiles

Chapter 2

Introduction to Homogenization

Homogenization of partial differential equations is an asymptotic method for studying problems with highly oscillating and usually periodic coefficients or geometry. It is an efficient way to find a representative model for complex problems, thereby reducing the effort in analyzing or computing the whole structure. Since the method provides simpler macroscopic models with additional corrector problem containing the microscopic information, it is used whenever multiple scale arise in the problem. A typical homogenization problem is

$$\nabla \cdot \left(A \left(\frac{x}{\varepsilon} \right) \nabla u_\varepsilon \right) = f, \tag{2.1}$$

where ε is a small parameter, $A(y)$ a 1-periodic coefficient, u_ε the solution and f the force. Usually, the variable x is referred to as the slow or macroscopic variable, while $y = \frac{x}{\varepsilon}$ is called fast or microscopic variable. It is obvious that such problems heavily depend on ε and the smaller ε the more oscillations arise. However, in the field of homogenization the idea is to obtain a representative problem without this parameter by investigating the limit $\varepsilon \to 0$. The homogenized model is then a good approximation of the original problem for small ε, yet without depending on ε. For introductory literature see for instance [15, 16].

Chronologically speaking, the first homogenization method is the asymptotic expansion, which is a formal method to obtain a hierarchy of problems corresponding to the original problem. For this approach the actual solution is replaced by a series

$$u_\varepsilon = u_0 \left(x, \frac{x}{\varepsilon} \right) + \varepsilon u_1 \left(x, \frac{x}{\varepsilon} \right) + \varepsilon^2 u_2 \left(x, \frac{x}{\varepsilon} \right) \cdots . \tag{2.2}$$

This substitution is inserted into the problem and by collecting the coefficients for each order of ε a hierarchy of problems arises. This hierarchy allows to successively compute the solutions u_i to determine the full problem.

The mathematical treatment began with Bensoussan et al. [2] and Giorgi and Spagnolo [22]. In the year 1989 as Nguetseng [37] introduced the two-scale convergence as notion for homogenization problems.

Definition 2.0.1 (Two-Scale-convergence [17])**.** *Let $p \in (1,\infty)$. Then a bounded sequence $\{w_\varepsilon\} \subset L^p(\Omega)$ two-scale converges to some $w \in L^p(\Omega, Y)$, if*

$$\int_\Omega w_\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right)\, dx \to \int_{\Omega\times Y} w(x,y)\varphi\,(x,y)\ dxdy, \qquad \forall\varphi \in \mathcal{D}(\Omega\times Y). \tag{2.3}$$

Obviously the two-scale convergence is similar to a weak convergence, yet special test-functions are necessary to account for the oscillating scale. In the following decade it was further investigated and developed by Allaire [1] and many others.

In 2002 the periodic unfolding method was introduced by Cioranescu et al. [17] and later extended and refined in [16, 25]. This method gives rise to an operator calculus incorporating the splitting of the scales, i.e., slow and fast variable. The so called unfolding operator $\mathcal{T}_\varepsilon$ is a mapping from $L^p(\Omega)$ into $L^p(\Omega \times Y)$.

Definition 2.0.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set and Y a reference cell. Furthermore, let w be a measurable function on Ω. Then the unfolding operator is defined as*

$$\mathcal{T}_\varepsilon(w)(x,y) = \begin{cases} w(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y) & a.e.\ for\ (x,y) \in \widehat{\Omega}_\varepsilon \times Y \\ 0 & a.e.\ for\ (x,y) \in \Lambda_\varepsilon \times Y \end{cases} \tag{2.4}$$

where $\widehat{\Omega}_\varepsilon = int\,\{\varepsilon(\xi+Y) \mid \varepsilon(\xi+Y) \subset \Omega, \xi \in \mathbb{Z}^n\}$ denotes the interior cells and $\Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon$ the boundary cells.

This splitting of the variables via an operator translates the two-scale convergence to a more general weak convergence of the unfolded sequences. In fact, the two scale convergence and the weak convergence under unfolding are equivalent, see [18, Prop. 2.14]. Hence, for bounded sequences $\{\varphi_\varepsilon\} \subset L^p(\Omega)$, $p \in (1,\infty)$ the convergences

$$\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup \varphi \quad \text{weakly in } L^p(\Omega\times Y) \quad \Longleftrightarrow \quad \{w_\varepsilon\} \text{ two-scale converges to } w \tag{2.5}$$

are equivalent. Note that an advantage of the unfolding method is that the weak convergence works with the direct dual space $L^{p'}(\Omega \times Y)$. Moreover, it is possible to use smooth functions $\mathcal{D}(\Omega \times Y)$ together with density arguments.

Chapter 3

Homogenization of the textile in linear elasticity with contact

3.1 Homogenization of a textile

The homogenization of the textile begins with general results for single periodically oscillating beams in the textile. The beam is analyzed with the help of the decomposition of displacements (see [6, 16, 25]). Especially for the oscillating behavior in the textile new displacements are introduced to simplify estimates and asymptotic behavior. The results are transfered to the textile structure and a splitting into global and local displacements is introduced. The global fields are defined by the displacements on the contact areas and extended to the plate domain $\Omega = (0, L)^2$ by extension. This special definition allows to improve the primal estimations by the regularity coming from the contact condition. The derived estimates give the bounds and compactness results necessary for the asymptotic analysis.

The limit $\varepsilon \to 0$ is investigated with the unfolding operator. For the textile an adapted unfolding-rescaling operator is introduced, which includes the unfolding operator on the textile structure and a dimension reduction. With this definition it is possible to study the simultaneous limit of both, homogenization and dimension reduction. The properties of this unfolding operator together with the estimates on the displacements yield the limit of the displacements and the symmetric strain tensor. A special attention is drawn to limit contact condition for which the unfolding operator is adjusted, similar to a boundary unfolding operator (see [16, 27]).

The unfolded limit problem arising from the convergences is investigated in this chapters last section. First the cell problems are identified, where due to the contact condition of order $g_\varepsilon \sim \varepsilon^3$ an additional nonlinear corrector is introduced to obtain the homogenized coefficients. This implies that the final homogenized problem is of *Leray-Lions*-type. The existence of solutions for this problem is shown, the uniqueness, though, can not be proven. However, a contact of order $g_\varepsilon \sim \varepsilon^3$ results in the vanishing of the additional nonlinear corrector and the typical homogenized linear plate is recovered.

Further literature on homogenization and dimension reduction of beam-structures we refer to [26–28, 30, 43]. For general literature on elasticity and plate theory we refer to [10–13, 45] and many others. The work presented in this chapter is published in [29].

Hereafter, the notion of beams, fibers and yarns is used equivalently.

3.2 Periodically curved beams

In this section, we introduce the curved beams of which the textile consists. We concentrate on the geometry, the displacements and the orders of the different displacement components.

The basic parameters of the yarns are the length L and the radius r and in this case the periodicity ε. Observe that the fibers have a beam character, i.e., $l \gg r$, which is important for the subsequent treatment. Furthermore, assume $r \leq \kappa\varepsilon$ with $\kappa \leq \widehat{\kappa} = \frac{1}{3}$. This assumption assures that the fibers do not overlap and that the curved character does not imply a penetration of beams. Although we later assume $r = \kappa\varepsilon$ we keep for the sake of generality both parameters r and ε in the displacement-estimates. This allows to transfer the results to other choices for the parameters (e.g. $r = \kappa\varepsilon^2$).

The textile with the typical canvas structure admits two main-directions for the beams: the $\mathbf{e}_1$-direction and $\mathbf{e}_2$-direction. Due to their similarity of definition and the further treatment we only consider here the $\mathbf{e}_1$-directed beams and transfer the results.

To describe the reference domain of a single curved beam in the textile, define the 2-periodic function

$$\Phi(z) = \begin{cases} -\kappa, & \text{if } z \in [0, \kappa], \\ \kappa\Big(6\frac{(z-\kappa)^2}{(1-2\kappa)^2} - 4\frac{(z-\kappa)^3}{(1-2\kappa)^3} - 1\Big) & \text{if } z \in [\kappa, 1-\kappa], \\ \kappa & \text{if } z \in [1-\kappa, 1], \\ \Phi(2-z) & \text{if } z \in [1, 2]. \end{cases} \tag{3.1}$$

Rescaling $\Phi_\varepsilon(z) = \varepsilon\Phi(\frac{z}{\varepsilon})$ yields the oscillating middle line within the textile. Note, this function is almost everywhere $\mathcal{C}^2(\mathbb{R})$ and overall $\mathcal{C}^1(\mathbb{R})$ and satisfies by definition

$$\|\Phi_\varepsilon\|_{L^\infty(0,2\varepsilon)} \leq Cr, \qquad \|\Phi'_\varepsilon\|_{L^\infty(0,2\varepsilon)} \leq C\frac{r}{\varepsilon}. \tag{3.2}$$

Remark 3.2.1. *If it is necessary to keep r and ε separated, use*

$$\Phi_\varepsilon(z) = \begin{cases} -r & \textit{if } z \in [0, r], \\ r\Big(6\frac{(z-r)^2}{(\varepsilon-2r)^2} - 4\frac{(z-r)^3}{(\varepsilon-2r)^3} - 1\Big) & \textit{if } z \in [r, \varepsilon - r], \\ r & \textit{if } z \in [\varepsilon - r, \varepsilon], \\ \Phi_\varepsilon(2\varepsilon - z) & \textit{if } z \in [\varepsilon, 2\varepsilon], \end{cases} \tag{3.3}$$

instead. The estimates (3.2) *remain true.*

Hence, define the oscillating beam with the main-direction $\mathbf{e}_1$ by the parametrization

$$M(z_1) = z_1\mathbf{e}_1 + \Phi_\varepsilon(z_1)\mathbf{e}_3. \tag{3.4}$$

Note that the keyword main-direction for this beam is justified by the fact that for negligible oscillations we arrive at a straight beam with direction $\mathbf{e}_1$.

This parametrization gives rise to the local Frenet-Serret frame (or TNB-frame) consisting of the three vectors $(\mathbf{t}_\varepsilon(z_1), \mathbf{e}_2, \mathbf{n}_\varepsilon(z_1))$. The arc-length s_1 and the curvature c_ε of the curve are easily computed

$$s_1(0) = 0, \qquad \frac{ds_1}{dz_1} = \gamma_\varepsilon, \qquad c_\varepsilon(z_1) = \frac{\Phi''_\varepsilon(z_1)}{\gamma^3_\varepsilon(z_1)} \tag{3.5}$$

where

$$\gamma_\varepsilon(z_1) = \sqrt{1 + (\Phi_\varepsilon(z_1))^2}. \tag{3.6}$$

Although, the parametrization with respect to the arc-length (3.5) has some advantages, we choose here to parametrize the beams with respect to the length of the textile $(0, L)$ to simplify the limiting behavior.

Then, the Frenet-Serret vectors can be expressed by

$$\begin{aligned}
\mathbf{t}_\varepsilon &= \frac{1}{\gamma_\varepsilon}\left(\mathbf{e}_1 + \Phi'_\varepsilon(z_1)\mathbf{e}_3\right), & \frac{d\mathbf{t}_\varepsilon}{ds_1} &= c_\varepsilon\mathbf{n}_\varepsilon, & \frac{d\mathbf{t}_\varepsilon}{dz_1} &= c_\varepsilon\gamma_\varepsilon\mathbf{n}_\varepsilon,\\
\mathbf{n}_\varepsilon &= \frac{1}{\gamma_\varepsilon}\left(\mathbf{e}_3 - \Phi'_\varepsilon(z_1)\mathbf{e}_1\right), & \frac{d\mathbf{n}_\varepsilon}{ds_1} &= -c_\varepsilon\mathbf{t}_\varepsilon, & \frac{d\mathbf{n}_\varepsilon}{dz_1} &= -c_\varepsilon\gamma_\varepsilon\mathbf{t}_\varepsilon,
\end{aligned}$$

while the binormal vector $\mathbf{e}_2$ obviously remains constant. These vector fields are almost everywhere $\mathbf{t}_\varepsilon, \mathbf{n}_\varepsilon \in \mathcal{C}^1(0, L)$, thus $H^1(0, L)$. For simplicity we treat hereafter $\mathbf{t}_\varepsilon, \mathbf{n}_\varepsilon \in \mathcal{C}^1(0, L)$, which can also be achieved by a smoother Φ but keep in mind that this is not necessary and does not disturb the following analysis.

To end this section about the curved reference beam and the necessary differential geometry denote the straight beams of length L and cross-section $\omega_r = [-r, r]^2$ or the curved beam respectively by

$$P_r = (0, L) \times \omega_r, \tag{3.7}$$

$$\mathcal{P}_\varepsilon = \left\{x \in \mathbb{R}^3 \mid x = \psi_\varepsilon(z), \quad z \in P_r\right\} \tag{3.8}$$

with the transition map

$$\psi_\varepsilon(z) = M_\varepsilon(z_1) + z_2\mathbf{e}_2 + z_3\mathbf{n}_\varepsilon, \tag{3.9}$$

from a straight to a curved domain. In fact, for obvious reasons the arc-length of the curved fibers actually do not coincide with the length of the straight reference beam.

The mobile Frenet-frame $(\mathbf{t}_\varepsilon(z_1), \mathbf{e}_2, \mathbf{n}_\varepsilon(z_1))$ is the natural one for the beam and thus mainly used. Nevertheless, switching between this mobile frame and the global one $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, to which the whole textile is referred to, requires that the transition map ψ_ε is a diffeomorphism, which is proven in the following Lemma.

Lemma 3.2.2. *Consider the functional determinant* $\det(\nabla\psi_\varepsilon)$ *and define*

$$\eta_\varepsilon(z_1) \doteq \det\big(\nabla\psi_\varepsilon(z)\big) = \boldsymbol{\gamma}_\varepsilon(z_1)\big(1 - z_3 c_\varepsilon(z_1)\big), \qquad \forall z_1 \in [0, L],$$

where c_ε *is the curvature.*

Furthermore, if $\dfrac{r}{\varepsilon} = \kappa < \widehat{\kappa}$ *the transformation* ψ_ε *from* P_r *onto* $\mathcal{P}_\varepsilon$ *is a diffeomorphism with*

$$\nabla\psi_\varepsilon = \big(\eta_\varepsilon \mathbf{t}_\varepsilon|\mathbf{e}_2|\mathbf{n}_\varepsilon\big) = \mathbf{C}_\varepsilon \begin{pmatrix} \eta_\varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\big(\nabla\psi_\varepsilon\big)^{-1} = \Big(\frac{1}{\eta_\varepsilon}\mathbf{t}_\varepsilon|\mathbf{e}_2|\mathbf{n}_\varepsilon\Big)^T = \mathbf{C}_\varepsilon^T \begin{pmatrix} \frac{1}{\eta_\varepsilon} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $\mathbf{C}_\varepsilon = \big(\mathbf{t}_\varepsilon \,|\, \mathbf{e}_2 \,|\, \mathbf{n}_\varepsilon\big)$. *The functional determinant* η_ε *is bounded from below and above*

$$\frac{1}{C} \leq \|\eta_\varepsilon\|_{L^\infty(0,L)} \leq C.$$

The constant C *is independent of* ε *and* r.

Proof. By differentiating with the help of the Frenet formulas the Jacobian

$$\nabla\psi_\varepsilon(z) = \Big(\frac{dM(z_1)}{dz_1} + z_3\frac{d\mathbf{n}_\varepsilon(z_1)}{dz_1}\Big|\mathbf{e}_2\Big|\mathbf{n}_\varepsilon(z_1)\Big) = \big(\boldsymbol{\gamma}_\varepsilon(z_1)\big(1 - z_3 c_\varepsilon(z_1)\big)\mathbf{t}_\varepsilon(z_1)\big|\mathbf{e}_2\big|\mathbf{n}_\varepsilon(z_1)\big)$$

is easily obtained as well as the Jacobian determinant

$$\det(\nabla\psi_\varepsilon)(z) = \eta_\varepsilon(z_1) = \boldsymbol{\gamma}_\varepsilon(z_1)\big(1 - z_3 c_\varepsilon(z_1)\big).$$

One has $1 \leq \boldsymbol{\gamma}_\varepsilon(z_1) \leq 1 + Cr$ for every $z_1 \in [0, L]$ (see (3.2)).
For the diffeomorphism condition it is now left to show that $0 < 1 - z_3 c_\varepsilon(z_1) \leq C$. The boundedness follows immediately from the boundedness of Φ'_ε and $r\Phi''_\varepsilon$ in L^∞ for fixed κ small enough. Recall that

$$c_\varepsilon(z_1) = \frac{\Phi''_\varepsilon(z_1)}{(\boldsymbol{\gamma}_\varepsilon(z_1))^3}.$$

Hence

$$1 - z_3 c_\varepsilon(z_1) \geq 1 - r\|\Phi''_\varepsilon\|_{L^\infty(0,L)} \geq 1 - \frac{12\kappa^2}{(1 - 2\kappa)^2}$$

and thereby $\kappa < \widehat{\kappa} = \frac{\sqrt{3}-1}{4}$. Here it suffices that η_ε is piecewise $\mathcal{C}^2$ on a compact interval. $\square$

Lemma 3.2.3. *Suppose $s \in [1, +\infty)$. There exist two constants C_0, C_1 independent of ε and r such that for every $\varphi \in L^s(\mathcal{P}_\varepsilon)$*

$$C_0 \|\varphi \circ \psi_\varepsilon\|_{L^s(P_r)} \leq \|\varphi\|_{L^s(\mathcal{P}_\varepsilon)} \leq C_1 \|\varphi \circ \psi_\varepsilon\|_{L^s(P_r)}. \tag{3.10}$$

Proof. The equivalence of norms is a simple application of the transformation theorem. Then the claim follows by the identity

$$\int_{\mathcal{P}_\varepsilon} |\varphi(x)|^s dx = \int_{P_r} |\varphi(\psi_\varepsilon(z))|^s \, |\det\big(\nabla \psi_\varepsilon(z)\big)| dz$$

and the diffeomorphism-property of ψ_ε from Lemma 3.2.2. □

Consequently, we henceforth write indifferently φ in place of $\varphi \circ \psi_\varepsilon$ for all functions.

For the rest of the thesis, we restrict ε such that the number of oscillations for one beam is given by $N_\varepsilon = \frac{L}{2\varepsilon} \in \mathbb{N}$. This is important to address the contact correctly and even more it yields that there are cells intersecting the boundary. The latter implies that Λ_ε in Definition 2.0.2 is a set of measure zero and is thus not considered in the analysis.

3.2.1 Decomposition of displacements

A crucial ingredient for the subsequent investigation of the textile are estimates on the displacement fields of the beams and the full textile. Typically, these are Korn-type inequalities. For the derivation of the estimates we introduce the decomposition of displacements following the paper [25]. In particular, we consider beam-displacement for every fiber, which is necessary to resolve the structure and contact between the fibers.

During the remaining part of this section we omit the indices ε and r because the additional indices lead to overloading and a reduction of comprehensibility. Only in some cases they remain to clarify certain issues. Keep in mind, though, that almost all fields and functions have such a dependence.

Now, let $u \in H^1(\mathcal{P}_\varepsilon; \mathbb{R}^3)$ be a displacement for a beam. Then we recall the decomposition obtained in [25, Theorem 3.1] and in [23, Lemma 3.2]

$$u(x) = \mathcal{U}(z_1) + \mathcal{R}(z_1) \wedge (z_2 \mathbf{e}_2 + z_3 \mathbf{n}_\varepsilon(z_1)) + \overline{u}(z), \qquad \text{for a.e. } x = \psi_\varepsilon(z) \in \mathcal{P}_\varepsilon, \ \ z \in P_r, \tag{3.11}$$

where $\mathcal{U}, \mathcal{R} \in H^1(0, L; \mathbb{R}^3)$ and $\overline{u} \in H^1(P_r; \mathbb{R}^3)$. The warping $\overline{u}$ satisfies for a.e. $z_1 \in (0, L)$ (see [25])

$$\int_{\omega_r} \overline{u}(z_1, z_2, z_3) dz_2 dz_3 = 0, \qquad \int_{\omega_r} \overline{u}(z_1, z_2, z_3) \wedge \big(z_2 \mathbf{e}_2 + z_3 \mathbf{n}_\varepsilon(z_1)\big) dz_2 dz_3 = 0. \tag{3.12}$$

For the understanding of the decomposed fields we recall the definition from [25].

Definition 3.2.4. *The elementary displacement associated to a beam-displacement $u \in H^1(\mathcal{P}_r)$ is defined as*

$$U^e(\cdot, z_2, z_3) = \mathcal{U} + \mathcal{R} \wedge (z_2\mathbf{e}_2 + z_3\mathbf{n}), \qquad (z_2, z_3) \in \omega_r$$

with the local frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ of the beam. The functions are defined via the original displacement:

$$\begin{aligned}
\mathcal{U}(z_1) &= \frac{1}{|\omega_r|}\int_{\omega_r} u(z_1, z_2, z_3)\, dz_2dz_3,\\
\mathcal{R}(z_1)\cdot\mathbf{t} &= \frac{1}{(I_1+I_2)r^4}\int_{\omega_r}\Big[(z_2\mathbf{e}_2 + z_3\mathbf{n}_2)\wedge u(z_1,z_2,z_3)\Big]\cdot\mathbf{t}\; dz_2dz_3,\\
\mathcal{R}(z_1)\cdot\mathbf{e}_2 &= \frac{1}{I_2r^4}\int_{\omega_r}\Big[(z_2\mathbf{e}_2 + z_3\mathbf{n}_2)\wedge u(z_1,z_2,z_3)\Big]\cdot\mathbf{e}_2\; dz_2dz_3,\\
\mathcal{R}(z_1)\cdot\mathbf{n} &= \frac{1}{I_1r^4}\int_{\omega_r}\Big[(z_2\mathbf{e}_2 + z_3\mathbf{n}_2)\wedge u(z_1,z_2,z_3)\Big]\cdot\mathbf{n}\; dz_2dz_3,
\end{aligned}$$

with the area moments $I_\alpha = \int_\omega z_\alpha^2\, dz_1dz_2 = \frac{4}{3}$ and $|\omega_r| = 4r^2$. The warping denotes the residual displacement

$$\overline{u} = u - U^e \in H^1(\mathcal{P}_r). \tag{3.13}$$

The warping captures remaining displacements, which are not represented via the elementary displacement. Although the warping is negligible for the global behavior of the structure, it is crucial for the cell problems.

The strength of this decomposition lays in the possibility to separate the middle line displacements and the rotation of a beam. Treating these two separately grants more control for the asymptotic analysis, since each field admits different orders with respect to the radius of the beam as it is shown in the next section.

3.2.2 Estimates for a curved beam

For one single curved beam the fields introduced by the decomposition of displacements satisfy Korn-type estimates depending on the geometry. As shown in [25, Theorem 3.1] the decomposed fields satisfy

$$\|\overline{u}\|_{L^2(P_r;\mathbb{R}^3)} \le Cr\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \qquad \|\nabla\overline{u}\|_{L^2(P_r;\mathbb{R}^{3\times3})} \le C\|e(u)\|_{L^2(\mathcal{P}_\varepsilon;\mathbb{R}^{3\times3})}, \tag{3.14}$$

and

$$\Big\|\frac{d\mathcal{R}}{ds_1}\Big\|_{L^2(0,L)} \le \frac{C}{r^2}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \qquad \Big\|\frac{d\mathcal{U}}{ds_1} - \mathcal{R}\wedge\mathbf{t}_\varepsilon\Big\|_{L^2(0,L)} \le \frac{C}{r}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}. \tag{3.15}$$

All constants are independent of r and ε (recall that $2r$ is the thickness of the beam).

In (3.14) and (3.15) it is possible to consider the gradients with respect to both sets of variables (z_1, z_2, z_3) or (s_1, z_2, z_3). Indeed, since the Jacobian determinant η_ε of the change of variables is bounded (see Lemma 3.2.2) and the estimates only change in the constant. Hence, we replace (3.15) by

$$\Big\|\frac{d\mathcal{R}}{dz_1}\Big\|_{L^2(0,L)} \leq \frac{C}{r^2}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \qquad \Big\|\frac{d\mathcal{U}}{dz_1} - \mathcal{R}\wedge\frac{dM_\varepsilon}{dz_1}\Big\|_{L^2(0,L)} \leq \frac{C}{r}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}. \tag{3.16}$$

Hereafter, we write d_i instead of $\frac{d}{dz_i}$ for the sake of simplicity.

Additionally to the above decomposition we define another splitting of the displacement $\mathcal{U}$, cf. (3.11).

Definition 3.2.5. *Splitting of the middle-line displacement. Set*

$$\mathcal{U} = \mathbb{U} + \Phi_\varepsilon\mathcal{R}\wedge\mathbf{e}_3. \tag{3.17}$$

The reason to define this additional field for the beam is provided in the following Lemma, which simplifies estimate $(3.16)_2$ by eliminating the high oscillations therein. It may also be interpreted as pulling the middle-line displacement back to the middle-plane of the textile.

Lemma 3.2.6. *The field* $\mathbb{U}$ *satisfies*

$$\big\|d_1\mathbb{U} - \mathcal{R}\wedge\mathbf{e}_1\big\|_{L^2(0,L)} \leq \frac{C}{r}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \quad \|\mathcal{U}_\alpha - \mathbb{U}_\alpha\|_{L^2(0,L)} \leq Cr\|\mathcal{R}\|_{L^2(0,L)}. \tag{3.18}$$

The constants does not depend on ε *and* r*.*

Proof. Estimates (3.18) are the immediate consequences of the L^∞-norm of Φ_ε and (3.16). Indeed, inserting the definition and using the estimates for the remaining parts yields

$$\begin{aligned}\|d_1\mathbb{U} - \mathcal{R}\wedge\mathbf{e}_1\|_{L^2(0,L)} &\leq \Big\|d_1\mathcal{U} - \mathcal{R}\wedge\frac{dM_\varepsilon}{dz_1}\Big\|_{L^2(0,L)} + \|\Phi_\varepsilon d_1\mathcal{R}\wedge\mathbf{e}_3\|_{L^2(0,L)}\\ &\leq \frac{C}{r}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)} + \|\Phi_\varepsilon\|_{L^\infty(0,L)}\|d_1\mathcal{R}\|_{L^2(0,L)} \leq \frac{C}{r}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}.\end{aligned}$$

□

Note that there exist discrete versions of the estimates $(3.16)_1$ and (3.18), which are necessary to establish global estimates.

Lemma 3.2.7. *The fields* $\mathcal{R}$ *and* $\mathbb{U}$ *defined above satisfy*

$$\sum_{p=0}^{2N_\varepsilon-1}\big|\mathcal{R}\big((p+1)\varepsilon\big)-\mathcal{R}(p\varepsilon)\big|^2 + \sum_{p=0}^{2N_\varepsilon-1}\Big|\frac{\mathbb{U}\big((p+1)\varepsilon\big)-\mathbb{U}(p\varepsilon)}{\varepsilon} - \mathcal{R}(p\varepsilon)\wedge\mathbf{e}_1\Big|^2 \leq \frac{C\varepsilon}{r^4}\|e(u)\|^2_{L^2(\mathcal{P}_\varepsilon)}. \tag{3.19}$$

Proof. Consider the left hand side and transform the expression using the fundamental theorem of calculus and the Jensen inequality:

$$\sum_{p=0}^{2N_\varepsilon-1}\big|\mathcal{R}\big((p+1)\varepsilon\big)-\mathcal{R}(p\varepsilon)\big|^2 \leq \varepsilon\sum_{p=0}^{2N_\varepsilon-1}\int_{p\varepsilon}^{(p+1)\varepsilon}\big|d_1\mathcal{R}(z_1)\big|^2dz_1 \leq \varepsilon\|d_1\mathcal{R}\|^2_{L^2(0,L)}.$$

Then use the estimate (3.16) to conclude the first inequality.
By the same means we obtain

$$\sum_{p=0}^{2N_\varepsilon-1}\Big|\frac{\mathbb{U}\big((p+1)\varepsilon\big)-\mathbb{U}(p\varepsilon)}{\varepsilon}-\mathcal{R}(p\varepsilon)\wedge\mathbf{e}_1\Big|^2\le\sum_{p=0}^{2N_\varepsilon-1}\frac{1}{\varepsilon}\int_{p\varepsilon}^{(p+1)\varepsilon}\big|d_1\mathbb{U}-\mathcal{R}(p\varepsilon)\wedge\mathbf{e}_1\big|^2dz_1. \tag{3.20}$$

Additionally, note that by introducing now the function $\mathcal{R}$ we obtain

$$\begin{aligned}\int_{p\varepsilon}^{(p+1)\varepsilon}\big|d_1\mathbb{U}-\mathcal{R}(p\varepsilon)\wedge\mathbf{e}_1\big|^2dz&\le\int_{p\varepsilon}^{(p+1)\varepsilon}\big|d_1\mathbb{U}-\mathcal{R}\wedge\mathbf{e}_1\big|^2dz_1+\int_{p\varepsilon}^{(p+1)\varepsilon}\big|\mathcal{R}-\mathcal{R}(p\varepsilon)\big|^2dz_1\\&\le\int_{p\varepsilon}^{(p+1)\varepsilon}\big|d_1\mathbb{U}-\mathcal{R}\wedge\mathbf{e}_1\big|^2dz_1+\varepsilon^2\int_{p\varepsilon}^{(p+1)\varepsilon}\big|\partial_1\mathcal{R}\big|^2dz_1,\end{aligned}$$

in every interval $(p\varepsilon,(p+1)\varepsilon)$. The second inequality is an application of the Poincaré inequality. Finally, we conclude the claim by inserting this into (3.20), where the remaining two terms are covered by the estimates (3.16) and (3.18). □

Finally, introduce a splitting of the decomposed displacements. Specifically, any function can be decomposed into a piecewise linear function and an additional function capturing the higher orders. To do so, note that a function φ defined on the set $\{p\varepsilon\mid p=1,\dots,2N_\varepsilon-1\}$ is easily extended to $\varphi^{(nod)}\in W^{1,\infty}$ by linear interpolation. Hence, define the linear interpolations $\mathcal{R}^{(nod)},\mathbb{U}^{(nod)}\in W^{1,\infty}$ with the values in the vertices

$$\mathcal{R}^{(nod)}(p\varepsilon)=\mathcal{R}(p\varepsilon)\qquad\text{and}\qquad\mathbb{U}^{(nod)}(p\varepsilon)=\mathbb{U}(p\varepsilon).$$

Then, the original displacement admits the decomposition

$$\mathcal{R}(z)=\mathcal{R}^{(nod)}(z)+\mathcal{R}^{(0)}(z)\qquad\text{and}\qquad\mathbb{U}(z)=\mathbb{U}^{(nod)}(z)+\mathbb{U}^{(0)}(z). \tag{3.21}$$

Here the functions $\mathcal{R}^{(0)}$ and $\mathbb{U}^{(0)}$ capture the high oscillations and are by definition zero on the nodes, i.e., $\mathcal{R}^{(0)}(p\varepsilon)=\mathbb{U}^{(0)}(p\varepsilon)=0$ for all $p\in 0,\dots,2N_\varepsilon$.

Lemma 3.2.8. *The functions $\mathcal{R}^{(0)}$, $\mathbb{U}^{(0)}$, $\mathcal{R}^{(nod)}$ and $\mathbb{U}^{(nod)}$ satisfy for $i=2,3$*

$$\begin{aligned}&\|\mathcal{R}^{(0)}\|_{L^2(0,L)}+\varepsilon\|d\mathcal{R}^{(0)}\|_{L^2(0,L)}+\varepsilon\|d\mathcal{R}^{(nod)}\|_{L^2(0,L)}\le\frac{C\varepsilon}{r^2}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)},\\&\|\mathbb{U}_1^{(0)}\|_{L^2(0,L)}+\varepsilon\|d\mathbb{U}_1^{(0)}\|_{L^2(0,L)}+\frac{r}{\varepsilon}\|\mathbb{U}_i^{(0)}\|_{L^2(0,L)}+r\|d\mathbb{U}_i^{(0)}\|_{L^2(0,L)}\le C\frac{\varepsilon}{r}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)},\\&\|d\mathbb{U}^{(nod)}-\mathcal{R}^{(nod)}\wedge\mathbf{e}_1\|_{L^2(0,L)}+\|d\mathbb{U}^{(0)}-\mathcal{R}^{(0)}\wedge\mathbf{e}_1\|_{L^2(0,L)}\le\frac{C\varepsilon}{r^2}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}.\end{aligned} \tag{3.22}$$

Proof. Note that $d\mathcal{R}^{(nod)}$ is constant in every interval $\big(p\varepsilon,(p+1)\varepsilon\big)$ and that $\mathcal{R}^{(0)}$ and $\mathbb{U}^{(0)}$ are zero on the nodes. Thus, $d\mathcal{R}^{(nod)}$ and $d\mathcal{R}^{(0)}$ are orthogonal to each other in the L^2-sense. Indeed, we have

$$\langle d\mathcal{R}^{(nod)},d\mathcal{R}^{(0)}\rangle_{L^2(p\varepsilon,(p+1)\varepsilon)}=d\mathcal{R}_p^{(nod)}\int_{p\varepsilon}^{(p+1)\varepsilon}d\mathcal{R}^{(0)}dz_1=0$$

where $d\mathcal{R}_p^{(nod)} = \Big(\mathcal{R}^{(nod)}((p+1)\varepsilon) - \mathcal{R}^{(nod)}(p\varepsilon)\Big)/\varepsilon$. The integral is zero due to the values on the nodes $\mathcal{R}^{(0)}(p\varepsilon) = \mathcal{R}^{(0)}((p+1)\varepsilon) = 0$. With this orthogonality and summing over all cells we obtain

$$\|d\mathcal{R}^{(nod)}\|^2_{L^2(0,L)} + \|d\mathcal{R}^{(0)}\|^2_{L^2(0,L)} = \|d\mathcal{R}\|^2_{L^2(0,L)} \leq \frac{C}{r^4}\|e(u)\|^2_{L^2(\mathcal{P}_\varepsilon)}.$$

Then, the Poincaré-inequality yields

$$\|\mathcal{R} - \mathcal{R}^{(nod)}\|_{L^2(0,L)} = \|\mathcal{R}^{(0)}\|_{L^2(0,L)} \leq \varepsilon\|d\mathcal{R}^{(0)}\|_{L^2(0,L)} \leq C\frac{\varepsilon}{r^2}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}.$$

For estimate $(3.22)_2$ similar considerations lead to

$$\|d\mathbb{U}_1^{(nod)}\|^2_{L^2(0,L)} + \|d\mathbb{U}_1^{(0)}\|^2_{L^2(0,L)} = \|d\mathbb{U}_1\|^2_{L^2(0,L)} \leq \frac{C}{r^2}\|e(u)\|^2_{L^2(\mathcal{P}_\varepsilon)}.$$

From this it is easy to obtain

$$\begin{aligned}\|d\mathbb{U}^{(0)}\|_{L^2(0,L)} \leq \|d\mathbb{U} - \mathcal{R}\wedge\mathbf{e}_1\|_{L^2(0,L)}& \\ &+ \|d\mathbb{U}^{(nod)} - \mathcal{R}^{(nod)}\wedge\mathbf{e}_1\|_{L^2(0,L)} + \|\mathcal{R} - \mathcal{R}^{(nod)}\|_{L^2(0,L)}.\end{aligned}$$

Together with

$$\begin{aligned}\|d\mathbb{U}^{(nod)} - \mathcal{R}^{(nod)}\wedge\mathbf{e}_1\|^2_{L^2(0,L)} \leq \sum_{p=0}^{2N_\varepsilon-1}\varepsilon\Big|\frac{\mathbb{U}\big((p+1)\varepsilon\big) - \mathbb{U}(p\varepsilon)}{\varepsilon} - \mathcal{R}(p\varepsilon)\wedge\mathbf{e}_1\Big|^2& \\ + C\varepsilon^2\|d\mathcal{R}\|_{L^2(0,L)} \leq \frac{C\varepsilon^2}{r^4}\|e(u)\|^2_{L^2(\mathcal{P}_\varepsilon)},&\end{aligned} \tag{3.23}$$

this yields

$$\|\mathbb{U}^{(0)}\|_{L^2(0,L)} \leq \varepsilon\|d\mathbb{U}^{(0)}\|_{L^2(0,L)} \leq C\frac{\varepsilon^2}{r^2}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}.$$

The last estimate in (3.22) is a consequence of (3.23) as well. □

3.2.3 Symmetric gradient for one beam

The gradient with respect to the set of variables (z_1, z_2, z_3) of the whole displacement u is split

$$\nabla_z u = \nabla_z U^e + \nabla_z \overline{u},$$

with the elementary displacement $U^e = u - \overline{u}$ and the warping $\overline{u}$. First, consider only the gradient of the elementary displacement:

$$\begin{aligned}\nabla_z U^e &= \big(\partial_{z_1}U^e \mid \partial_{z_2}U^e \mid \partial_{z_3}U^e\big) \\ &= \Big(d_1\mathcal{U} + d_1\mathcal{R}\wedge\big(z_2\mathbf{e}_2 + z_3\mathbf{n}_\varepsilon\big) - z_3c_\varepsilon\gamma_\varepsilon\mathcal{R}\wedge\mathbf{t}_\varepsilon\Big|\mathcal{R}\wedge\mathbf{e}_2\Big|\mathcal{R}\wedge\mathbf{n}_\varepsilon\Big).\end{aligned}$$

Obviously, this is in the local coordinate system of the parametrized beam $\mathcal{P}_\varepsilon$. This is not sufficient for the problem, since the Cartesian system of the composed textile is needed.

The transition between the reference systems is done via a change of basis and variables. First, one has $\nabla_z u = \nabla_x u \nabla \psi_\varepsilon$. Hence

$$\mathbf{C}_\varepsilon^T\,\nabla_x u\,\mathbf{C}_\varepsilon = \mathbf{C}_\varepsilon^T\,\nabla_z u \begin{pmatrix} \frac{1}{\eta_\varepsilon} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\eta_\varepsilon}\frac{\partial u}{\partial z_1}\cdot \mathbf{t}_\varepsilon & \frac{\partial u}{\partial z_2}\cdot \mathbf{t}_\varepsilon & \frac{\partial u}{\partial z_3}\cdot \mathbf{t}_\varepsilon \\ \frac{1}{\eta_\varepsilon}\frac{\partial u}{\partial z_1}\cdot \mathbf{e}_2 & \frac{\partial u}{\partial z_2}\cdot \mathbf{e}_2 & \frac{\partial u}{\partial z_3}\cdot \mathbf{e}_2 \\ \frac{1}{\eta_\varepsilon}\frac{\partial u}{\partial z_1}\cdot \mathbf{n}_\varepsilon & \frac{\partial u}{\partial z_2}\cdot \mathbf{n}_\varepsilon & \frac{\partial u}{\partial z_3}\cdot \mathbf{n}_\varepsilon \end{pmatrix}.$$

Recall that $e_x(u) = \frac{1}{2}\left(\nabla_x u + (\nabla_x u)^T\right)$ and define the symmetric tensor $e_z(u)$ by

$$e_z(u) = \mathbf{C}_\varepsilon^T\,e_x(u)\,\mathbf{C}_\varepsilon = \begin{pmatrix} \frac{1}{\eta_\varepsilon}\frac{\partial u}{\partial z_1}\cdot \mathbf{t}_\varepsilon & * & * \\ \frac{1}{2}\left(\frac{1}{\eta_\varepsilon}\frac{\partial u}{\partial z_1}\cdot \mathbf{e}_2 + \frac{\partial u}{\partial z_2}\cdot \mathbf{t}_\varepsilon\right) & \frac{\partial u}{\partial z_2}\cdot \mathbf{e}_2 & * \\ \frac{1}{2}\left(\frac{1}{\eta_\varepsilon}\frac{\partial u}{\partial z_1}\cdot \mathbf{n}_\varepsilon + \frac{\partial u}{\partial z_3}\cdot \mathbf{t}_\varepsilon\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z_2}\cdot \mathbf{n}_\varepsilon + \frac{\partial u}{\partial z_3}\cdot \mathbf{e}_2\right) & \frac{\partial u}{\partial z_3}\cdot \mathbf{n}_\varepsilon \end{pmatrix}. \tag{3.24}$$

Now, we change the notation for the symmetric strain tensor. Due to the symmetry of this tensor, it can be rewritten as a vector with six entries. Hence, define

$$E_x(u) = \left(e_{x,11}, e_{x,22}, e_{x,33}, \sqrt{2}e_{x,12}, \sqrt{2}e_{x,13}, \sqrt{2}e_{x,23}\right)^T,$$
$$E_z(u) = \left(e_{z,11}, e_{z,22}, e_{z,33}, \sqrt{2}e_{z,12}, \sqrt{2}e_{z,13}, \sqrt{2}e_{z,23}\right)^T.$$

Similar to (3.24), there exists a matrix $\widetilde{\mathbf{C}}_\varepsilon \in C^1(\mathcal{P}_\varepsilon)^{(6\times 6)}$ such that

$$E_z(u) = \widetilde{\mathbf{C}}_\varepsilon\,E_x(u), \tag{3.25}$$

where

$$\widetilde{\mathbf{C}}_\varepsilon = \begin{pmatrix} \frac{1}{\gamma_\varepsilon^2} & 0 & \frac{(\Phi_\varepsilon')^2}{\gamma_\varepsilon^2} & 0 & \frac{\sqrt{2}\Phi_\varepsilon'}{\gamma_\varepsilon^2} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{(\Phi_\varepsilon')^2}{\gamma_\varepsilon^2} & 0 & \frac{1}{\gamma_\varepsilon^2} & 0 & \frac{-\sqrt{2}\Phi_\varepsilon'}{\gamma_\varepsilon^2} & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma_\varepsilon} & 0 & \frac{\Phi_\varepsilon'}{\gamma_\varepsilon} \\ \frac{-\sqrt{2}\Phi_\varepsilon'}{\gamma_\varepsilon^2} & 0 & \frac{\sqrt{2}\Phi_\varepsilon'}{\gamma_\varepsilon^2} & 0 & \frac{1-(\Phi_\varepsilon')^2}{\gamma_\varepsilon^2} & 0 \\ 0 & 0 & 0 & \frac{-\Phi_\varepsilon'}{\gamma_\varepsilon} & 0 & \frac{1}{\gamma_\varepsilon} \end{pmatrix}. \tag{3.26}$$

Observe that $\widetilde{\mathbf{C}}_\varepsilon$ is an orthogonal matrix.

The gradient for the elementary displacement is a straight forward computation and composed of

$$
\begin{aligned}
&\frac{\partial U^e}{\partial z_1}\cdot \mathbf{t}_\varepsilon = d_1\mathcal{U}\cdot \mathbf{t}_\varepsilon - d_1\mathcal{R}\cdot(z_2\mathbf{n}_\varepsilon - z_3\mathbf{e}_2), && \frac{\partial U^e}{\partial z_2}\cdot \mathbf{t}_\varepsilon = -\mathcal{R}\cdot\mathbf{n}_\varepsilon, && \frac{\partial U^e}{\partial z_3}\cdot \mathbf{t}_\varepsilon = \mathcal{R}\cdot\mathbf{e}_2,\\
&\frac{\partial U^e}{\partial z_1}\cdot \mathbf{e}_2 = d_1\mathcal{U}\cdot \mathbf{e}_2 - z_3 d_1\mathcal{R}\cdot\mathbf{t}_\varepsilon - z_3 c_\varepsilon \boldsymbol{\gamma}_\varepsilon \mathcal{R}\cdot\mathbf{n}_\varepsilon, && \frac{\partial U^e}{\partial z_2}\cdot \mathbf{e}_2 = 0, && \frac{\partial U^e}{\partial z_3}\cdot \mathbf{e}_2 = -\mathcal{R}\cdot\mathbf{t}_\varepsilon,\\
&\frac{\partial U^e}{\partial z_1}\cdot \mathbf{n}_\varepsilon = d_1\mathcal{U}\cdot \mathbf{n}_\varepsilon + z_2 d_1\mathcal{R}\cdot\mathbf{t}_\varepsilon + z_3 c_\varepsilon \boldsymbol{\gamma}_\varepsilon \mathcal{R}\cdot\mathbf{e}_2, && \frac{\partial U^e}{\partial z_2}\cdot \mathbf{n}_\varepsilon = \mathcal{R}\cdot\mathbf{t}_\varepsilon, && \partial U^e \partial z_3\cdot\mathbf{n}_\varepsilon = 0.
\end{aligned}
$$

To compute the complete strain tensor note, that it is also a linear operation and we can consider the elementary displacement and the warping again separately. The symmetric gradient for the elementary displacement (given by (3.24)) is obtained by combining the respective terms and yields $e_{z,22}(U^e) = e_{z,33}(U^e) = e_{z,23}(U^e) = 0$ and the nonzero components

$$
\begin{aligned}
e_{z,11}(U^e) &= \frac{1}{\eta_\varepsilon}\left[\big(d_1\mathcal{U} - \boldsymbol{\gamma}_\varepsilon\mathcal{R}\wedge\mathbf{t}_\varepsilon\big)\cdot\mathbf{t}_\varepsilon - d_1\mathcal{R}\cdot(z_2\mathbf{n}_\varepsilon - z_3\mathbf{e}_2)\right],\\
e_{z,12}(U^e) &= \frac{1}{2\eta_\varepsilon}\left[\big(d_1\mathcal{U} - \boldsymbol{\gamma}_\varepsilon\mathcal{R}\wedge\mathbf{t}_\varepsilon\big)\cdot\mathbf{e}_2 - z_3 d_1\mathcal{R}\cdot\mathbf{t}_\varepsilon\right],\\
e_{z,13}(U^e) &= \frac{1}{2\eta_\varepsilon}\left[\big(d_1\mathcal{U} - \boldsymbol{\gamma}_\varepsilon\mathcal{R}\wedge\mathbf{t}_\varepsilon\big)\cdot\mathbf{n}_\varepsilon + z_2 d_1\mathcal{R}\cdot\mathbf{t}_\varepsilon\right].
\end{aligned}
$$

In the following, we pass over to the new displacement defined in Definition 3.2.5 and with the identity

$$
d_1\mathcal{U} - \boldsymbol{\gamma}_\varepsilon\mathcal{R}\cdot\mathbf{t}_\varepsilon = \big(d_1\mathbb{U} - \mathcal{R}\wedge\mathbf{e}_1\big) + \Phi_\varepsilon d_1\mathcal{R}\wedge\mathbf{e}_3, \qquad \text{a.e. in } (0,L)
$$

the strain tensor is transformed to

$$
\begin{aligned}
\eta_\varepsilon e_{z,11} &= \big(d_1\mathbb{U} - \mathcal{R}\wedge\mathbf{e}_1\big)\cdot\mathbf{t}_\varepsilon + d_1\mathcal{R}\cdot\Big(\big(\frac{\Phi_\varepsilon}{\boldsymbol{\gamma}_\varepsilon} + z_3\big)\mathbf{e}_2 - z_2\mathbf{n}_\varepsilon\Big),\\
2\eta_\varepsilon e_{z,12} &= \big(d_1\mathbb{U} - \mathcal{R}\wedge\mathbf{e}_1\big)\cdot\mathbf{e}_2 - d_1\mathcal{R}\cdot\big(z_3\mathbf{t}_\varepsilon + \Phi_\varepsilon\mathbf{e}_1\big),\\
2\eta_\varepsilon e_{z,13} &= \big(d_1\mathbb{U} - \mathcal{R}\wedge\mathbf{e}_1\big)\cdot\mathbf{n}_\varepsilon + d_1\mathcal{R}\cdot\big(z_2\mathbf{t}_\varepsilon - \frac{\Phi_\varepsilon\Phi'_\varepsilon}{\boldsymbol{\gamma}_\varepsilon}\mathbf{e}_2\big).
\end{aligned}
\tag{3.27}
$$

The completion of the strain tensor for the full displacement $e_z(u) = e_z(U^e) + e_z(\overline{u})$ includes the warping terms again and with $e_z(\overline{u}) = \mathbf{C}_\varepsilon^T\, e_x(\overline{u})\, \mathbf{C}_\varepsilon$ given by (3.24).

3.3 The textile structure

In the remaining work, we drop the index r if there is a dependence on ε as well. This is for comprehensibility and prevention of index-overloading.

Set

$$
\begin{aligned}
P_r^{(1)} &\doteq \big\{z\in\mathbb{R}^3 \mid z_1\in(0,L),\ \ (z_2,z_3)\in\omega_r\big\},\\
P_r^{(2)} &\doteq \big\{z\in\mathbb{R}^3 \mid z_2\in(0,L),\ \ (z_1,z_3)\in\omega_r\big\},
\end{aligned}
$$

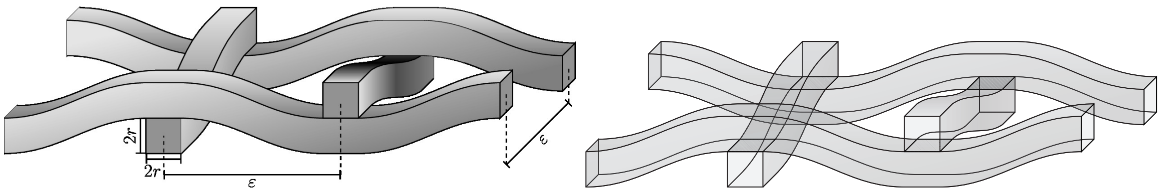


Figure 3.1: The left picture shows a portion of the textile structure containing a part of the periodicity cell. The right picture shows the contact areas $\mathbf{C}_{pq}$ of the beams as gray-striped rectangles.

for the reference beams in the two directions. Then the curved beams are defined by

$$\mathcal{P}_\varepsilon^{(1,q)} \doteq \Big\{x \in \mathbb{R}^3 \mid x = \psi_\varepsilon^{(1,q)}(z),\ \ z \in P_r^{(1)}\Big\},$$
$$\mathcal{P}_\varepsilon^{(2,p)} \doteq \Big\{x \in \mathbb{R}^3 \mid x = \psi_\varepsilon^{(2,p)}(z),\ \ z \in P_r^{(2)}\Big\},$$

with the diffeomorphisms

$$\psi_\varepsilon^{(1,q)}(z) \doteq M_\varepsilon^{(1,q)}(z_1) + z_2\mathbf{e}_2 + z_3\mathbf{n}_\varepsilon^{(1,q)}(z_1),$$
$$\psi_\varepsilon^{(2,p)}(z) \doteq M_\varepsilon^{(2,p)}(z_2) + z_1\mathbf{e}_1 + z_3\mathbf{n}_\varepsilon^{(2,p)}(z_2),$$

and the corresponding middle lines

$$M_\varepsilon^{(1,q)}(z_1) \doteq z_1\mathbf{e}_1 + q\varepsilon\mathbf{e}_2 + (-1)^{q+1}\Phi_\varepsilon(z_1)\mathbf{e}_3,$$
$$M_\varepsilon^{(2,p)}(z_2) \doteq p\varepsilon\mathbf{e}_1 + z_2\mathbf{e}_2 + (-1)^{p}\Phi_\varepsilon(z_2)\mathbf{e}_3.$$

The whole textile structure is then given by

$$\mathcal{S}_\varepsilon \doteq \bigcup_{q=1}^{2N_\varepsilon} \mathcal{P}_\varepsilon^{(1,q)} \cup \bigcup_{p=0}^{2N_\varepsilon} \mathcal{P}_\varepsilon^{(2,p)}. \tag{3.28}$$

Moreover, observe the respective local Frenet-frames $\Big(\mathbf{t}_\varepsilon^{(1,q)}, \mathbf{e}_2, \mathbf{n}_\varepsilon^{(1,q)}\Big)$ and $\Big(\mathbf{e}_1, \mathbf{t}_\varepsilon^{(2,p)}, \mathbf{n}_\varepsilon^{(2,p)}\Big)$ with

$$\mathbf{t}_\varepsilon^{(1,q)}(z_1) = \frac{dM_\varepsilon^{(1,q)}}{ds_1}(z_1), \qquad \mathbf{n}_\varepsilon^{(1,q)}(z_1) = \mathbf{t}_\varepsilon^{(1,q)}(z_1) \wedge \mathbf{e}_2, \qquad z_1 \in [0,L],$$
$$\mathbf{t}_\varepsilon^{(2,p)}(z_2) = \frac{dM_\varepsilon^{(2,p)}}{rds_2}(z_2), \qquad \mathbf{n}_\varepsilon^{(2,p)}(z_2) = \mathbf{e}_1 \wedge \mathbf{t}_\varepsilon^{(2,p)}(z_2), \qquad z_2 \in [0,L].$$

Denote by $\mathbf{C}^{(1,q)} = \Big(\mathbf{t}_\varepsilon^{(1,q)}, \mathbf{e}_2, \mathbf{n}_\varepsilon^{(1,q)}\Big)$ and $\mathbf{C}^{(2,p)} = \Big(\mathbf{e}_1, \mathbf{t}_\varepsilon^{(2,p)}, \mathbf{n}_\varepsilon^{(2,p)}\Big)$ the respective basis-transformation matrix. Note that we work mostly on the straight reference beams, i.e., with respect to (z_1, z_2, z_3). Thus denote $\mathbf{P}_\varepsilon^{[1]} \doteq \bigcup_{q=1}^{2N_\varepsilon} \big(q\varepsilon\mathbf{e}_2 + P_r^{(1)}\big)$ and $\mathbf{P}_\varepsilon^{[2]} \doteq \bigcup_{p=0}^{2N_\varepsilon} \big(p\varepsilon\mathbf{e}_1 + P_r^{(2)}\big)$ the collection of the straight refence beams. Then, for every $\varphi \in L^1(\mathcal{S}_\varepsilon)$, the couple $\big(\varphi^{[1]}, \varphi^{[2]}\big) \in L^1\big(\mathbf{P}_\varepsilon^{[1]}\big) \times L^1\big(\mathbf{P}_\varepsilon^{[2]}\big)$ is associated, with

$$\varphi^{[1]}(q\varepsilon\mathbf{e}_2 + z) = \varphi\big(q\varepsilon\mathbf{e}_2 + \psi_\varepsilon^{(1,q)}(z)\big), \qquad \text{for } q \in \{1,\dots,2N_\varepsilon\} \text{ and a.e. } z \in P_r^{(1)},$$
$$\varphi^{[2]}(p\varepsilon\mathbf{e}_1 + z) = \varphi\big(p\varepsilon\mathbf{e}_1 + \psi_\varepsilon^{(2,p)}(z)\big), \qquad \text{for } p \in \{0,\dots,2N_\varepsilon\} \text{ and a.e. } z \in P_r^{(2)}.$$

Then, the integral over the whole structure is split to the single beams by

$$\begin{aligned}\int_{\mathcal{S}_\varepsilon}\varphi(x)dx &= \int_{\mathbf{P}_\varepsilon^{[1]}}\varphi^{[1]}(z)|\det\big(\nabla\psi_\varepsilon^{[1]}(z)\big)|dz + \int_{\mathbf{P}_\varepsilon^{[2]}}\varphi^{[2]}(z)|\det\big(\nabla\psi_\varepsilon^{[2]}(z)\big)|dz \\ &= \sum_{q=1}^{2N_\varepsilon}\int_{P_r^{(1)}}\varphi^{[1]}(q\varepsilon\mathbf{e}_2+z)|\det\big(\nabla\psi_\varepsilon^{(1,q)}(z)\big)|dz \\ &\qquad\qquad + \sum_{p=0}^{2N_\varepsilon}\int_{P_r^{(2)}}\varphi^{[2]}(p\varepsilon\mathbf{e}_1+z)|\det\big(\nabla\psi_\varepsilon^{(2,p)}(z)\big)|dz.\end{aligned} \tag{3.29}$$

Finally, given this structure define the displacements for the respective beams as

$$u^{(1,q)}\in H^1(\mathcal{P}^{(1,q)}) \qquad \text{and} \qquad u^{(2,p)}\in H^1(\mathcal{P}^{(2,p)}).$$

Remark 3.3.1. *Note that for the beams in $\mathbf{e}_2$-direction we have similarly a diffeomorphism with the same condition as in Lemma 3.2.2. Indeed, consider*

$$\nabla\psi_\varepsilon^{(2)}(z)=\eta_\varepsilon(z_2)=\Big(\mathbf{e}_1 \;\big|\; \gamma_\varepsilon(z_2)\big(1-z_3c_\varepsilon(z_2)\big)\mathbf{t}_\varepsilon^{(2)}(z_2) \;\big|\; \mathbf{n}_\varepsilon^{(2)}(z_2)\Big), \tag{3.30}$$

where we denoted all functions with the index (2) *to distinguish them from the beam considered before.*

3.3.1 Boundary conditions

The only assumption applied on the textile-structure is a clamp-condition on its lateral boundary $z_2=0$ and every displacement there equals zero. In fact, due to the structure (3.28) only the displacements $u^{(2,p)}$ are affected by this condition, i.e., $u^{(2,p)}_{|z_2=0}=0$ for every $p\in\{0,\dots,2N_\varepsilon\}$.

3.3.2 The contact condition

The contact between the fibers is restricted to the portions, where the beams are right above each other. Define the contact domains as small surfaces included in the lateral boundary of the beams

$$\mathbf{C}_{pq}\doteq C_{pq}\times\{0\}, \qquad C_{pq}\doteq(p\varepsilon,q\varepsilon)+\omega_r, \qquad (p,q)\in\mathcal{K}_\varepsilon,$$

with

$$\mathcal{K}_\varepsilon=\big\{(p,q)\in\mathbb{N}\times\mathbb{N}\mid(p\varepsilon,q\varepsilon)\in\overline{\Omega}\big\}=\{0,\dots,2N_\varepsilon\}^2. \tag{3.31}$$

Observe that in these contact domains the centerlines of the beams reduce to

$$\begin{aligned}M^{(1,q)}(z_1)&=z_1\mathbf{e}_1+q\varepsilon\mathbf{e}_2+(-1)^{p+q}r\mathbf{e}_3,\\ M^{(2,p)}(z_2)&=p\varepsilon\mathbf{e}_1+z_2\mathbf{e}_2+(-1)^{p+q+1}r\mathbf{e}_3,\end{aligned} \qquad \text{for a.e. } (z_1,z_2)\in C_{pq}.$$

Then, the beam-to-beam interaction is characterized by the gap-function $g_\varepsilon : \mathcal{K}_\varepsilon \to [0,+\infty)^3$ and the condition

$$|u_\alpha^{(1,q)} - u_\alpha^{(2,p)}| \leq g_{\varepsilon,\alpha}, \qquad \text{a.e in } \mathbf{C}_{pq}, \ (p,q) \in \mathcal{K}_\varepsilon,$$

for in-plane displacements, while the third direction

$$0 \leq \big(u_3^{(1,q)} - u_3^{(2,p)}\big)(-1)^{p+q} \leq g_{\varepsilon,3}, \qquad \text{a.e in } \mathbf{C}_{pq}, \ (p,q) \in \mathcal{K}_\varepsilon.$$

needs to account for the oscillating manner of the beams switching the vertical positions. Further restrictions and specifications on the contact are given later in the work.

3.3.3 The admissible displacements of the structure

Given the structure, the boundary condition and the contact, the convex set of the admissible displacements is denoted by

$$\begin{aligned}\mathcal{V}_\varepsilon \doteq \Big\{ u = \big(u^{(1,1)}, \dots, u^{(1,2N_\varepsilon)}, u^{(2,0)} \dots, u^{(2,2N_\varepsilon)}\big) \in \prod_{q=1}^{2N_\varepsilon} H^1(\mathcal{P}_\varepsilon^{(1,q)})^3 \times \prod_{p=0}^{2N_\varepsilon} H^1(\mathcal{P}_\varepsilon^{(2,p)})^3 \ \Big| \\ \text{such that } 0 \leq \big(u_3^{(1,q)}(x) - u_3^{(2,p)}(x)\big)(-1)^{p+q} \leq g_{\varepsilon,3}(p\varepsilon, q\varepsilon), \\ |u_\alpha^{(1,q)}(x) - u_\alpha^{(2,p)}(x)| \leq g_{\varepsilon,\alpha}(p\varepsilon, q\varepsilon), \quad \text{for a.e } x \in \mathbf{C}_{pq} \text{ and } (p,q) \in \mathcal{K}_\varepsilon, \\ u^{(2,0)}_{|z_2=0} = u^{(2,1)}_{|z_2=0} = \ldots = u^{(2,2N_\varepsilon)}_{|z_2=0} = 0 \Big\},\end{aligned} \tag{3.32}$$

where $g_{\varepsilon,i}$, $i \in \{1,2,3\}$, is a non-negative function belonging to $\mathcal{C}^0(\overline{\Omega})$. The space $\mathcal{V}_\varepsilon$ is equipped with the semi-norm

$$\forall u \in \mathcal{V}_\varepsilon, \quad \|e(u)\|^2_{L^2(\mathcal{S}_\varepsilon)} = \sum_{q=1}^{2N_\varepsilon} \|e(u^{(1,q)})\|^2_{L^2(\mathcal{P}_\varepsilon^{(1,q)})} + \sum_{p=0}^{2N_\varepsilon} \|e(u^{(2,p)})\|^2_{L^2(\mathcal{P}_\varepsilon^{(2,p)})}.$$

3.3.4 The elasticity problem

The original problem of the textile is stated as the three dimensional elasticity problem on the given space. Thus, for a complete description a material law a is needed. Hereafter, we consider the usual Hooke's law satisfying for $(i,j,k,l) \in \{1,2,3\}^4$ that

- a_ε is bounded: $a_{\varepsilon,ijkl} \in L^\infty(\mathcal{S}_\varepsilon)$
- a_ε is symmetric: $a_{\varepsilon,ijkl} = a_{\varepsilon,jikl} = a_{\varepsilon,klij}$
- a_ε is positive definite: $\exists c_0,\ C_0 > 0 \ : \ c_0\xi_{ij}\xi_{kl} \leq a_{\varepsilon,jikl}(x)\xi_{ij}\xi_{kl} \leq C_0\xi_{ij}\xi_{kl}$ for a.e. $x \in \mathcal{S}_\varepsilon$, where $\xi \in \mathbb{R}^{3\times 3}$ is symmetric

It is also convenient to use the stress tensor σ_ε instead of the material law a_ε and the stress tensor is defined as $\sigma_{\varepsilon,ij}(u) = a_{\varepsilon,ijkl}e_{kl}(u)$. The textile problem in variational form reads as

$$\begin{cases} \text{Find } u_\varepsilon \in \mathcal{V}_\varepsilon \text{ such that:} \\ \displaystyle\int_{\mathcal{S}_\varepsilon} a_\varepsilon e(u_\varepsilon) : e(u_\varepsilon - \varphi)\, dx - \int_{\mathcal{S}_\varepsilon} f_\varepsilon \cdot (u_\varepsilon - \varphi)\, dx \leq 0, \qquad \forall \varphi \in \mathcal{V}_\varepsilon. \end{cases} \tag{3.33}$$

Later on the vectorial notation of the problem is used. Thus, recall

$$\begin{cases} \text{Find } u_\varepsilon \in \mathcal{V}_\varepsilon \text{ such that:} \\ \displaystyle\int_{\mathcal{S}_\varepsilon} A_\varepsilon E_x(u_\varepsilon) \cdot E_x(u_\varepsilon - \varphi)\, dx - \int_{\mathcal{S}_\varepsilon} f_\varepsilon \cdot (u_\varepsilon - \varphi)\, dx \leq 0, \qquad \forall \varphi \in \mathcal{V}_\varepsilon. \end{cases} \tag{3.34}$$

where $A_\varepsilon \in L^\infty(\mathcal{S}_\varepsilon)^{6\times 6}$ is bounded, symmetric and positive definite, which is easily deduced from the properties of a_ε. Furthermore, it satisfies

$$c_0|\zeta|^2 \leq A_\varepsilon(x)\,\zeta \cdot \zeta \leq C_0|\zeta|^2, \qquad \text{for a.e. } x \in \mathcal{S}_\varepsilon \text{ and } \forall \zeta \in \mathbb{R}^6.$$

Remark 3.3.2. *Note that the problem in the current form is solvable but not unique. This follows from the boundary conditions, which allow rigid motions, i.e., motions in the kernel of the symmetric strain tensor. Namely the displacements $u^{(1,q)}$ can have an in-plane rigid motion, since they are only subjected to the rather loose contact condition in $\mathcal{V}_\varepsilon$. To circumvent this ambiguity equip the space with a glued contact at $z_1 = 0$ whereby the $\mathbf{e}_1$-directed beams inherit the clamped condition at $z_2 = 0$. This does not change the limit behavior in the following hence w.l.o.g. we omit this condition below in the estimates and the limit. It is only necessary for the uniqueness of the original problem.*

With the additional condition (glued contact at $z_1 = 0$) existence and uniqueness of this problem is ensured by Stampacchia-Lemma (see [33]).

3.4 Preliminary estimates

This section is dedicated to define a splitting of local and global fields for the textile, which are necessary to investigate the asymptotic behavior of the textile structure. The splitting here is comparable to the technique of the scale-splitting-operators in [18, Section 4]. The first step is to define global fields by interpolating the existing decomposed displacement fields in a suitable way to obtain plate-like displacement fields. That they really resemble a plate displacement is shown in the subsequent estimates for the fields and verified at the end of the section.

For the derivation of the estimates we focus especially on the contact within the structure. As for real specimen the entanglement of the fibers generates stability, which is necessary to obtain suitable global displacements and different contact condition lead to very different limits. At some point we show where such differences of the contact plays an important role. However, we do not present all possible cases, since this would go beyond the scope of this work.

3.4.1 An extension operator for the textile structure

In this section we characterize the extension for the textile structure. The definition of global fields on $\Omega = (0, L)^2$ is characterized by an extension of the fields between the contact midpoints $(p\varepsilon, q\varepsilon)$. First, for the general procedure consider a function φ defined on $\mathcal{K}_\varepsilon$. We extend φ as a function belonging to $W^{1,\infty}(\Omega)$, denoted $\boldsymbol{\varphi}$, in the following way: in the cell $\varepsilon(p,q) + \varepsilon Y$, $(p,q) \in \{0, \dots, 2N_\varepsilon - 1\}^2$ and $Y = (0,2)^2$ we define $\boldsymbol{\varphi}$ as the Q_1-interpolate of its values on the vertices of the cell $\varepsilon(p,q) + \varepsilon Y$.

Lemma 3.4.1. *Let φ be a function defined on $\mathcal{K}_\varepsilon$ and extended as above to a function denoted $\boldsymbol{\varphi}$ and belonging to $W^{1,\infty}(\Omega)$. One has*

$$\|\boldsymbol{\varphi}\|^2_{L^2(\Omega)} \le C\varepsilon^2 \sum_{(p,q)\in\mathcal{K}_\varepsilon} |\varphi(p\varepsilon, q\varepsilon)|^2. \tag{3.35}$$

The constants do not depend on ε and r.
Moreover, $\boldsymbol{\varphi}$ satisfies

$$\begin{aligned}
\boldsymbol{\varphi}(z_1, z_2) &= \boldsymbol{\varphi}(z_1, q\varepsilon) + (z_2 - q\varepsilon)\frac{\partial \boldsymbol{\varphi}}{\partial z_2}(z_1, z_2) \\
&\qquad \forall z_1 \in [0, L], \;\; \textit{for a.e.}\;\; z_2 \in \big((q-1)\varepsilon, (q+1)\varepsilon\big) \cap [0, L], \;\; q \in \{0, \dots, 2N_\varepsilon\}, \\
\boldsymbol{\varphi}(z_1, z_2) &= \boldsymbol{\varphi}(p\varepsilon, z_2) + (z_1 - p\varepsilon)\frac{\partial \boldsymbol{\varphi}}{\partial z_1}(z_1, z_2) \\
&\qquad \forall z_2 \in [0, L], \;\; \textit{for a.e.}\;\; z_1 \in \big((p-1)\varepsilon, (p+1)\varepsilon\big) \cap [0, L], \;\; p \in \{0, \dots, 2N_\varepsilon\}.
\end{aligned} \tag{3.36}$$

Proof. Since the function $\boldsymbol{\varphi}$ is a Q_1-interpolate, it is decomposed using the four Q_1-basis functions $\{N_i(x,y)\}_{i=1,\dots,4}$ in the cell Y. Then, with $\boldsymbol{\varphi}_i$ denoting the four values on the vertices, we obtain

$$\int_Y |\boldsymbol{\varphi}|^2 dxdy = \int_Y \Big|\sum_{i=1}^4 \boldsymbol{\varphi}_i N_i(x,y)\Big|^2 dxdy \le 4\sum_{i=1}^4 |\boldsymbol{\varphi}_i|^2 \|N_i\|^2_{L^2(Y)} = \frac{4}{9}\sum_{i=1}^4 |\boldsymbol{\varphi}_i|^2 \tag{3.37}$$

Consequently, with a rescaling argument transfer this to the cell εY and the fact that every node is part of four cells we obtain the claim by summing over all the cells.
A straightforward calculation gives (3.36). □

This estimation of the interpolant is crucial for the upcoming estimates of the extended fields. Furthermore, the defined extension leaves a function on Ω linear on the edges of the cells $\varepsilon(p,q) + \varepsilon Y$ and thereby on the middle lines of the beams, which is desirable as shown in the next section.

Henceforth, denote by $\mathbf{g}_\varepsilon$ the extension of g_ε, defined in section 3.3.2.

3.4.2 Decomposition of the displacements of the beams structure

Throughout this section the fields depend on ε, which is not further indicated to simplify the notation.

We decompose the displacements $u^{(1,q)}$, $q \in \{1, \dots, 2N_\varepsilon\}$, $u^{(2,p)}$, $p \in \{0, \dots, 2N_\varepsilon\}$, as in section3.2 (see (3.11))

$$\begin{aligned} u^{(1,q)}(x) &= \mathcal{U}^{(1,q)}(z_1) + \mathcal{R}^{(1,q)}(z_1) \wedge (z_2 \mathbf{e}_2 + z_3 \mathbf{n}_\varepsilon^{(1,q)}(z_1)) + \overline{u}^{(1,q)}(z), \\ &\qquad \text{for a.e. } x \in \mathcal{P}_\varepsilon^{(1,q)},\ z \in P_r^{(1)}, \\ u^{(2,p)}(x) &= \mathcal{U}^{(2,p)}(z_2) + \mathcal{R}^{(2,p)}(z_2) \wedge (z_1 \mathbf{e}_1 + z_3 \mathbf{n}_\varepsilon^{(2,p)}(z_2)) + \overline{u}^{(2,p)}(z), \\ &\qquad \text{for a.e. } x \in \mathcal{P}_\varepsilon^{(2,p)},\ z \in P_r^{(2)}. \end{aligned} \tag{3.38}$$

Following (3.17), set for every $(p_\varepsilon, q_\varepsilon) \in \{0, \dots, 2N_\varepsilon\} \times \{1, \dots, 2N\varepsilon\}$

$$\mathbb{U}^{(1,q)} = \mathcal{U}^{(1,q)} - (-1)^{q+1} \Phi_\varepsilon \mathcal{R}^{(1,q)} \wedge \mathbf{e}_3, \tag{3.39}$$

$$\mathbb{U}^{(2,p)} = \mathcal{U}^{(2,p)} - (-1)^{p} \Phi_\varepsilon \mathcal{R}^{(2,p)} \wedge \mathbf{e}_3. \tag{3.40}$$

Denote $\mathcal{U}^{(\alpha)}$, $\mathbb{U}^{(\alpha)}$ and $\mathcal{R}^{(\alpha)}$, $\alpha = 1, 2$ the functions defined on $\mathcal{K}_\varepsilon$, by

$$\begin{aligned} \mathcal{U}^{(1)}(p\varepsilon, q\varepsilon) &= \mathcal{U}^{(1,q)}(p\varepsilon), & \mathcal{U}^{(2)}(p\varepsilon, q\varepsilon) &= \mathcal{U}^{(2,p)}(q\varepsilon), \\ \mathbb{U}^{(1)}(p\varepsilon, q\varepsilon) &= \mathbb{U}^{(1,q)}(p\varepsilon), & \mathbb{U}^{(2)}(p\varepsilon, q\varepsilon) &= \mathbb{U}^{(2,p)}(q\varepsilon), \qquad (p,q) \in \mathcal{K}_\varepsilon, \\ \mathcal{R}^{(1)}(p\varepsilon, q\varepsilon) &= \mathcal{R}^{(1,q)}(p\varepsilon), & \mathcal{R}^{(2)}(p\varepsilon, q\varepsilon) &= \mathcal{R}^{(2,p)}(q\varepsilon), \end{aligned} \tag{3.41}$$

and then extended, without changing the notation, to functions belonging to $W^{1,\infty}(\Omega)$ as defined in the previous subsection 3.4.1.

Moreover, it is necessary to identify the remaining displacement covering the fast oscillations on the middle lines. Thus, similar to (3.21) set

$$\begin{aligned} \widetilde{\mathcal{U}}^{(1)}(\cdot, q\varepsilon) &= \mathcal{U}^{(1,q)}(\cdot) - \mathcal{U}^{(1)}(\cdot, q\varepsilon), & \widetilde{\mathcal{U}}^{(2)}(p\varepsilon, \cdot) &= \mathcal{U}^{(2,p)}(\cdot) - \mathcal{U}^{(2)}(p\varepsilon, \cdot), \\ \widetilde{\mathbb{U}}^{(1)}(\cdot, q\varepsilon) &= \mathbb{U}^{(1,q)}(\cdot) - \mathbb{U}^{(1)}(\cdot, q\varepsilon), & \widetilde{\mathbb{U}}^{(2)}(p\varepsilon, \cdot) &= \mathbb{U}^{(2,p)}(\cdot) - \mathbb{U}^{(2)}(p\varepsilon, \cdot), \qquad (p,q) \in \mathcal{K}_\varepsilon. \\ \widetilde{\mathcal{R}}^{(1)}(\cdot, q\varepsilon) &= \mathcal{R}^{(1,q)}(\cdot) - \mathcal{R}^{(1)}(\cdot, q\varepsilon), & \widetilde{\mathcal{R}}^{(2)}(p\varepsilon, \cdot) &= \mathcal{R}^{(2,p)}(\cdot) - \mathcal{R}^{(2)}(p\varepsilon, \cdot), \end{aligned} \tag{3.42}$$

These fields denoted by $\widetilde{\ }$ are only defined on the lines

$$\begin{aligned} L_\varepsilon^{(1)} &= \bigcup_q \{q\varepsilon \mathbf{e}_2 + z_1 \mid z_1 \in (0, L)\}, \\ L_\varepsilon^{(2)} &= \bigcup_p \{p\varepsilon \mathbf{e}_1 + z_2 \mid z_2 \in (0, L)\}, \end{aligned} \tag{3.43}$$

and are equal to zero on any knot $(p\varepsilon, q\varepsilon) \in \mathcal{K}_\varepsilon$. Furthermore, they coincide with the fields $\mathbb{U}^{(0)}$ and $\mathcal{R}^{(0)}$ defined in (3.21).

Note that $\mathbf{P}_\varepsilon^{[\alpha]} = L_\varepsilon^{(\alpha)} \times \omega_r$ and that $\widetilde{\mathcal{U}}^{(\alpha)}, \widetilde{\mathcal{R}}^{(\alpha)}, \widetilde{\mathbb{U}}^{(\alpha)} \in H^1(L_\varepsilon^{(\alpha)})$. The following Lemma recalls the results of Lemma 3.2.8 for the new setting.

Lemma 3.4.2. *The fields $\widetilde{\mathbb{U}}^{(\alpha)}$, $\widetilde{\mathcal{R}}^{(\alpha)}$ satisfy the estimates*

$$\begin{aligned}
&\|\widetilde{\mathbb{U}}^{(\alpha)}_\alpha\|_{L^2(L^{(\alpha)}_\varepsilon)} + \varepsilon\|\partial_\alpha \widetilde{\mathbb{U}}^{(\alpha)}_\alpha\|_{L^2(L^{(\alpha)}_\varepsilon)} \le C\frac{\varepsilon}{r}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)},\\
&\|\widetilde{\mathbb{U}}^{(\alpha)}_i\|_{L^2(L^{(\alpha)}_\varepsilon)} + \varepsilon\|\partial_\alpha \widetilde{\mathbb{U}}^{(\alpha)}_i\|_{L^2(L^{(\alpha)}_\varepsilon)} \le C\frac{\varepsilon^2}{r^2}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)}, \qquad \text{for } i \neq \alpha\\
&\|\widetilde{\mathcal{R}}^{(\alpha)}\|_{L^2(L^{(\alpha)}_\varepsilon)} + \varepsilon\|\partial_\alpha \widetilde{\mathcal{R}}^{(\alpha)}\|_{L^2(L^{(\alpha)}_\varepsilon)} \le C\frac{\varepsilon}{r^2}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)},
\end{aligned} \tag{3.44}$$

or equivalently

$$\begin{aligned}
&\|\widetilde{\mathbb{U}}^{(\alpha)}_\alpha\|_{L^2(\mathbf{P}^{(\alpha)}_\varepsilon)} + \varepsilon\|\partial_\alpha \widetilde{\mathbb{U}}^{(\alpha)}_\alpha\|_{L^2(\mathbf{P}^{(\alpha)}_\varepsilon)} \le C\varepsilon\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)},\\
&\|\widetilde{\mathbb{U}}^{(\alpha)}_i\|_{L^2(\mathbf{P}^{(\alpha)}_\varepsilon)} + \varepsilon\|\partial_\alpha \widetilde{\mathbb{U}}^{(\alpha)}_i\|_{L^2(\mathbf{P}^{(\alpha)}_\varepsilon)} \le C\frac{\varepsilon^2}{r}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)}, \qquad \text{for } i \neq \alpha\\
&\|\widetilde{\mathcal{R}}^{(\alpha)}\|_{L^2(\mathbf{P}^{(\alpha)}_\varepsilon)} + \varepsilon\|\partial_\alpha \widetilde{\mathcal{R}}^{(\alpha)}\|_{L^2(\mathbf{P}^{(\alpha)}_\varepsilon)} \le C\frac{\varepsilon}{r}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)}.
\end{aligned} \tag{3.45}$$

The constants do not depend on ε and r.

Proof. A direct consequence of Lemma 3.2.8. □

Concluding this section, note that the estimate on the warping (3.14) is easily ported onto the complete structure and we obtain

$$\begin{aligned}
&\sum_{q=1}^{2N_\varepsilon}\big(\|\overline{u}^{(1,q)}\|^2_{L^2(P^{(1)}_r)} + r^2\|\nabla\overline{u}^{(1,q)}\|^2_{L^2(P^{(1)}_r)}\big)\\
&\qquad + \sum_{p=0}^{2N_\varepsilon}\big(\|\overline{u}^{(2,p)}\|^2_{L^2(P^{(2)}_r)} + r^2\|\nabla\overline{u}^{(2,p)}\|^2_{L^2(P^{(2)}_r)}\big) \le Cr^2\|e(u)\|^2_{L^2(\mathcal{S}_\varepsilon)}.
\end{aligned} \tag{3.46}$$

3.4.3 First global estimates

Below we estimate the extended fields $\mathbb{U}^{(1)}$, $\mathbb{U}^{(2)}$, $\mathcal{U}^{(1)}$, $\mathcal{U}^{(2)}$, $\mathcal{R}^{(1)}$, $\mathcal{R}^{(2)}$. From Lemma 3.2.6 and estimate (3.16), we get the following Lemma:

Lemma 3.4.3. *Suppose $r \le \kappa\varepsilon$ with $\kappa < \frac{1}{3}$, then*

$$\begin{aligned}
&\|\partial_\alpha \mathcal{R}^{(\alpha)}\|_{L^2(\Omega)} \le \frac{C\sqrt{\varepsilon}}{r^2}\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})},\\
&\|\partial_\alpha \mathbb{U}^{(\alpha)} - \mathcal{R}^{(\alpha)} \wedge \mathbf{e}_\alpha\|_{L^2(\Omega)} \le C\frac{\varepsilon\sqrt{\varepsilon}}{r^2}\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})}.
\end{aligned} \tag{3.47}$$

The constants do not depend on ε and r.

Proof. Recall the definition (3.41) of the global fields. Then consider $(3.47)_1$ for $\alpha = 1$:

$$\begin{aligned}\left\|\partial_1\mathcal{R}^{(1)}\right\|^2_{L^2(\Omega)} &\leq 2\varepsilon\sum_q \left\|\partial_1\mathcal{R}^{(1)}(\cdot,q\varepsilon)\right\|^2_{L^2(0,L)}\\ &\leq 2\varepsilon^2\sum_{p,q}\left|\frac{\mathcal{R}^{(1)}((p+1)\varepsilon,q\varepsilon)-\mathcal{R}^{(1)}(p\varepsilon,q\varepsilon)}{\varepsilon}\right|^2\\ &\leq 2\sum_{p,q}\left|\mathcal{R}^{(1)}((p+1)\varepsilon,q\varepsilon)-\mathcal{R}^{(1)}(p\varepsilon,q\varepsilon)\right|^2,\end{aligned}$$

where upon we apply $(3.19)_1$ and obtain $(3.47)_1$. The case $\alpha = 2$ is analogous.
The second estimate $(3.47)_2$ is a consequence of $(3.19)_2$ by the same means as for $(3.47)_1$. □

Corollary 3.4.4. *Furthermore, the fields satisfy*

$$\begin{aligned}&\|\mathbb{U}_2^{(2)}\|_{L^2(\Omega)} \leq C\frac{\sqrt{\varepsilon}}{r}\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})},\\ &\left\|\mathcal{R}^{(2)}\right\|_{L^2(\Omega)} + \left\|\mathbb{U}_3^{(2)}\right\|_{L^2(\Omega)} \leq C\frac{\sqrt{\varepsilon}}{r^2}\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})}.\end{aligned} \tag{3.48}$$

Proof. Use (3.47), the Poincaré inequality and the boundary condition in order to get (3.48). □

3.4.4 Estimates on contact

The contact between the fibers gives rise to estimates on the global fields and their difference. The latter is very important to determine the limits and whether or not they coincide. First recall that for a.e. $x \in \mathbf{C}_{pq}$ (note that $|z_3| = r$ in $\mathbf{C}_{pq}$) the displacements reduce to

$$\begin{aligned}u^{(1,q)}(x) &= \mathcal{U}^{(1,q)}(p\varepsilon+z_1) + \mathcal{R}^{(1,q)}(p\varepsilon+z_1)\wedge\left(z_2\mathbf{e}_2 + (-1)^{p+q+1}r\mathbf{e}_3\right) + \overline{u}^{(1,q)}(x),\\ &= \mathbb{U}^{(1,q)}(p\varepsilon+z_1) + \mathcal{R}^{(1,q)}(p\varepsilon+z_1)\wedge z_2\mathbf{e}_2 + \overline{u}^{(1,q)}(x),\\ u^{(2,p)}(x) &= \mathcal{U}^{(2,p)}(q\varepsilon+z_2) + \mathcal{R}^{(2,p)}(q\varepsilon+z_2)\wedge\left(z_1\mathbf{e}_1 + (-1)^{p+q}r\mathbf{e}_3\right) + \overline{u}^{(2,p)}(x),\\ &= \mathbb{U}^{(2,p)}(q\varepsilon+z_2) + \mathcal{R}^{(2,p)}(q\varepsilon+z_2)\wedge z_1\mathbf{e}_1 + \overline{u}^{(2,p)}(x),\end{aligned} \tag{3.49}$$

with $(z_1, z_2) \in \omega_r$.

To estimate the fields independently, it is necessary to start with the warping.

Lemma 3.4.5. *Let u be in V_ε, then we have*

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon}\left(\|\overline{u}^{(1,q)}\|^2_{L^2(\mathbf{C}_{pq})} + \|\overline{u}^{(2,p)}\|^2_{L^2(\mathbf{C}_{pq})}\right) \leq Cr\|e(u)\|^2_{L^2(\mathcal{S}_\varepsilon)}. \tag{3.50}$$

The constant does not depend on ε and r.

Proof. First, we recall a classical inequality. For any $\varphi \in H^1(0,r)$ we have

$$r|\varphi(0)|^2 \leq 2\|\varphi\|^2_{L^2(0,r)} + r^2\|\varphi'\|^2_{L^2(0,r)}.$$

This inequality applied in the third direction yields

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} r\big(\|\overline{u}^{(1,q)}\|^2_{L^2(\mathbf{C}_{pq})}+\|\overline{u}^{(2,p)}\|^2_{L^2(\mathbf{C}_{pq})}\big)\le C\sum_{q=1}^{2N_\varepsilon}\big(\|\overline{u}^{(1,q)}\|^2_{L^2(P_r^{(1)})}+r^2\|\nabla\overline{u}^{(1,q)}\|^2_{L^2(P_r^{(1)})}\big)$$
$$+C\sum_{p=0}^{2N_\varepsilon}\big(\|\overline{u}^{(2,p)}\|^2_{L^2(P_r^{(2)})}+r^2\|\nabla\overline{u}^{(2,p)}\|^2_{L^2(P_r^{(2)})}\big).$$

Then (3.46) implies (3.50). □

Lemma 3.4.6. *We have*

$$\begin{aligned}\big\|\mathcal{R}^{(1)}-\mathcal{R}^{(2)}\big\|_{L^2(\Omega)}&\le\frac{C}{r}\big(\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)}+\frac{\varepsilon}{\sqrt{r}}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)}\big),\\ \big\|\mathbb{U}^{(1)}-\mathbb{U}^{(2)}\big\|_{L^2(\Omega)}&\le C\big(\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)}+\frac{\varepsilon}{\sqrt{r}}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)}\big).\end{aligned}\tag{3.51}$$

Proof. From the equalities (3.49), the jump conditions on $\mathbf{C}_{pq}$ in the definition of V_ε (see subsection 3.3.3) one first obtains

$$\begin{aligned}\sum_{p,\,q}\int_{C_{pq}}&\big|\mathbb{U}^{(1,q)}(p\varepsilon+z_1)+\mathcal{R}^{(1,q)}(p\varepsilon+z_1)\wedge z_2\mathbf{e}_2\\ &-\mathbb{U}^{(2,p)}(q\varepsilon+z_2)-\mathcal{R}^{(2,p)}(q\varepsilon+z_2)\wedge z_1\mathbf{e}_1\big|^2dz_1dz_2\\ &\le\sum_{p,\,q}\big(\|\overline{u}^{(1,q)}\|^2_{L^2(C_{pq})}+\|\overline{u}^{(2,p)}\|^2_{L^2(C_{pq})}+r^2\big|g_\varepsilon(p\varepsilon,q\varepsilon)\big|^2\big).\end{aligned}\tag{3.52}$$

Then estimates (3.19) and (3.50) lead to

$$\sum_{p,\,q}\big|\mathbb{U}^{(1,q)}(p\varepsilon)-\mathbb{U}^{(2,p)}(q\varepsilon)\big|^2+r^2\sum_{p,\,q}\big|\mathcal{R}^{(1,q)}(p\varepsilon)-\mathcal{R}^{(2,p)}(q\varepsilon)\big|^2$$
$$\le C\sum_{p,\,q}\big|g_\varepsilon(p\varepsilon,q\varepsilon)\big|^2+\frac{C}{r}\|e(u)\|^2_{L^2(\mathcal{S}_\varepsilon)}.$$

Now, applying Lemma 3.4.1 yields the claim. □

As a consequence of Lemma 3.4.6 and (3.48), we get

Corollary 3.4.7. *The fields satisfy*

$$\begin{aligned}&\|\mathbb{U}_2^{(1)}\|_{L^2(\Omega)}\le C\Big(\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)}+\frac{\sqrt{\varepsilon}}{r}\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})}\Big),\\ &\|\mathcal{R}^{(1)}\|_{L^2(\Omega)}\le\frac{C}{r}\Big(\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)}+\frac{\sqrt{\varepsilon}}{r}\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})}\Big),\\ &\|\mathbb{U}_3^{(1)}\|_{L^2(\Omega)}\le C\Big(\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)}+\frac{\sqrt{\varepsilon}}{r^2}\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})}\Big).\end{aligned}\tag{3.53}$$

Proof. A direct consequence of Lemma 3.4.6 and Corollary 3.4.4. □

3.4.5 The bending and the membrane displacements

In this section, the estimates on the global displacements are revisited and improved by the estimates on the contact areas.

Lemma 3.4.8. *The rotations $\mathcal{R}^{(\alpha)}$ and bending displacement $\mathbb{U}_3^{(\alpha)}$ fulfill*

$$\begin{aligned}\|\mathcal{R}^{(\alpha)}\|_{H^1(\Omega)} &\leq C\frac{\sqrt{\varepsilon}}{r^2}\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})} + \frac{C}{\varepsilon r}\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)},\\ \|\mathbb{U}_3^{(\alpha)}\|_{H^1(\Omega)} &\leq C\frac{\sqrt{\varepsilon}}{r^2}\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})} + \frac{C}{r}\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)}.\end{aligned} \tag{3.54}$$

Proof. First, note that the global fields are piecewise bilinear and any such function satisfies

$$\|\nabla\varphi\|_{L^2(\Omega)} \leq \frac{C}{\varepsilon}\|\varphi\|_{L^2(\Omega)}. \tag{3.55}$$

This yields together with (3.51):

$$\begin{aligned}\|\partial_\beta\mathcal{R}^{(1)} - \partial_\beta\mathcal{R}^{(2)}\|_{L^2(\Omega)} &\leq \frac{C}{\varepsilon}\|\mathcal{R}^{(1)} - \mathcal{R}^{(2)}\|_{L^2(\Omega)} \leq \frac{C}{\varepsilon r}\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)} + \frac{C}{r\sqrt{r}}\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})},\\ \|\partial_\beta\mathbb{U}_3^{(1)} - \partial_\beta\mathbb{U}_3^{(2)}\|_{L^2(\Omega)} &\leq \frac{C}{\varepsilon}\|\mathbb{U}_3^{(1)} - \mathbb{U}_3^{(2)}\|_{L^2(\Omega)} \leq \frac{C}{\varepsilon}\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)} + \frac{C}{\sqrt{r}}\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})}.\end{aligned}$$

Collecting the estimates (3.47), $(3.48)_2$ and $(3.53)_{2,3}$ yields the claim.

□

The next Lemma estimates the membrane displacements in the strain tensor.

Lemma 3.4.9. *One has*

$$\|e_{12}(\mathbb{U}^{(1)})\|_{L^2(\Omega)} + \|e_{12}(\mathbb{U}^{(2)})\|_{L^2(\Omega)} \leq C\frac{\varepsilon\sqrt{\varepsilon}}{r^2}\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})} + \frac{C}{r}\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)}.$$

Proof. Observe that

$$\begin{aligned}\|\partial_1\mathbb{U}_2^{(1)} + \partial_2\mathbb{U}_1^{(1)}\|_{L^2(\Omega)} \leq{}& \|\big(\partial_1\mathbb{U}^{(1)} - \mathcal{R}^{(1)}\wedge\mathbf{e}_1\big)\cdot\mathbf{e}_2\|_{L^2(\Omega)} + \|\big(\partial_2\mathbb{U}^{(2)} - \mathcal{R}^{(2)}\wedge\mathbf{e}_2\big)\cdot\mathbf{e}_1\|_{L^2(\Omega)}\\ &+ \|\partial_2\mathbb{U}_1^{(2)} - \partial_2\mathbb{U}_1^{(1)}\|_{L^2(\Omega)} + \|\mathcal{R}_3^{(1)} - \mathcal{R}_3^{(2)}\|_{L^2(\Omega)}.\end{aligned}$$

Then from $(3.47)_2$ and $(3.51)_{1,2}$, it yields

$$\|e_{12}(\mathbb{U}^{(1)})\|_{L^2(\Omega)} \leq C\big(\frac{\varepsilon}{r\sqrt{r}} + \frac{\varepsilon\sqrt{\varepsilon}}{r^2}\big)\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})} + \frac{C}{r}\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)}$$

and together with $r \leq \kappa\varepsilon$ we recover the claim. In the same way $\|e_{12}(\mathbb{U}^{(2)})\|_{L^2(\Omega)}$ is estimated.

□

The estimate on the symmetric gradient allows to transfer the estimation onto the membrane displacements itself. The next Corollary uses the 2D-Korn-inequality to obtain the H^1-estimates on $\mathbb{U}^{(\alpha)}$.

Corollary 3.4.10. *The membrane displacements and* $\mathcal{R}_3^{(\alpha)}$ *satisfy*

$$\begin{aligned} \|\mathbb{U}_\beta^{(\alpha)}\|_{H^1(\Omega)} &\leq C\frac{\varepsilon\sqrt{\varepsilon}}{r^2}\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})} + \frac{C}{r}\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)}, \\ \|\mathcal{R}_3^{(\alpha)}\|_{L^2(\Omega)} &\leq C\frac{\varepsilon\sqrt{\varepsilon}}{r^2}\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})} + \frac{C}{r}\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)}. \end{aligned} \tag{3.56}$$

Proof. By the clamp-condition at $z_2 = 0$, the estimates in Lemmas 3.4.3-3.4.9 and the 2D-Korn inequality we deduce $(3.56)_1$. The second estimate is a consequence of the first one and $(3.47)_2$. □

Remark 3.4.11 (The complements)**.** *From Lemmas 3.2.6, 3.4.3 and 3.4.7 we obtain*

$$\begin{aligned} \big\|\mathbb{U}^{(1)} - \mathcal{U}^{(1)}\big\|_{L^2(\Omega)^3} &\leq Cr\|\mathcal{R}^{(1)}\|_{L^2(\Omega)^3} \leq C\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)^3} + \frac{C\sqrt{\varepsilon}}{r}\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})}, \\ \big\|\mathbb{U}^{(2)} - \mathcal{U}^{(2)}\big\|_{L^2(\Omega)^3} &\leq Cr\|\mathcal{R}^{(2)}\|_{L^2(\Omega)^3} \leq \frac{C\sqrt{\varepsilon}}{r}\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})}. \end{aligned} \tag{3.57}$$

3.4.6 Final decomposition

Since $\mathbb{U}^{(1)}$ and $\mathbb{U}^{(2)}$ (respectively $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$) converge to the same limit, it is convenient to define a combined field. Hence, set

$$\begin{aligned} \mathbb{U} &= \frac{1}{2}\big(\mathbb{U}^{(1)} + \mathbb{U}^{(2)}\big), & \mathcal{R} &= \frac{1}{2}\big(\mathcal{R}^{(1)} + \mathcal{R}^{(2)}\big), \\ \mathbb{U}^{(g)} &= \frac{1}{2}\big(\mathbb{U}^{(1)} - \mathbb{U}^{(2)}\big), & \mathcal{R}^{(g)} &= \frac{1}{2}\big(\mathcal{R}^{(1)} - \mathcal{R}^{(2)}\big). \end{aligned} \tag{3.58}$$

Observe that these fields vanish on $z_2 = 0$ by definition and moreover one has

$$\begin{aligned} \mathbb{U}^{(1)} &= \mathbb{U} + \mathbb{U}^{(g)}, & \mathbb{U}^{(2)} &= \mathbb{U} - \mathbb{U}^{(g)}, \\ \mathcal{R}^{(1)} &= \mathcal{R} + \mathcal{R}^{(g)}, & \mathcal{R}^{(2)} &= \mathcal{R} - \mathcal{R}^{(g)}, \end{aligned} \tag{3.59}$$

and for the original beam-displacements

$$\begin{aligned} \mathbb{U}^{(1,q)}(z_1) &= \mathbb{U}(z_1, q\varepsilon) + \mathbb{U}^{(g)}(z_1, q\varepsilon) + \widetilde{\mathbb{U}}^{(1,q)}(z_1), \\ \mathbb{U}^{(2,p)}(z_2) &= \mathbb{U}(p\varepsilon, z_2) - \mathbb{U}^{(g)}(p\varepsilon, z_2) + \widetilde{\mathbb{U}}^{(2,p)}(z_2), \\ \mathcal{R}^{(1,q)}(z_1) &= \mathcal{R}(z_1, q\varepsilon) + \mathcal{R}^{(g)}(z_1, q\varepsilon) + \widetilde{\mathcal{R}}^{(1,q)}(z_1), \\ \mathcal{R}^{(2,p)}(z_2) &= \mathcal{R}(p\varepsilon, z_2) - \mathcal{R}^{(g)}(p\varepsilon, z_2) + \widetilde{\mathcal{R}}^{(2,p)}(z_2). \end{aligned} \tag{3.60}$$

The Lemma below is an immediate consequence of the above results for global fields.

Lemma 3.4.12. *One has*

$$
\begin{aligned}
&\|\mathcal{R}\|_{H^1(\Omega)} \le C\frac{\sqrt{\varepsilon}}{r^2}\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})} + \frac{C}{\varepsilon r}\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)},\\
&\|\mathcal{R}_3\|_{L^2(\Omega)} \le C\frac{\varepsilon\sqrt{\varepsilon}}{r^2}\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})} + \frac{C}{r}\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)},\\
&\|\mathbb{U}_1\|_{H^1(\Omega)} + \|\mathbb{U}_2\|_{H^1(\Omega)} \le C\frac{\varepsilon\sqrt{\varepsilon}}{r^2}\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})} + \frac{C}{r}\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)},\\
&\|\mathbb{U}_3\|_{H^1(\Omega)} \le C\frac{\sqrt{\varepsilon}}{r^2}\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})} + \frac{C}{r}\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)},\\
&\|\partial_\alpha\mathbb{U} - \mathcal{R}\wedge\mathbf{e}_\alpha\|_{L^2(\Omega)} \le C\frac{\varepsilon\sqrt{\varepsilon}}{r^2}\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})} + \frac{C}{r}\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)},
\end{aligned}
\tag{3.61}
$$

and

$$
\begin{aligned}
&\big\|\mathcal{R}^{(g)}\big\|_{L^2(\Omega)} + \varepsilon\|\nabla\mathcal{R}^{(g)}\|_{L^2(\Omega)} \le \frac{C}{r}\big(\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)} + \frac{\varepsilon}{\sqrt{r}}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)}\big),\\
&\big\|\mathbb{U}^{(g)}\big\|_{L^2(\Omega)} + \varepsilon\big\|\nabla\mathbb{U}^{(g)}\big\|_{L^2(\Omega)} \le C\big(\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)} + \frac{\varepsilon}{\sqrt{r}}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)}\big).
\end{aligned}
\tag{3.62}
$$

The constants do not depend on ε.

Proof. The estimates of the gradients in (3.62) are the consequences of the fact that all these fields are piecewise linear between two knots. □

3.4.7 Estimate on the right hand side

The elastic energy corresponds directly to the force applied to the structure and with the estimates on the displacements one can show, that the force $f^{(\alpha)} \in H^1(\Omega)$ $(\alpha = 1, 2)$ with

$$
f_\varepsilon^{(\alpha)} = \varepsilon^\tau f_{1,\varepsilon}^{(\alpha)}\mathbf{e}_1 + \varepsilon^\tau f_{2,\varepsilon}^{(\alpha)}\mathbf{e}_2 + \varepsilon^{\tau+1} f_{3,\varepsilon}^{(\alpha)}\mathbf{e}_3 \tag{3.63}
$$

and then restricting to the mid-line of every beam, i.e., for $(p, q) \in \mathcal{K}_\varepsilon$:

$$
\begin{aligned}
f_\varepsilon^{(1,q)}(z_1) &= f_\varepsilon^{(1)}(z_1, q\varepsilon) \quad \text{for a.e. } z_1 \in (0, L),\\
f_\varepsilon^{(2,p)}(z_2) &= f_\varepsilon^{(2)}(p\varepsilon, z_2) \quad \text{for a.e. } z_2 \in (0, L).
\end{aligned}
$$

is sufficient to estimate the elastic energy. Henceforth, write indifferently f_ε for the collection of the forces $f^{(1,q)}$ and $f^{(2,p)}$ in the beams, since the difference is in most cases obvious and a distinction is not necessary.

Indeed, the estimates from section 3.4 lead to

$$
\Big|\int_{\mathcal{S}_\varepsilon} f_\varepsilon\cdot u_\varepsilon dx\Big| \le C\varepsilon^{\tau+1/2}\left(\|f^{(1)}\|_{H^1(\Omega)} + \|f^{(2)}\|_{H^1(\Omega)}\right)\left[\|e(u_\varepsilon)\|_{L^2(\mathcal{S}_{\varepsilon,r})} + \Big(\frac{1}{\sqrt{\varepsilon}} + \sqrt{\varepsilon}\Big)\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)}\right].
$$

By the coercivity of the problem, we obtain for $\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)} \le C\varepsilon^{\tau+1}$:

$$
C_0\|e(u_\varepsilon)\|^2_{L^2(\mathcal{S}_{\varepsilon,r})} \le C\varepsilon^{\tau+1/2}\left(\|f^{(1)}\|_{H^1(\Omega)} + \|f^{(2)}\|_{H^1(\Omega)}\right)\left[\|e(u_\varepsilon)\|_{L^2(\mathcal{S}_{\varepsilon,r})} + \varepsilon^{\tau+1/2}\right]
$$

and thus

$$\|e(u_\varepsilon)\|_{L^2(\mathcal{S}_{\varepsilon,r})} \le C\varepsilon^{\tau+1/2}(\|f^{(1)}\|_{H^1(\Omega)} + \|f^{(2)}\|_{H^1(\Omega)}) \le C\varepsilon^{\tau+1/2}, \tag{3.64}$$

which finalizes the estimates on the fields.

3.5 Asymptotic behavior of the macroscopic fields

From now on, we assume that $r = \kappa\varepsilon$ with $\kappa \le \widehat{\kappa}$ and to identify the problem completely we assume the gap-function $g_\varepsilon = \varepsilon^3 g$ with $g \in \mathcal{C}(\overline{\Omega})^3$ such that g_ε satisfies

$$g_\varepsilon = \varepsilon^3 g, \quad g \in \mathcal{C}(\overline{\Omega})^3, \text{ hence } g_\varepsilon \in \mathcal{C}(\overline{\Omega})^3 \text{ and } \quad \|g_\varepsilon\|_{L^2(\Omega)} \le C\varepsilon^3 \|g\|_{L^\infty(\Omega)}. \tag{3.65}$$

This condition is important and bequeaths much information and regularity for the whole problem.

Furthermore, we assume for the elastic energy

$$\|e(u_\varepsilon)\|_{L^2(\mathcal{S}_\varepsilon)} \le C\varepsilon^{5/2}, \tag{3.66}$$

which is achieved by estimate (3.64) for a right-hand side in the form (3.63) and $\tau = 2$.

3.5.1 First limit of the macroscopic fields

Lemma 3.5.1. *Let $\{u_\varepsilon\}$ be a sequence of displacements belonging to V_ε and satisfying* (3.66)*. Then there exists a subsequence of ε, still denoted ε, and functions $\mathbb{U}_1, \mathbb{U}_2, \mathcal{R}_\alpha \in H^1(\Omega)$, $\mathbb{U}_3 \in H^2(\Omega)$and $\mathcal{Z}_\alpha \in L^2(\Omega)^3$ such that the following convergences hold ($(\alpha,\beta) \in \{1,2\}^2$):*

$$\frac{1}{\varepsilon^2}\mathbb{U}_\varepsilon^{(g)} \rightharpoonup 0 \qquad \text{weakly in } H^1(\Omega)^3, \tag{3.67}$$

$$\frac{1}{\varepsilon}\mathcal{R}_\varepsilon^{(g)} \rightharpoonup 0 \qquad \text{weakly in } H^1(\Omega)^3, \tag{3.68}$$

and

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathbb{U}_{\varepsilon,\alpha},\ \frac{1}{\varepsilon^2}\mathbb{U}_{\varepsilon,\alpha}^{(\beta)} \rightharpoonup \mathbb{U}_\alpha \qquad \text{weakly in } H^1(\Omega),\\
&\frac{1}{\varepsilon}\mathbb{U}_{\varepsilon,3},\ \frac{1}{\varepsilon}\mathbb{U}_{\varepsilon,3}^{(\beta)} \rightharpoonup \mathbb{U}_3 \qquad \text{weakly in } H^1(\Omega),\\
&\frac{1}{\varepsilon}\mathcal{R}_{\varepsilon,\alpha},\ \frac{1}{\varepsilon}\mathcal{R}_{\varepsilon,\alpha}^{(\beta)} \rightharpoonup \mathcal{R}_\alpha \qquad \text{weakly in } H^1(\Omega),\\
&\frac{1}{\varepsilon^2}\Big(\partial_\alpha \mathbb{U}_\varepsilon - \mathcal{R}_\varepsilon \wedge \mathbf{e}_\alpha\Big) \rightharpoonup \mathcal{Z}_\alpha \qquad \text{weakly in } L^2(\Omega)^3.
\end{aligned} \tag{3.69}$$

The fields satisfy the boundary conditions $\mathbb{U}(\cdot,0) = \mathcal{R}(\cdot,0) = 0$. Moreover, in the limit the identity

$$\partial_1\mathbb{U}_3 = -\mathcal{R}_2, \qquad \partial_2\mathbb{U}_3 = \mathcal{R}_1 \tag{3.70}$$

holds true.

Proof. Lemma 3.4.12 gives

$$\begin{aligned}
&\|\mathcal{R}_\varepsilon\|_{H^1(\Omega)} + \|\mathbb{U}_3\|_{H^1(\Omega)} \le C\varepsilon,\\
&\|\mathbb{U}_{\varepsilon,1}\|_{H^1(\Omega)} + \|\mathbb{U}_{\varepsilon,2}\|_{H^1(\Omega)} + \|\mathcal{R}_{\varepsilon,3}\|_{L^2(\Omega)} \le C\varepsilon^2,\\
&\|\partial_\alpha \mathbb{U}_\varepsilon - \mathcal{R}_\varepsilon \wedge \mathbf{e}_\alpha\|_{L^2(\Omega)} \le C\varepsilon^2,
\end{aligned}$$

and

$$\begin{aligned}
&\big\|\mathcal{R}^{(g)}_\varepsilon\big\|_{L^2(\Omega)} + \varepsilon\|\nabla\mathcal{R}^{(g)}_\varepsilon\|_{L^2(\Omega)} \le C\varepsilon^2,\\
&\big\|\mathbb{U}^{(g)}_\varepsilon\big\|_{L^2(\Omega)} + \varepsilon\big\|\nabla\mathbb{U}^{(g)}_\varepsilon\big\|_{L^2(\Omega)} \le C\varepsilon^3.
\end{aligned}$$

Hence, there exist a subsequence of ε, still denoted ε, and functions $\mathbb{U}_1$, $\mathbb{U}_2$, $\mathbb{U}_3$, $\mathcal{R}_1$ and $\mathcal{R}_2$ in $H^1(\Omega)$ such that the convergences (3.67)-(3.69) hold. Moreover, one has

$$\frac{1}{\varepsilon}(\partial_\alpha\, \mathbb{U}_\varepsilon - \mathcal{R}_\varepsilon \wedge \mathbf{e}_\alpha)\cdot \mathbf{e}_3 \to 0 \qquad \text{strongly in } L^2(\Omega),$$

from which we obtain (3.70). Hence $\mathbb{U}_3$ belongs to $H^2(\Omega)$. For the boundary conditions we refer to the definition of the fields, then $\mathbb{U}_i(\cdot,0) = \mathcal{R}_\alpha(\cdot,0) = 0$ is an immediate consequence. □

Beside the weak convergence of the rotation field $\mathcal{R}$, we have in the limit that $\mathcal{R}_3 = 0$ by estimate $(3.56)_2$ in Corollary 3.4.10.

3.5.2 The unfolding operator for the middle-lines

In this section, we introduce the unfolding operator $\mathcal{T}_\varepsilon$ especially for the global fields $\mathbb{U}$, $\mathcal{R}$, $\mathbb{U}^{(g)}$, $\mathcal{R}^{(g)}$, $\mathbb{U}^{(\alpha)}$ and $\mathcal{R}^{(\alpha)}$.

Set $\mathcal{Y} = (0,2)^2$, the periodicity cell of the global fields. Furthermore, set

$$\begin{aligned}
\mathcal{Y}^{\ell s} &= \bigcup_{(a,b)\in\{0,1\}^2} \Big\{(z_1,b)\;\big|\; z_1 \in (a,a+1)\Big\} \cup \Big\{(a,z_2)\;\big|\; z_2\in(b,b+1)\Big\},\\
\mathcal{Y}^K &= \Big\{(a,b)\;\big|\;(a,b)\in\{0,1,2\}^2\Big\},
\end{aligned}$$

for set of lines and set of contact-points in $\mathcal{Y}$.

Definition 3.5.2. *For every measurable function φ in the domain Ω define the measurable function $\mathcal{T}_\varepsilon(\varphi)$ on $\Omega\times\mathcal{Y}$ by*

$$\mathcal{T}_\varepsilon(\varphi)(s,X') = \varphi\big(2p\varepsilon\mathbf{e}_1 + 2q\varepsilon\mathbf{e}_2 + \varepsilon X'\big) \quad \textit{for a.e.}\;\; s\in(2p\varepsilon,2q\varepsilon)+\varepsilon\mathcal{Y}, \quad X'\in\mathcal{Y}.$$

Note that $\mathcal{T}_\varepsilon$ maps $L^p(\Omega)$ into $L^p(\Omega\times\mathcal{Y})$. The properties of the unfolding operator can be found in [18]. The most important property here is described by the next Lemma.

Lemma 3.5.3. *The unfolding operator $\mathcal{T}_\varepsilon : L^p(\Omega)\to L^p(\Omega\times\mathcal{Y})$ satisfies*

$$\|\mathcal{T}_\varepsilon(\varphi)\|_{L^2(\Omega\times\mathcal{Y})} \le C\|\varphi\|_{L^2(\Omega)},$$

where C is a constant only depending on $\mathcal{Y}$.

Proof. Consequence of [18]. □

For the determination of the limits, especially for the limit-contact, a special property of the unfolding operator is needed.

Lemma 3.5.4 (see [16, Lemma 11.11]). *Let $\{(u_\varepsilon, v_\varepsilon)\}_\varepsilon$ be a sequence converging weakly to (u, v) in the space $H^1(\Omega) \times H^1(\Omega)^2$. Assume furthermore that there exist $\mathcal{Z}$ in $L^2(\Omega)^2$ and $\widehat{v}$ in $L^2(\Omega; H^1_{per,0}(\mathcal{Y}))^2$ such that*

$$\begin{aligned}
&\frac{1}{\varepsilon}\big(\nabla u_\varepsilon + v_\varepsilon\big) \rightharpoonup \mathcal{Z} \quad \text{weakly in } \ L^2(\Omega)^2,\\
&\mathcal{T}_\varepsilon(\nabla v_\varepsilon) \rightharpoonup \nabla v + \nabla_X \widehat{v} \quad \text{weakly in } \ L^2(\Omega \times \mathcal{Y})^{2\times 2}.
\end{aligned}$$

Then u belongs to $H^2(\Omega)$ and there exists $\mathfrak{u} \in L^2(\Omega; H^1_{per,0}(\mathcal{Y}))$ such that, up to a subsequence,

$$\frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\nabla u_\varepsilon + v_\varepsilon\big) \rightharpoonup \mathcal{Z} + \nabla_X \mathfrak{u} + \widehat{v} \quad \text{weakly in } \ L^2(\Omega \times \mathcal{Y})^2.$$

To conclude this subsection define the spaces of special Q^1-interpolates by

$$\begin{aligned}
&Q^1(\mathcal{Y}) = \Big\{\varphi \in W^{1,\infty}(\mathcal{Y}) \mid \varphi \text{ is the } Q_1 \text{ interpolated of the values on the points in } \mathcal{Y}^K\Big\},\\
&Q^1_{per}(\mathcal{Y}) = Q^1(\mathcal{Y}) \cap H^1_{per}(\mathcal{Y}).
\end{aligned}$$

Note that the macroscopic fields are in $Q^1(\mathcal{Y})$ by definition.

3.5.3 Unfolded limits of the macroscopic fields

Lemma 3.5.5. *There exist $\widehat{\mathbb{U}}_\alpha$, $\widehat{\mathbb{U}}_3$, $\widehat{\mathcal{R}}_\alpha \in L^2(\Omega; Q^1_{per}(\mathcal{Y}))$, $\widehat{\mathcal{R}}_3 \in L^2(\Omega; Q^1_{per}(\mathcal{Y}))$ and $\widehat{\mathbb{U}}^{(g)}, \mathcal{R}^{(g)} \in L^2(\Omega; Q^1_{per}(\mathcal{Y}))^3$ such that*

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon(\nabla \mathbb{U}_{\varepsilon,\alpha}) \rightharpoonup \nabla \mathbb{U}_\alpha + \nabla_X \widehat{\mathbb{U}}_\alpha \qquad \text{weakly in } L^2(\Omega \times \mathcal{Y})^2,\\
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon(\nabla \mathcal{R}_{\varepsilon,\alpha}) \rightharpoonup \nabla \mathcal{R}_\alpha + \nabla_X \widehat{\mathcal{R}}_\alpha \qquad \text{weakly in } L^2(\Omega \times \mathcal{Y})^2,\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon(\mathcal{R}_{\varepsilon,3}) \rightharpoonup \widehat{\mathcal{R}}_3 \qquad \text{weakly in } L^2(\Omega; Q^1(\mathcal{Y})),\\
&\frac{1}{\varepsilon^3}\mathcal{T}_\varepsilon(\mathbb{U}^{(g)}_\varepsilon) \rightharpoonup \widehat{\mathbb{U}}^{(g)} \qquad \text{weakly in } L^2(\Omega; Q^1(\mathcal{Y}))^3,\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon(\mathcal{R}^{(g)}_\varepsilon) \rightharpoonup \widehat{\mathcal{R}}^{(g)} \qquad \text{weakly in } L^2(\Omega; Q^1(\mathcal{Y}))^3,
\end{aligned} \tag{3.71}$$

Moreover, one has

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_1\mathbb{U}_\varepsilon-\mathcal{R}_\varepsilon\wedge\mathbf{e}_1\big)\cdot\mathbf{e}_1\rightharpoonup\partial_1\mathbb{U}_1+\partial_{X_1}\widehat{\mathbb{U}}_1 \qquad \textit{weakly in } L^2(\Omega\times\mathcal{Y}),\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_1\mathbb{U}_\varepsilon-\mathcal{R}_\varepsilon\wedge\mathbf{e}_1\big)\cdot\mathbf{e}_2\rightharpoonup\partial_1\mathbb{U}_2+\partial_{X_1}\widehat{\mathbb{U}}_2-\widehat{\mathcal{R}}_3 \qquad \textit{weakly in } L^2(\Omega\times\mathcal{Y}),\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_1\mathbb{U}_\varepsilon-\mathcal{R}_\varepsilon\wedge\mathbf{e}_1\big)\cdot\mathbf{e}_3\rightharpoonup\mathcal{Z}_{13}+\partial_{X_1}\widehat{\mathbb{U}}_3+\widehat{\mathcal{R}}_2 \qquad \textit{weakly in } L^2(\Omega\times\mathcal{Y}),
\end{aligned}\tag{3.72}$$

and

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_2\mathbb{U}_\varepsilon-\mathcal{R}_\varepsilon\wedge\mathbf{e}_2\big)\cdot\mathbf{e}_1\rightharpoonup\partial_2\mathbb{U}_1+\partial_{X_2}\widehat{\mathbb{U}}_1+\widehat{\mathcal{R}}_3 \qquad \textit{weakly in } L^2(\Omega\times\mathcal{Y}),\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_2\mathbb{U}_\varepsilon-\mathcal{R}_\varepsilon\wedge\mathbf{e}_2\big)\cdot\mathbf{e}_2\rightharpoonup\partial_2\mathbb{U}_2+\partial_{X_2}\widehat{\mathbb{U}}_2 \qquad \textit{weakly in } L^2(\Omega\times\mathcal{Y}),\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_2\mathbb{U}_\varepsilon-\mathcal{R}_\varepsilon\wedge\mathbf{e}_2\big)\cdot\mathbf{e}_3\rightharpoonup\mathcal{Z}_{23}+\partial_{X_2}\widehat{\mathbb{U}}_3-\widehat{\mathcal{R}}_1 \qquad \textit{weakly in } L^2(\Omega\times\mathcal{Y}).
\end{aligned}\tag{3.73}$$

Proof. Convergences (3.71) are the consequences of the estimates in Lemma 3.4.12 and the convergences in Lemma 3.5.1 (see [18]). Convergences $(3.72)_{1,2}$-$(3.73)_{1,2}$ are the immediate consequences of (3.71), while convergences $(3.72)_3$-$(3.73)_3$ come from the convergence $(3.69)_4$, Lemma 3.5.4 and denoting $\widehat{\mathbb{U}}_3=\mathfrak{u}$. ☐

Set

$$\mathcal{Y}_{ab}=(a,a+1)\times(b,b+1),\qquad (a,b)\in\{0,1\}^2.$$

In Lemma below we specify the function $\widehat{\mathbb{U}}_3$.

Lemma 3.5.6. *There exists $\widetilde{\mathbb{U}}_3\in L^2(\Omega;Q^1_{per}(\mathcal{Y}))$ such that*

$$\widehat{\mathbb{U}}_3(\cdot,X_1,X_2)=\widetilde{\mathbb{U}}_3(\cdot,X_1,X_2)-\frac{1}{2}(X_1-1)^2\partial_{11}\mathbb{U}_3-\frac{1}{2}(X_2-1)^2\partial_{22}\mathbb{U}_3. \tag{3.74}$$

Proof. Write

$$\mathcal{R}_{\varepsilon,\alpha}=\mathcal{M}_\varepsilon(\mathcal{R}_{\varepsilon,\alpha})+\big(\mathcal{R}_{\varepsilon,\alpha}-\mathcal{M}_\varepsilon(\mathcal{R}_{\varepsilon,\alpha})\big),$$

where

$$\mathcal{M}_\varepsilon(\mathcal{R}_{\varepsilon,\alpha})=\frac{1}{|\mathcal{Y}|}\int_{\mathcal{Y}}\mathcal{T}_\varepsilon(\mathcal{R}_{\varepsilon,\alpha})(\cdot,X_1,X_2)dX_1dX_2.$$

One has from the estimate of $\mathcal{R}_{\varepsilon,\alpha}$, convergence $(3.71)_2$ and Theorem 3.5 in [18]

$$\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\mathcal{R}_{\varepsilon,\alpha}-\mathcal{M}_\varepsilon(\mathcal{R}_{\varepsilon,\alpha})\big)\rightharpoonup(X_1-1)\frac{\partial\mathcal{R}_\alpha}{\partial z_1}+(X_2-1)\frac{\partial\mathcal{R}_\alpha}{\partial z_2}+\widehat{\mathcal{R}}_\alpha\quad\text{weakly in }\ L^2(\Omega\times\mathcal{Y}).$$

Hence, due to $(3.72)_3$-$(3.73)_3$ and the above convergences, we obtain the following weak convergences in $L^2(\Omega\times\mathcal{Y})$ (recall that $\partial_1\mathbb{U}_3=-\mathcal{R}_2$, and $\partial_2\mathbb{U}_3=\mathcal{R}_1$):

$$\begin{aligned}
\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_1\mathbb{U}_\varepsilon-\mathcal{M}_\varepsilon(\mathcal{R}_\varepsilon)\wedge\mathbf{e}_1\big)\cdot\mathbf{e}_3 &\rightharpoonup \mathcal{Z}_{13}+\partial_{X_1}\widehat{\mathbb{U}}_3-(X_1-1)\frac{\partial\mathcal{R}_2}{\partial z_1}-(X_2-1)\frac{\partial\mathcal{R}_2}{\partial z_2}\\
&=\mathcal{Z}_{13}+\partial_{X_1}\widehat{\mathbb{U}}_3+(X_1-1)\partial_{11}\mathbb{U}_3+(X_2-1)\partial_{12}\mathbb{U}_3,\\
\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_2\mathbb{U}_\varepsilon-\mathcal{M}_\varepsilon(\mathcal{R}_\varepsilon)\wedge\mathbf{e}_2\big)\cdot\mathbf{e}_3 &\rightharpoonup \mathcal{Z}_{23}+\partial_{X_2}\widehat{\mathbb{U}}_3+(X_1-1)\frac{\partial\mathcal{R}_1}{\partial z_1}+(X_2-1)\frac{\partial\mathcal{R}_1}{\partial z_2}\\
&=\mathcal{Z}_{23}+\partial_{X_2}\widehat{\mathbb{U}}_3+(X_1-1)\partial_{12}\mathbb{U}_3+(X_2-1)\partial_{22}\mathbb{U}_3.
\end{aligned}$$

Set

$$\widetilde{\mathbb{U}}_3(\cdot,X_1,X_2)=\widehat{\mathbb{U}}_3(\cdot,X_1,X_2)+\frac{1}{2}(X_1-1)^2\partial_{11}\mathbb{U}_3+\frac{1}{2}(X_2-1)^2\partial_{22}\mathbb{U}_3.$$

This function belongs to $L^2(\Omega;H^1_{per}(\mathcal{Y}))$.
Now, observe that by construction $\mathcal{T}_\varepsilon(\partial_1\mathbb{U}_\varepsilon)(\cdot,X_1,X_2)$ is piecewise constant with respect to X_1 and linear with respect to X_2 in each domain $\mathcal{Y}_{ab}$, conversely $\mathcal{T}_\varepsilon(\partial_2\mathbb{U}_\varepsilon)(\cdot,X_1,X_2)$ is piecewise constant with respect to X_2 and linear with respect to X_1 in each domain $\mathcal{Y}_{ab}$. As a consequence, the function $\partial_1\widetilde{\mathbb{U}}_3(\cdot,X_1,X_2)$ is piecewise constant with respect to X_1 and linear with respect to X_2 in each domain $\mathcal{Y}_{ab}$, and $\partial_2\widetilde{\mathbb{U}}_3(\cdot,X_1,X_2)$ is piecewise constant with respect to X_2 and linear with respect to X_1 in each domain $\mathcal{Y}_{ab}$. It means that $\widetilde{\mathbb{U}}_3$ belongs to $L^2(\Omega;\mathcal{Q}^1_{per}(\mathcal{Y}))$. □

3.6 Asymptotic behavior of the unfolded fields

3.6.1 Unfolding for the Textile

Set for $(a,b)\in\{0,1\}^2$ the beam-periodicity cell $Cyls\doteq Cyls^{(1)}\cup Cyls^{(2)}$ with

$$\begin{aligned}
&Cyl^{(1,b)}\doteq b\mathbf{e}_2+(0,2)\times(-\kappa,\kappa)^2, \qquad Cyl^{(2,a)}\doteq a\mathbf{e}_1+(-\kappa,\kappa)\times(0,2)\times(-\kappa,\kappa),\\
&Cyls^{(1)}\doteq Cyl^{(1,0)}\cup Cyl^{(1,1)}, \quad Cyls^{(2)}\doteq Cyl^{(2,0)}\cup Cyl^{(2,1)}.
\end{aligned}$$

Definition 3.6.1. *For every measurable function φ in the domain $\mathbf{P}^{[\alpha]}_\varepsilon$, one defines the measurable function $\Pi^{[\alpha]}_\varepsilon(\varphi)$ in $\Omega\times Cyls^{(\alpha)}$ by ($\alpha\in\{1,2\}$)*

$$\Pi^{[\alpha]}_\varepsilon(\varphi)(z,X)=\varphi\big(2p\varepsilon\mathbf{e}_1+2q\varepsilon\mathbf{e}_2+\varepsilon X\big),\qquad z\in 2p\varepsilon\mathbf{e}_1+2q\varepsilon\mathbf{e}_2+\varepsilon\mathcal{Y},\quad X\in Cyls^{(\alpha)}.$$

Furthermore, for every $\varphi\in L^s(\mathcal{S}_\varepsilon)$ ($s\in[1,+\infty)$) we define the unfolding operator

$$\Pi_\varepsilon(\varphi)=\big(\Pi^{[1]}_\varepsilon(\varphi^{[1]}),\Pi^{[2]}_\varepsilon(\varphi^{[2]})\big)$$

as a mapping from $L^s(\mathcal{S}_\varepsilon)$ into $L^s(\Omega\times Cyls^{(1)})\times L^s(\Omega\times Cyls^{(2)})$ and we set

$$\|\Pi_\varepsilon(\varphi)\|_{L^s(\Omega\times Cyls)}=\Big(\|\Pi^{[1]}_\varepsilon(\varphi^{(1)})\|^s_{L^s(\Omega\times Cyls^{(1)})}+\|\Pi^{[2]}_\varepsilon(\varphi^{(2)})\|^s_{L^s(\Omega\times Cyls^{(2)})}\Big)^{1/s}.$$

To characterize the unfolded functions, it is necessary to state a relation to the original function. Note that this unfolding operator changes the convergence-rate, since a dimension reduction is directly incorporated. To address this individually, it is possible to define it as a composition of an unfolding operator and a rescaling operator, see [16].

Lemma 3.6.2. *For every $\varphi \in L^1(\mathbf{P}_\varepsilon^{[\alpha]})$, one has*

$$\int_{P_\varepsilon^{[\alpha]}} \varphi(z)dz = \frac{\varepsilon}{4}\int_\Omega \int_{Cyls^{[\alpha]}} \Pi_\varepsilon^{[\alpha]}(\varphi)(s,X)dsdX, \qquad \alpha \in \{1,2\}.$$

Proof. It is an easy consequence of the transformation of integrals and the definitions above. Indeed, we have for $\alpha = 1$

$$\begin{aligned}
\int_{\mathbf{P}_\varepsilon^{[1]}} \varphi(z)dz &= \sum_q^{N_\varepsilon}\sum_p^{N_\varepsilon} \int_{\varepsilon Cyls^{(1)}} \varphi(2q\varepsilon\mathbf{e}_2 + 2p\varepsilon\mathbf{e}_1 + z)dz \\
&= \varepsilon^3 \sum_q^{N_\varepsilon}\sum_p^{N_\varepsilon} \int_{Cyls^{(1)}} \varphi(2q\varepsilon\mathbf{e}_2 + 2p\varepsilon\mathbf{e}_1 + \varepsilon X)dX \\
&= \frac{\varepsilon^3}{4\varepsilon^2} \sum_q^{N_\varepsilon}\sum_p^{N_\varepsilon} \int_{2p\varepsilon\mathbf{e}_1+2q\varepsilon\mathbf{e}_2+(0,2\varepsilon)^2} \int_{Cyls^{(1)}} \varphi(2q\varepsilon\mathbf{e}_2 + 2p\varepsilon\mathbf{e}_1 + \varepsilon X)dXdz \\
&= \frac{\varepsilon}{4}\int_\Omega \int_{Cyls^{(1)}} \Pi_\varepsilon^{[1]}(\varphi)(z,X)dXdz.
\end{aligned}$$

Analogously for $\alpha = 2$ which yields then the claim. $\square$

Lemma 3.6.3. *For every $\varphi \in L^s(\mathcal{S}_\varepsilon)$, $s \in [1,+\infty]$ one has*

$$C_0\varepsilon^{1/s}\|\Pi_\varepsilon(\varphi)\|_{L^s(\Omega\times Cyls)} \leq \|\varphi\|_{L^s(\mathcal{S}_\varepsilon)} \leq C_1\|\Pi_\varepsilon(\varphi)\|_{L^s(\Omega\times Cyls)}. \tag{3.75}$$

The constants do not depend on ε and r.

Proof. First assume $s \in [1,+\infty)$. As a consequence of the above Lemma and (3.29), one gets for every $\varphi \in L^s(\mathcal{S}_\varepsilon)$

$$\begin{aligned}
\|\varphi\|^s_{L^s(\mathcal{S}_\varepsilon)} = \int_{\mathcal{S}_\varepsilon} |\varphi(x)|^s dx = &\sum_{q=0}^{2N_\varepsilon} \int_{\mathbf{P}_\varepsilon^{[1]}} |\varphi^{[1]}(q\varepsilon\mathbf{e}_2 + z)|^s |\det\big(\nabla\psi_\varepsilon^{(1,q)}(z)\big)|dz \\
&+ \sum_{p=1}^{2N_\varepsilon} \int_{\mathbf{P}_\varepsilon^{[2]}} |\varphi^{[2]}(p\varepsilon\mathbf{e}_1 + z)|^s |\det\big(\nabla\psi_\varepsilon^{(2,p)}(z)\big)|dz \\
\leq &C\varepsilon\Big(\|\Pi_\varepsilon^{[1]}(\varphi^{[1]})\|^s_{L^s(\Omega\times Cyls^{(1)})} + \|\Pi_\varepsilon^{[2]}(\varphi^{[2]})\|^s_{L^s(\Omega\times Cyls^{(2)})}\Big).
\end{aligned}$$

Since the Jacobians are bounded from below, we also obtain

$$C\varepsilon\Big(\|\Pi_\varepsilon^{[1]}(\varphi^{[1]})\|^s_{L^s(\Omega\times Cyls^{(1)})} + \|\Pi_\varepsilon^{[2]}(\varphi^{[2]})\|^s_{L^s(\Omega\times Cyls^{(2)})}\Big) \leq \|\varphi\|^s_{L^s(\mathcal{S}_\varepsilon)}.$$

Hence (3.75) is proved for any $s \in [1,+\infty)$. The case $s = +\infty$ is obvious. $\square$

Actually, the case $s=2$, where

$$\|\Pi_\varepsilon(\varphi)\|_{L^2(\Omega\times Cyls)} \leq \frac{C}{\sqrt{\varepsilon}}\|\varphi\|_{L^2(\mathcal{S}_\varepsilon)}, \qquad \forall\varphi\in L^2(\mathcal{S}_\varepsilon).$$

is the most important for the following analysis.

In fact, the unfolding operator $\mathcal{T}_\varepsilon$, defined in Definition 3.5.2, is a restriction of the unfolding operator $\Pi^{[\alpha]}$ of the complete textile. Indeed, we find for a function φ defined on $\mathcal{K}_\varepsilon$ and extended as in subsection 3.4.1 into a function belonging to $W^{1,\infty}(\Omega)$, denoted $\boldsymbol{\varphi}$, then

$$\begin{aligned}\Pi^{[1]}_\varepsilon(\boldsymbol{\varphi}_{|\mathcal{Y}^{\ell s}})(s,X) &= \boldsymbol{\varphi}\big((2p\mathbf{e}_1+2q\mathbf{e}_2)\varepsilon+\varepsilon b\mathbf{e}_2+\varepsilon X_1\mathbf{e}_1\big) = \mathcal{T}_\varepsilon(\boldsymbol{\varphi})(s,X_1,b),\\ &s\in(2p\mathbf{e}_1+2q\mathbf{e}_2)+\varepsilon\mathcal{Y}, \quad b\in\{0,1\}, \quad X_1\in(0,2), \quad (p,q)\in\{0,\dots N_\varepsilon-1\}^2. \end{aligned}\tag{3.76}$$

The second direction

$$\begin{aligned}\Pi^{[2]}_\varepsilon(\boldsymbol{\varphi}_{|\mathcal{Y}^{\ell s}})(s,X) &= \boldsymbol{\varphi}\big((2p\mathbf{e}_1+2q\mathbf{e}_2)\varepsilon+\varepsilon a\mathbf{e}_1+\varepsilon X_2\mathbf{e}_2\big) = \mathcal{T}_\varepsilon(\boldsymbol{\varphi})(s,a,X_2),\\ &s\in(2p\mathbf{e}_1+2q\mathbf{e}_2)+\varepsilon\mathcal{Y}, \quad a\in\{0,1\}, \quad X_2\in(0,2), \quad (p,q)\in\{0,\dots N_\varepsilon-1\}^2, \end{aligned}\tag{3.77}$$

is derived analogously.

First consider the limit of the unfolded basis-frame in the following Lemma.

Lemma 3.6.4. *The oscillating function Φ_ε converges strongly*

$$\frac{1}{\varepsilon}\Pi^{(\alpha)}(\Phi^{[\alpha]}_\varepsilon)\to\Phi^{(\alpha,b)}, \qquad \textit{strongly in } L^2(\Omega,H^1(\mathcal{Y}^{\ell s,\alpha}))\tag{3.78}$$

with $\Phi^{(\alpha,b)}=(-1)^{\alpha+b}\Phi(X_\alpha)$. Moreover, we have the following strong convergences

$$\begin{aligned}
&\Pi^{[\alpha]}_\varepsilon(c^{[\alpha]}_\varepsilon)\to \widehat{c}^{(\alpha,c)}(X_\alpha)= \frac{d^2_{X_\alpha}\Phi^{(\alpha,c)}(X_\alpha-c)}{\boldsymbol{\gamma}(X_\alpha-c)^3} \quad \textit{strongly in } L^2(\Omega;H^1(0,2))\\
&\Pi^{[\alpha]}_\varepsilon(\eta^{[\alpha]}_\varepsilon)\to \eta^{(\alpha,c)}(X_\alpha,X_3)= \boldsymbol{\gamma}(X_\alpha-c)\big(1-X_3\widehat{c}^{(\alpha,c)}(X_\alpha-c)\big) \quad \textit{strongly in } L^2(\Omega\times Cyls^{(\alpha)})\\
&\Pi^{[\alpha]}_\varepsilon(\mathbf{t}^{[\alpha]}_\varepsilon)\to \mathbf{t}^{(\alpha,c)}(X_\alpha)= \frac{1}{\boldsymbol{\gamma}(X_\alpha-c)}\big(\mathbf{e}_\alpha+d_{X_\alpha}\Phi^{(\alpha,c)}(X_\alpha-c)\mathbf{e}_3\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{strongly in } [L^2(\Omega;H^1(0,2))]^3\\
&\Pi^{[\alpha]}_\varepsilon(\mathbf{n}^{[\alpha]}_\varepsilon)\to \mathbf{n}^{(\alpha,c)}(X_\alpha)= \frac{1}{\boldsymbol{\gamma}(X_\alpha-c)}\big(-d_{X_\alpha}\Phi^{(\alpha,c)}(X_\alpha-c)\mathbf{e}_\alpha+\mathbf{e}_3\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{strongly in } [L^2(\Omega;H^1(0,2))]^3
\end{aligned}$$

with $\boldsymbol{\gamma}(t)=\sqrt{1+\big(\Phi'(t)\big)^2}$ and thereby

$$\Pi^{[1]}_\varepsilon(\nabla\psi^{[1]}_\varepsilon)\to\big(\eta^{(1,b)}\mathbf{t}^{(1,b)}\mid\mathbf{e}_2\mid\mathbf{n}^{(1,b)}\big), \qquad \Pi^{[2]}_\varepsilon(\nabla\psi^{[2]}_\varepsilon)\to\big(\mathbf{e}_2\mid\eta^{(2,a)}\mathbf{t}^{(2,a)}\mid\mathbf{n}^{(2,a)}\big).$$

Proof. The proof is trivial and uses just the properties of the unfolding operator and the fact that the unfolded functions only depend on the microscopic set of variables. $\square$

Remark 3.6.5. *Note that the identity* $\mathbf{n}_\varepsilon = \mathbf{t}_\varepsilon \wedge \mathbf{e}_2$ *remains for all the original, the unfolded and the limit vectors. Additionally, the curvature fulfills the identity* $\dfrac{d\mathbf{n}^{(1,b)}}{dX_1} = -\widehat{c}^{(1,b)}\,\boldsymbol{\gamma}\,\mathbf{t}^{(1,b)}$ *at the limit.*

3.6.2 Limits of the unfolded elementary displacements

Lemma 3.6.6. *Under the assumptions of Lemma 3.5.1, the following convergences hold* ($(\alpha,\beta) \in \{1,2\}^2$)*:*

$$\begin{aligned}
&\frac{1}{\varepsilon}\Pi_\varepsilon^{[\alpha]}\big(\mathbb{U}^{(\alpha)}_{\varepsilon,3\,|\mathcal{Y}^{\ell s}}\big) \rightharpoonup \mathbb{U}_3 \qquad \text{weakly in } L^2(\Omega; H^1(Cyls^{(\alpha)})),\\
&\frac{1}{\varepsilon}\Pi_\varepsilon^{[\alpha]}\big(\mathcal{R}^{(\alpha)}_{\varepsilon,\beta\,|\mathcal{Y}^{\ell s}}\big) \rightharpoonup \mathcal{R}_\beta \qquad \text{weakly in } L^2(\Omega; H^1(Cyls^{(\alpha)})),\\
&\frac{1}{\varepsilon^2}\Pi_\varepsilon^{[\alpha]}\big(\mathbb{U}^{(\alpha)}_{\varepsilon,\beta\,|\mathcal{Y}^{\ell s}}\big) \longrightarrow \mathbb{U}_\beta \qquad \text{weakly in } L^2(\Omega; H^1(Cyls^{(\alpha)})),
\end{aligned}$$

where the above macroscopic limit fields are given by Lemma 3.5.1.

Proof. These convergences are the consequences of the definitions (3.58), the convergences in Lemmas 3.5.1 and the definitions of the unfolding operators $\Pi_\varepsilon^{[\alpha]}$, $\mathcal{T}_\varepsilon$ and (3.76)-(3.77). $\square$

Denote

$$H^1_{00}(0,2) \doteq \Big\{ \quad \in H^1(0,2) \ \ \big| \quad (0) = \psi(1) = \psi(2) = 0\Big\}.$$

Recall, that the fields $\mathbb{U}_\varepsilon$, $\mathcal{R}_\varepsilon$ and $\mathbb{U}_\varepsilon^{(g)}$, $\mathcal{R}_\varepsilon^{(g)}$ have to be restricted to $L^{(\alpha)}$, the center lines of the beams, to build the actual beam displacements, cf. (3.60).

Lemma 3.6.7. *Under the assumptions of Lemma 3.5.1, there exist a subsequence of* $\{\varepsilon\}$ *(still denoted* $\{\varepsilon\}$*) and* $\widehat{\mathcal{R}}^{(\alpha,b)}$, $\widehat{\mathbb{U}}^{(\alpha,b)} \in L^2(\Omega; H^1_{00}(0,2))^3$ *such that the following convergences hold* ($(a,b) \in \{0,1\}^2$, $(\alpha,\beta) \in \{1,2\}^2$)*:*

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\Pi_\varepsilon^{[1]}\big(\mathcal{R}^{[1]}_{\varepsilon,3}\big) \rightharpoonup \widehat{\mathcal{R}}_{3|X_2=b} + \widehat{\mathcal{R}}^{(g)}_{3|X_2=b} + \widehat{\mathcal{R}}^{(1,b)}_3 \qquad \text{weakly in } L^2(\Omega; H^1(Cyl^{(1,b)})),\\
&\frac{1}{\varepsilon^2}\Pi_\varepsilon^{[2]}\big(\mathcal{R}^{[2]}_{\varepsilon,3}\big) \rightharpoonup \widehat{\mathcal{R}}_{3|X_1=a} - \widehat{\mathcal{R}}^{(g)}_{3|X_1=a} + \widehat{\mathcal{R}}^{(2,a)}_3 \qquad \text{weakly in } L^2(\Omega; H^1(Cyl^{(2,a)})),\\
&\frac{1}{\varepsilon}\Pi_\varepsilon^{[1]}\big(\partial_1\mathcal{R}^{[1]}_{\varepsilon,\beta}\big) \rightharpoonup \partial_1\mathcal{R}_\beta + \partial_{X_1}\widehat{\mathcal{R}}_{\beta|X_2=b} + \partial_{X_1}\widehat{\mathcal{R}}^{(g)}_{\beta|X_2=b} + \partial_{X_1}\widehat{\mathcal{R}}^{(1,b)}_\beta \ \text{weakly in } L^2(\Omega\times Cyl^{(1,b)}),\\
&\frac{1}{\varepsilon}\Pi_\varepsilon^{[2]}\big(\partial_2\mathcal{R}^{[2]}_{\varepsilon,\beta}\big) \rightharpoonup \partial_2\mathcal{R}_\beta + \partial_{X_2}\widehat{\mathcal{R}}_{\beta|X_1=a} - \partial_{X_2}\widehat{\mathcal{R}}^{(g)}_{\beta|X_1=a} + \partial_{X_2}\widehat{\mathcal{R}}^{(2,a)}_\beta \ \text{weakly in } L^2(\Omega\times Cyl^{(2,a)}),
\end{aligned} \tag{3.79}$$

and

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\Pi_\varepsilon^{[1]}\big(\partial_1\,\mathbb{U}^{[1]}_{\varepsilon,\beta}\big) \rightharpoonup \partial_1\mathbb{U}_\beta + \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{\beta|X_2=b} + \partial_{X_1}\widehat{\mathbb{U}}_{\beta|X_2=b} + \partial_{X_1}\widehat{\mathbb{U}}^{(1,b)}_\beta \ \text{weakly in } L^2(\Omega\times Cyl^{(1,b)}),\\
&\frac{1}{\varepsilon^2}\Pi_\varepsilon^{[2]}\big(\partial_2\,\mathbb{U}^{[2]}_{\varepsilon,\beta}\big) \rightharpoonup \partial_2\mathbb{U}_\beta - \partial_{X_2}\widehat{\mathbb{U}}^{(g)}_{\beta|X_1=a} + \partial_{X_2}\widehat{\mathbb{U}}_{\beta|X_1=a} + \partial_{X_2}\widehat{\mathbb{U}}^{(2,a)}_\beta \ \text{weakly in } L^2(\Omega\times Cyl^{(2,a)}).
\end{aligned} \tag{3.80}$$

Moreover,

$$\frac{1}{\varepsilon^2}\Pi_\varepsilon^{[1]}(\partial_1\mathbb{U}_\varepsilon^{[1]}-\mathcal{R}_\varepsilon^{[1]}\wedge\mathbf{e}_1)\rightharpoonup$$
$$\begin{pmatrix}\partial_1\mathbb{U}_1+\partial_{X_1}\widehat{\mathbb{U}}_{1|X_2=b}\\ \partial_1\mathbb{U}_2+\partial_{X_1}\widehat{\mathbb{U}}_{2|X_2=b}-\widehat{\mathcal{R}}_{3|X_2=b}\\ \mathcal{Z}_{13}+\partial_{X_1}\widehat{\mathbb{U}}_{3|X_2=b}+\widehat{\mathcal{R}}_{2|X_2=b}\end{pmatrix}+\begin{pmatrix}\partial_{X_1}\widehat{\mathbb{U}}_1^{(1,b)}\\ \partial_{X_1}\widehat{\mathbb{U}}_2^{(1,b)}-\widehat{\mathcal{R}}_3^{(1,b)}\\ \partial_{X_1}\widehat{\mathbb{U}}_3^{(1,b)}+\widehat{\mathcal{R}}_2^{(1,b)}\end{pmatrix}+\begin{pmatrix}\partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{1|X_2=b}\\ \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{2|X_2=b}-\widehat{\mathcal{R}}^{(g)}_{3|X_2=b}\\ \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{3|X_2=b}+\widehat{\mathcal{R}}^{(g)}_{2|X_2=b}\end{pmatrix}$$
$$\text{weakly in } L^2(\Omega\times Cyl^{(1,b)})^3, \tag{3.81}$$

and

$$\frac{1}{\varepsilon^2}\Pi_\varepsilon^{[2]}(\partial_2\mathbb{U}_\varepsilon^{[2]}-\mathcal{R}_\varepsilon^{[2]}\wedge\mathbf{e}_2)\rightharpoonup$$
$$\begin{pmatrix}\partial_2\mathbb{U}_1+\partial_{X_2}\widehat{\mathbb{U}}_{1|X_1=a}+\widehat{\mathcal{R}}_{3|X_1=a}\\ \partial_2\mathbb{U}_2+\partial_{X_2}\widehat{\mathbb{U}}_{2|X_1=a}\\ \mathcal{Z}_{23}+\partial_{X_2}\widehat{\mathbb{U}}_{3|X_1=a}-\widehat{\mathcal{R}}_{1|X_1=a}\end{pmatrix}+\begin{pmatrix}\partial_{X_2}\widehat{\mathbb{U}}_1^{(2,a)}+\widehat{\mathcal{R}}_3^{(2,a)}\\ \partial_{X_2}\widehat{\mathbb{U}}_2^{(2,a)}\\ \partial_{X_2}\widehat{\mathbb{U}}_3^{(2,a)}-\widehat{\mathcal{R}}_1^{(2,a)}\end{pmatrix}-\begin{pmatrix}\partial_{X_2}\widehat{\mathbb{U}}^{(g)}_{1|X_1=a}+\widehat{\mathcal{R}}^{(g)}_{3|X_1=a}\\ \partial_{X_2}\widehat{\mathbb{U}}^{(g)}_{2|X_1=a}\\ \partial_{X_2}\widehat{\mathbb{U}}^{(g)}_{3|X_1=a}-\widehat{\mathcal{R}}^{(g)}_{1|X_1=a}\end{pmatrix}$$
$$\text{weakly in } L^2(\Omega\times Cyl^{(2,a)})^3. \tag{3.82}$$

Proof. First, as a consequence of estimates (3.44), there exist a subsequence of $\{\varepsilon\}$ (still denoted $\{\varepsilon\}$) and $\widehat{\mathcal{R}}^{(\alpha,c)}$, $\widehat{\mathbb{U}}^{(\alpha,c)}\in L^2(\Omega;H^1_{00}(0,2))^3$ such that the following convergences hold ($c\in\{0,1\}$, $(\alpha,\beta)\in\{1,2\}^2$):

$$\begin{aligned}&\frac{1}{\varepsilon^2}\Pi_\varepsilon^{[\alpha]}\big(\widetilde{\mathcal{R}}_\varepsilon^{(\alpha)}\big)\rightharpoonup\widehat{\mathcal{R}}^{(\alpha,c)} && \text{weakly in } L^2(\Omega;H^1(Cyl^{(\alpha,b)}))^3,\\ &\frac{1}{\varepsilon^3}\Pi_\varepsilon^{[\alpha]}\big(\widetilde{\mathbb{U}}_\varepsilon^{(\alpha)}\big)\rightharpoonup\widehat{\mathbb{U}}^{(\alpha,c)} && \text{weakly in } L^2(\Omega;H^1(Cyl^{(\alpha,b)}))^3.\end{aligned} \tag{3.83}$$

Furthermore, note that the displacements are split according to (3.60), which is why Lemma B.1 is needed and the reason for the restrictions of the limit fields.

In fact, it is a priori not clear if the limit functions admit a trace. Actually, to obtain this result note that due to the piecewise-linear character of the functions, one has

$$\begin{aligned}&\|[\partial_1\mathbb{U}_\varepsilon-\mathcal{R}_\varepsilon\wedge\mathbf{e}_1]\cdot\mathbf{e}_3\|_{L^2(\Omega)}\le C\frac{\varepsilon\sqrt{\varepsilon}}{r^2}\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})}+\frac{C}{r}\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)}\le C\varepsilon^2,\\ &\|\partial_2\left[(\partial_1\mathbb{U}_\varepsilon-\mathcal{R}_\varepsilon\wedge\mathbf{e}_1)\cdot\mathbf{e}_3\right]\|_{L^2(\Omega)}\le C\frac{\sqrt{\varepsilon}}{r^2}\|e(u)\|_{L^2(\mathcal{S}_{\varepsilon,r})}+\frac{C}{\varepsilon r}\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)}\le C\varepsilon.\end{aligned}$$

As a consequence the restricted unfolded function equals the unfolded restricted function, i.e.,

$$\Pi_\varepsilon^{[1]}\big([\partial_1\mathbb{U}_\varepsilon-\mathcal{R}_\varepsilon\wedge\mathbf{e}_1]\cdot\mathbf{e}_3\big)_{|X_2=b}=\Pi_\varepsilon^{[1]}\big(([\partial_1\mathbb{U}_\varepsilon-\mathcal{R}_\varepsilon\wedge\mathbf{e}_1]\cdot\mathbf{e}_3)_{|L_\varepsilon^{(1)}}\big)$$

and we have by Lemma B.1

$$\|\Pi_\varepsilon^{[1]}\big([\partial_1\mathbb{U}_\varepsilon-\mathcal{R}_\varepsilon\wedge\mathbf{e}_1]\cdot\mathbf{e}_3\big)_{|X_2=b}\|_{L^2(\Omega\times(\overline{\mathcal{Y}}\cap\{X_2=b\})}\le C\varepsilon^2.$$

The second direction is derived analogously.

Observe that the resulting restrictions only apply to the variable in the "lateral" direction, i.e., $X_2 = b$ for the fields corresponding to the fields with index $(1,b)$ (or $X_1 = a$ for $(2,a)$ respectively).

Then, convergences (3.79)-(3.80) are the consequences of the above (3.83) and those in Lemma 3.5.5. From (3.79)-(3.80) we also derive $(3.81)_{1,2}$-$(3.82)_{1,2}$. For the convergences $(3.81)_3$-$(3.82)_3$ we use Lemma 3.5.5 together with again the convergences (3.79)-(3.80). $\square$

Remark 3.6.8. *The limit displacements themselves converge strongly in the unfolded spaces, i.e.,*

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\Pi_\varepsilon^{[\alpha]}(u_{\beta,\varepsilon}^{[\alpha]}) \to \mathbb{U}_\beta - \left(\frac{X_3}{\gamma} + \Phi\right)\partial_\beta \mathbb{U}_3 \qquad \text{strongly in } L^2(\Omega; H^1(Cyls^{(\alpha)})),\\
&\frac{1}{\varepsilon}\Pi_\varepsilon^{[\alpha]}(u_{3,\varepsilon}^{[\alpha]}) \to \mathbb{U}_3 \qquad \text{strongly in } L^2(\Omega; H^1(Cyls^{(\alpha)})).
\end{aligned}$$

Note that for $\Phi \equiv 0$ they coincide with the usual Kirchoff-Love *displacement for a plate.*

3.6.3 The limit of the warping

Now set for the convergences of the warpings

$$\begin{aligned}
\widehat{\mathbf{W}}^{(1)} &\doteq \left\{v^{(1)} = (v^{(1,0)}, v^{(1,1)}) \in H^1(Cyls^{(1)}) \;\middle|\; v^{(1,b)}(2, X_2 - b, X_3) = v^{(1,b)}(0, X_2 - b, X_3)\right\},\\
\widehat{\mathbf{W}}^{(2)} &\doteq \left\{v^{(2)} = (v^{(2,0)}, v^{(2,1)}) \in H^1(Cyls^{(2)}) \;\middle|\; v^{(2,a)}(X_1 - a, 2, X_3) = v^{(2,a)}(X_1 - a, 0, X_3)\right\}.
\end{aligned}$$

Lemma 3.6.9. *There exists a subsequence, still denoted by ε, and a $\overline{u}^{(\alpha)} \in L^2(\Omega; \widehat{\mathbf{W}}^{(\alpha)})^3$ such that the following convergence holds*

$$\frac{1}{\varepsilon^3}\Pi_\varepsilon^{[\alpha]}(\overline{u}_\varepsilon^{[\alpha]}) \rightharpoonup \overline{u}^{(\alpha,c)} \;\; \text{weakly in} \;\; L^2(\Omega; H^1(Cyl^{(\alpha,c)}))^3, \quad \alpha \in \{1,2\}, \;\; c \in \{0,1\}.$$

Furthermore, the fields $\overline{u}^{(1,b)}$, $b \in \{0,1\}$, satisfy a.e. in $\Omega \times (0,L)$

$$\begin{aligned}
&\int_\omega \overline{u}^{(1,b)}(\cdot, X)\,dX_2 dX_3 = 0,\\
&\int_\omega \overline{u}^{(1,b)}(\cdot, X) \wedge \big((X_2 - b)\mathbf{e}_2 + X_3\mathbf{n}(X_1)\big)\,dX_2 dX_3 = 0,
\end{aligned} \tag{3.84}$$

or respectively for $\overline{u}^{(2,a)}$, $a \in \{0,1\}$

$$\begin{aligned}
&\int_\omega \overline{u}^{(2,a)}(\cdot, X)\,dX_1 dX_3 = 0,\\
&\int_\omega \overline{u}^{(2,a)}(\cdot, X) \wedge \big((X_1 - a)\mathbf{e}_1 + X_3\mathbf{n}(X_2)\big)\,dX_1 dX_3 = 0.
\end{aligned} \tag{3.85}$$

Proof. From (3.14)-(3.75) we obtain the estimates for the warping terms

$$\begin{aligned}\|\Pi_\varepsilon^{[\alpha]}(\overline{u}_\varepsilon^{[\alpha]})\|_{L^2(\Omega\times Cyls^{(\alpha)})} &= \frac{2}{\sqrt{\varepsilon}}\|\overline{u}_\varepsilon^{[\alpha]}\|_{L^2(\mathcal{S}_\varepsilon)} \leq C\frac{r}{\sqrt{\varepsilon}}\|e(u_\varepsilon)\|_{L^2(\mathcal{S}_\varepsilon)} \leq C\varepsilon^3,\\ \Big\|\frac{\partial}{\partial X_i}\Pi_\varepsilon^{[\alpha]}(\overline{u}_\varepsilon^{[\alpha]})\Big\|_{L^2(\Omega\times Cyls^{(\alpha)})} &= \varepsilon\Big\|\Pi_\varepsilon^{[\alpha]}\Big(\frac{\partial}{\partial z_i}\overline{u}_\varepsilon^{[\alpha]}\Big)\Big\|_{L^2(\Omega\times Cyls^{(\alpha)})} \leq C\frac{\varepsilon}{\sqrt{\varepsilon}}\Big\|\frac{\partial}{\partial z_i}\overline{u}_\varepsilon^{[\alpha]}\Big\|_{L^2(\mathcal{S}_\varepsilon)}\\ &\leq C\sqrt{\varepsilon}\|e(u_\varepsilon)\|_{L^2(\mathcal{S}_\varepsilon)} \leq C\varepsilon^3.\end{aligned} \tag{3.86}$$

The conditions (3.84) and (3.85) are the result of the conditions (3.12) on the warping. □

For simplification define the spaces

$$\overline{\mathbf{W}}^{(1)} \doteq \big\{v \in \widehat{\mathbf{W}}^{(1)} \ | \ v \text{ satisfies } (3.84)\big\}, \qquad \overline{\mathbf{W}}^{(2)} \doteq \big\{v \in \widehat{\mathbf{W}}^{(2)} \ | \ v \text{ satisfies } (3.85)\big\}.$$

To conclude this section note that the limit of the corresponding strain tensor is directly inherited of (3.24), i.e., the symmetric gradient of one beam, resulting in

$$\frac{1}{\varepsilon^2}\Pi_\varepsilon^{[\alpha]}\big(e_z(\overline{u}_\varepsilon^{[\alpha]})\big) \rightharpoonup \mathcal{E}_X^{(\alpha,c)}(\overline{u}^{(\alpha,c)}) \quad \text{weakly in} \quad L^2(\Omega\times Cyl^{(\alpha,c)})^{3\times 3}, \quad c\in\{0,1\}$$

with

$$\mathcal{E}_X^{(1,b)}(\varphi)=\begin{pmatrix} \frac{1}{\eta^{(1,b)}}\partial_{X_1}\varphi\cdot\mathbf{t}^{(1,b)} & * & * \\ \frac{1}{2}\big(\frac{1}{\eta^{(1,b)}}\partial_{X_1}\varphi\cdot\mathbf{e}_2+\partial_{X_2}\varphi\cdot\mathbf{t}^{(1,b)}\big) & \partial_{X_2}\varphi\cdot\mathbf{e}_2 & * \\ \frac{1}{2}\big(\frac{1}{\eta^{(1,b)}}\partial_{X_1}\varphi\cdot\mathbf{n}^{(1,b)}+\partial_{X_3}\varphi\cdot\mathbf{t}^{(1,b)}\big) & \frac{1}{2}\big(\partial_{X_2}\varphi\cdot\mathbf{n}^{(1,b)}+\partial_{X_3}\varphi\cdot\mathbf{e}_2\big) & \partial_{X_3}\varphi\cdot\mathbf{n}^{(1,b)}\end{pmatrix}, \tag{3.87}$$

$$\mathcal{E}_X^{(2,a)}(\varphi)=\begin{pmatrix} \partial_{X_1}\varphi\cdot\mathbf{e}_1 & * & * \\ \frac{1}{2}\big(\partial_{X_1}\varphi\cdot\mathbf{t}^{(2,a)}+\frac{1}{\eta^{(2,a)}}\partial_{X_2}\varphi\cdot\mathbf{e}_1\big) & \frac{1}{\eta^{(2,a)}}\partial_{X_2}\varphi\cdot\mathbf{t}^{(2,a)} & * \\ \frac{1}{2}\big(\partial_{X_1}\varphi\cdot\mathbf{n}^{(2,a)}+\partial_{X_3}\varphi\cdot\mathbf{e}_1\big) & \frac{1}{2}\big(\frac{1}{\eta^{(2,a)}}\partial_{X_2}\varphi\cdot\mathbf{n}^{(2,a)}+\partial_{X_3}\varphi\cdot\mathbf{t}^{(2,a)}\big) & \partial_{X_3}\varphi\cdot\mathbf{n}^{(2,a)}\end{pmatrix}, \tag{3.88}$$

for the first and second direction respectively.

3.6.4 The limit strain tensor for the elementary displacement

First note that the strain-tensor admits a weak limit in form of a weak convergent subsequence. Indeed assumption (3.66) gives rise to the estimate:

$$\Big\|\frac{1}{\varepsilon^2}\Pi_\varepsilon^{[\alpha]}\big(e_z(u_\varepsilon^{[\alpha]})\big)\Big\|_{L^2(\Omega\times Cyls^{(\alpha)})} \leq \frac{1}{\varepsilon^{5/2}}\,\|e(u_\varepsilon)\|_{L^2(\mathcal{S}_{\varepsilon r})} \leq C.$$

Hence, there exists a weak convergent subsequence, but mere existence is not enough to give the limit problem. Thus, to obtain the actual form of the limit strain tensor the convergences of all fields in the section above are needed. To simplify the representation of the limit strain tensor we split the limit into two main parts

$$\frac{1}{\varepsilon^2}\Pi_\varepsilon^{[1]}\big(e_z(u_\varepsilon)\big) \rightharpoonup \mathbf{E}^{(1,b)} + \mathcal{E}_X^{(1,b)}(\overline{u}^{(1,b)}) \text{ weakly in } L^2(\Omega; H^1(Cyl^{(1,b)}))^9,\ b \in \{0,1\},$$

where $\mathbf{E}^{(1,b)}$ denotes the limit of the strain tensor for the elementary displacements.

Recall the form of the strain-tensor for one beam (3.27). Then for every field use the decomposition developed in sections 2-4 and with the convergences above we find for the elementary displacement the limit strain tensor entries

$$\begin{aligned}
\eta\mathbf{E}^{(1,b)}_{z,11} = &\begin{pmatrix} \partial_1\mathbb{U}_1 + \partial_{X_1}\widehat{\mathbb{U}}_{1|X_2=b} \\ \partial_1\mathbb{U}_2 + \partial_{X_1}\widehat{\mathbb{U}}_{2|X_2=b} - \widehat{\mathcal{R}}_{3|X_2=b} \\ \mathcal{Z}_{13} + \partial_{X_1}\widehat{\mathbb{U}}_{3|X_2=b} + \widehat{\mathcal{R}}_{2|X_2=b} \end{pmatrix}\cdot\mathbf{t}^{(1,b)} \\
&+ \begin{pmatrix} \partial_{X_1}\widehat{\mathbb{U}}^{(1,b)}_1 \\ \partial_{X_1}\widehat{\mathbb{U}}^{(1,b)}_2 - \widehat{\mathcal{R}}^{(1,b)}_3 \\ \partial_{X_1}\widehat{\mathbb{U}}^{(1,b)}_3 + \widehat{\mathcal{R}}^{(1,b)}_2 \end{pmatrix}\cdot\mathbf{t}^{(1,b)} + \begin{pmatrix} \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{1|X_2=b} \\ \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{2|X_2=b} - \widehat{\mathcal{R}}^{(g)}_{3|X_2=b} \\ \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{3|X_2=b} + \widehat{\mathcal{R}}^{(g)}_{2|X_2=b} \end{pmatrix}\cdot\mathbf{t}^{(1,b)} \\
&+ \big(\partial_1\mathcal{R} + \partial_{X_1}\widehat{\mathcal{R}}_{|X_2=b} + \partial_{X_1}\widehat{\mathcal{R}}^{(g)}_{|X_2=b} + \partial_{X_1}\widetilde{\mathcal{R}}^{(1,b)}\big)\cdot\big(\big(\frac{\Phi^{(1,b)}}{\gamma} + X_3\big)\mathbf{e}_2 - (X_2-b)\mathbf{n}^{(1,b)}\big),
\end{aligned}$$

$$\begin{aligned}
2\eta\mathbf{E}^{(1,b)}_{z,12} = &\begin{pmatrix} \partial_1\mathbb{U}_1 + \partial_{X_1}\widehat{\mathbb{U}}_{1|X_2=b} \\ \partial_1\mathbb{U}_2 + \partial_{X_1}\widehat{\mathbb{U}}_{2|X_2=b} - \widehat{\mathcal{R}}_{3|X_2=b} \\ \mathcal{Z}_{13} + \partial_{X_1}\widehat{\mathbb{U}}_{3|X_2=b} + \widehat{\mathcal{R}}_{2|X_2=b} \end{pmatrix}\cdot\mathbf{e}_2 \\
&+ \begin{pmatrix} \partial_{X_1}\widehat{\mathbb{U}}^{(1,b)}_1 \\ \partial_{X_1}\widehat{\mathbb{U}}^{(1,b)}_2 - \widehat{\mathcal{R}}^{(1,b)}_3 \\ \partial_{X_1}\widehat{\mathbb{U}}^{(1,b)}_3 + \widehat{\mathcal{R}}^{(1,b)}_2 \end{pmatrix}\cdot\mathbf{e}_2 + \begin{pmatrix} \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{1|X_2=b} \\ \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{2|X_2=b} - \widehat{\mathcal{R}}^{(g)}_{3|X_2=b} \\ \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{3|X_2=b} + \widehat{\mathcal{R}}^{(g)}_{2|X_2=b} \end{pmatrix}\cdot\mathbf{e}_2 \\
&- \big(\partial_1\mathcal{R} + \partial_{X_1}\widehat{\mathcal{R}}_{|X_2=b} + \partial_{X_1}\widehat{\mathcal{R}}^{(g)}_{|X_2=b} + \partial_{X_1}\widetilde{\mathcal{R}}^{(1,b)}\big)\cdot\big(X_3\mathbf{t}^{(1,b)} + \Phi^{(1,b)}\mathbf{e}_1\big),
\end{aligned}$$

$$\begin{aligned}
2\eta\mathbf{E}^{(1,b)}_{z,13} = &\begin{pmatrix} \partial_1\mathbb{U}_1 + \partial_{X_1}\widehat{\mathbb{U}}_{1|X_2=b} \\ \partial_1\mathbb{U}_2 + \partial_{X_1}\widehat{\mathbb{U}}_{2|X_2=b} - \widehat{\mathcal{R}}_{3|X_2=b} \\ \mathcal{Z}_{13} + \partial_{X_1}\widehat{\mathbb{U}}_{3|X_2=b} + \widehat{\mathcal{R}}_{2|X_2=b} \end{pmatrix}\cdot\mathbf{n}^{(1,b)} \\
&+ \begin{pmatrix} \partial_{X_1}\widehat{\mathbb{U}}^{(1,b)}_1 \\ \partial_{X_1}\widehat{\mathbb{U}}^{(1,b)}_2 - \widehat{\mathcal{R}}^{(1,b)}_3 \\ \partial_{X_1}\widehat{\mathbb{U}}^{(1,b)}_3 + \widehat{\mathcal{R}}^{(1,b)}_2 \end{pmatrix}\cdot\mathbf{n}^{(1,b)} + \begin{pmatrix} \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{1|X_2=b} \\ \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{2|X_2=b} - \widehat{\mathcal{R}}^{(g)}_{3|X_2=b} \\ \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{3|X_2=b} + \widehat{\mathcal{R}}^{(g)}_{2|X_2=b} \end{pmatrix}\cdot\mathbf{n}^{(1,b)} \\
&+ \big(\partial_1\mathcal{R} + \partial_{X_1}\widehat{\mathcal{R}}_{|X_2=b} + \partial_{X_1}\widehat{\mathcal{R}}^{(g)}_{|X_2=b} + \partial_{X_1}\widetilde{\mathcal{R}}^{(1,b)}\big)\cdot\big((X_2-b)\mathbf{t}^{(1,b)} - \frac{\Phi^{(1,b)}d_{X_1}\Phi^{(1,b)}}{\gamma}\mathbf{e}_2\big).
\end{aligned}$$

To simplify this expression define the purely microscopic displacement

$$\begin{aligned}
\widehat{u}^{(1,b)} = &\big(\widehat{\mathbb{U}}_{|X_2=b} + \widehat{\mathbb{U}}^{(g)}_{|X_2=b} + \widehat{\mathbb{U}}^{(1,b)}\big) \\
&+ \big(\mathbf{Z} + \widehat{\mathcal{R}}_{|X_2=b} + \widehat{\mathcal{R}}^{(g)}_{|X_2=b} + \widehat{\mathcal{R}}^{(1,b)}\big) \wedge \big(\Phi^{(1,b)}\mathbf{e}_3 + X_3\mathbf{n}^{(1,b)} + (X_2-b)\mathbf{e}_2\big) + \overline{u}^{(1,b)}, \quad (3.89)
\end{aligned}$$

where

$$\mathbf{Z} = -\mathcal{Z}_{23}\mathbf{e}_1 + \mathcal{Z}_{13}\mathbf{e}_2 - \frac{1}{2}\big(\partial_1\mathbb{U}_2 - \partial_2\mathbb{U}_1\big)\mathbf{e}_3.$$

Then the strain tensor limit for the elementary displacement can be written as

$$\mathbf{E}_z^{(1,b)} + \mathcal{E}_X^{(1,b)}(\overline{u}^{(1,b)}) = \mathcal{E}^{(1,b)} + \mathcal{E}_X^{(1,b)}(\widehat{u}^{(1,b)})$$

or equivalently write directly

$$\frac{1}{\varepsilon^2}\Pi_\varepsilon^{[1]}\big(e_z(u_\varepsilon)\big) \rightharpoonup \mathcal{E}^{(1,b)} + \mathcal{E}_X^{(1,b)}(\widehat{u}^{(1,b)}), \qquad \text{weakly in } \; L^2(\Omega; H^1(Cyl^{(1,b)}))^9, \; b \in \{0,1\}, \tag{3.90}$$

where

$$\begin{aligned}
\mathcal{E}_{11}^{(1,b)}(\mathbb{U}) =& \frac{1}{\eta^{(1,b)}}\left[\begin{pmatrix} e_{11}(\mathbb{U}) \\ e_{12}(\mathbb{U}) \\ 0 \end{pmatrix} \cdot \mathbf{t}^{(1,b)} + \begin{pmatrix} \partial_{12}\mathbb{U}_3 \\ -\partial_{11}\mathbb{U}_3 \\ 0 \end{pmatrix} \cdot \left(\left(X_3 + \frac{\Phi^{(1,b)}}{\gamma}\right)\mathbf{e}_2 - (X_2 - b)\,\mathbf{n}^{(1,b)}\right)\right], \\
\mathcal{E}_{12}^{(1,b)}(\mathbb{U}) =& \frac{1}{2\eta^{(1,b)}}\left[\begin{pmatrix} e_{11}(\mathbb{U}) \\ e_{12}(\mathbb{U}) \\ 0 \end{pmatrix} \cdot \mathbf{e}_2 - \begin{pmatrix} \partial_{12}\mathbb{U}_3 \\ -\partial_{11}\mathbb{U}_3 \\ 0 \end{pmatrix} \cdot \left(X_3\mathbf{t}^{(1,b)} + \Phi^{(1,b)}\mathbf{e}_1\right)\right], \\
\mathcal{E}_{13}^{(1,b)}(\mathbb{U}) =& \frac{1}{2\eta^{(1,b)}}\left[\begin{pmatrix} e_{11}(\mathbb{U}) \\ e_{12}(\mathbb{U}) \\ 0 \end{pmatrix} \cdot \mathbf{n}^{(1,b)} + \begin{pmatrix} \partial_{12}\mathbb{U}_3 \\ -\partial_{11}\mathbb{U}_3 \\ 0 \end{pmatrix} \cdot \left((X_2 - b)\,\mathbf{t}^{(1,b)} - \frac{\Phi^{(1,b)}d_1\Phi^{(1,b)}}{\gamma}\mathbf{e}_2\right)\right].
\end{aligned} \tag{3.91}$$

and $\mathcal{E}_{22}^{(1,b)} = \mathcal{E}_{33}^{(1,b)} = \mathcal{E}_{23}^{(1,b)} = 0$ include all macroscopic fields. Note that for this representation the identities (3.70) were used.

3.6.4.1 The limit strain tensor for the $\mathbf{e}_2$-direction

For the sake of completeness the limit strain tensor for the second directed beams is discussed hereafter. Nevertheless, due to the very similar character only the end result for the elementary displacement is shown. Besides the zero-components $\mathbf{E}_{z,11}^{(2,a)} = \mathbf{E}_{z,13}^{(2,a)} = \mathbf{E}_{z,33}^{(2,a)} = 0$ we have:

$$\begin{aligned}
\eta\mathbf{E}_{z,22}^{(2,a)} =& \begin{pmatrix} \partial_2\mathbb{U}_1 + \partial_{X_2}\widehat{\mathbb{U}}_{1|X_1=a} + \widehat{\mathcal{R}}_{3|X_1=a} \\ \partial_2\mathbb{U}_2 + \partial_{X_2}\widehat{\mathbb{U}}_{2|X_1=a} \\ \mathcal{Z}_{23} + \partial_{X_2}\widehat{\mathbb{U}}_{3|X_1=a} - \widehat{\mathcal{R}}_{1|X_1=a} \end{pmatrix} \cdot \mathbf{t}^{(2,a)} \\
&+ \begin{pmatrix} \partial_{X_2}\widehat{\mathbb{U}}_1^{(2,a)} + \widehat{\mathcal{R}}_3^{(2,a)} \\ \partial_{X_2}\widehat{\mathbb{U}}_2^{(2,a)} \\ \partial_{X_2}\widehat{\mathbb{U}}_3^{(2,a)} - \widehat{\mathcal{R}}_1^{(2,a)} \end{pmatrix} \cdot \mathbf{t}^{(2,a)} - \begin{pmatrix} \partial_{X_2}\widehat{\mathbb{U}}_{1|X_1=a}^{(g)} + \widehat{\mathcal{R}}_{3|X_1=a}^{(g)} \\ \partial_{X_2}\widehat{\mathbb{U}}_{2|X_1=a}^{(g)} \\ \partial_{X_2}\widehat{\mathbb{U}}_{3|X_1=a}^{(g)} - \widehat{\mathcal{R}}_{1|X_1=a}^{(g)} \end{pmatrix} \cdot \mathbf{t}^{(2,a)} \\
&- \big(\partial_2\mathcal{R} + \partial_{X_2}\widehat{\mathcal{R}}_{|X_1=a} - \partial_{X_2}\widehat{\mathcal{R}}_{|X_1=a}^{(g)} + \partial_{X_2}\widetilde{\mathcal{R}}^{(2,a)}\big) \cdot \big(\big(\frac{\Phi^{(2,a)}}{\gamma} + X_3\big)\mathbf{e}_1 - (X_1 - a)\mathbf{n}^{(2,a)}\big),
\end{aligned}$$

$$
\begin{aligned}
2\eta\mathbf{E}^{(2,a)}_{z,12} = &\begin{pmatrix}\partial_2\mathbb{U}_1+\partial_{X_2}\widehat{\mathbb{U}}_{1|X_1=a}+\widehat{\mathcal{R}}_{3|X_1=a}\\ \partial_2\mathbb{U}_2+\partial_{X_2}\widehat{\mathbb{U}}_{2|X_1=a}\\ \mathcal{Z}_{23}+\partial_{X_2}\widehat{\mathbb{U}}_{3|X_1=a}-\widehat{\mathcal{R}}_{1|X_1=a}\end{pmatrix}\cdot\mathbf{e}_1\\
&+\begin{pmatrix}\partial_{X_2}\widehat{\mathbb{U}}^{(2,a)}_1+\widehat{\mathcal{R}}^{(2,a)}_3\\ \partial_{X_2}\widehat{\mathbb{U}}^{(2,a)}_2\\ \partial_{X_2}\widehat{\mathbb{U}}^{(2,a)}_3-\widehat{\mathcal{R}}^{(2,a)}_1\end{pmatrix}\cdot\mathbf{e}_1-\begin{pmatrix}\partial_{X_2}\widehat{\mathbb{U}}^{(g)}_{1|X_1=a}+\widehat{\mathcal{R}}^{(g)}_{3|X_1=a}\\ \partial_{X_2}\widehat{\mathbb{U}}^{(g)}_{2|X_1=a}\\ \partial_{X_2}\widehat{\mathbb{U}}^{(g)}_{3|X_1=a}-\widehat{\mathcal{R}}^{(g)}_{1|X_1=a}\end{pmatrix}\cdot\mathbf{e}_1\\
&+\big(\partial_2\mathcal{R}+\partial_{X_2}\widehat{\mathcal{R}}_{|X_1=a}-\partial_{X_2}\widehat{\mathcal{R}}^{(g)}_{|X_1=a}+\partial_{X_2}\widehat{\mathcal{R}}^{(2,a)}\big)\cdot\big(X_3\mathbf{t}^{(2,a)}+\Phi^{(2,a)}\mathbf{e}_2\big),
\end{aligned}
$$

$$
\begin{aligned}
2\eta\mathbf{E}^{(2,a)}_{z,23} = &\begin{pmatrix}\partial_2\mathbb{U}_1+\partial_{X_2}\widehat{\mathbb{U}}_{1|X_1=a}+\widehat{\mathcal{R}}_{3|X_1=a}\\ \partial_2\mathbb{U}_2+\partial_{X_2}\widehat{\mathbb{U}}_{2|X_1=a}\\ \mathcal{Z}_{23}+\partial_{X_2}\widehat{\mathbb{U}}_{3|X_1=a}-\widehat{\mathcal{R}}_{1|X_1=a}\end{pmatrix}\cdot\mathbf{n}^{(2,a)}\\
&+\begin{pmatrix}\partial_{X_2}\widehat{\mathbb{U}}^{(2,a)}_1+\widehat{\mathcal{R}}^{(2,a)}_3\\ \partial_{X_2}\widehat{\mathbb{U}}^{(2,a)}_2\\ \partial_{X_2}\widehat{\mathbb{U}}^{(2,a)}_3-\widehat{\mathcal{R}}^{(2,a)}_1\end{pmatrix}\cdot\mathbf{n}^{(2,a)}-\begin{pmatrix}\partial_{X_2}\widehat{\mathbb{U}}^{(g)}_{1|X_1=a}+\widehat{\mathcal{R}}^{(g)}_{3|X_1=a}\\ \partial_{X_2}\widehat{\mathbb{U}}^{(g)}_{2|X_1=a}\\ \partial_{X_2}\widehat{\mathbb{U}}^{(g)}_{3|X_1=a}-\widehat{\mathcal{R}}^{(g)}_{1|X_1=a}\end{pmatrix}\cdot\mathbf{n}^{(2,a)}\\
&-\big(\partial_1\mathcal{R}+\partial_{X_2}\widehat{\mathcal{R}}_{|X_1=a}-\partial_{X_2}\widehat{\mathcal{R}}^{(g)}_{|X_1=a}+\partial_{X_2}\widehat{\mathcal{R}}^{(2,a)}\big)\cdot\Big((X_1-a)\mathbf{t}^{(2,a)}-\frac{\Phi^{(2,a)}d_{X_2}\Phi^{(2,a)}}{\gamma}\mathbf{e}_1\Big).
\end{aligned}
$$

Define analogously to (3.89), the microscopic displacement

$$
\begin{aligned}
\widehat{u}^{(2,a)}=&\big(\widehat{\mathbb{U}}_{|X_1=a}-\widehat{\mathbb{U}}^{(g)}_{|X_1=a}+\widehat{\mathbb{U}}^{(2,a)}\big)\\
&+\big(\mathbf{Z}^{(2,a)}+\widehat{\mathcal{R}}_{|X_1=a}-\widehat{\mathcal{R}}^{(g)}_{|X_1=a}+\widehat{\mathcal{R}}^{(2,a)}\big)\wedge\big(\Phi^{(2,a)}\mathbf{e}_3+X_3\mathbf{n}^{(2,a)}+(X_1-a)\mathbf{e}_1\big)+\overline{u}^{(2,a)}.
\end{aligned}
\tag{3.92}
$$

For the same reason the limit strain tensor splits into two parts

$$
\frac{1}{\varepsilon^2}\Pi^{[2]}_\varepsilon\big(e_z(u_\varepsilon)\big)\rightharpoonup\mathcal{E}^{(2,a)}+\mathcal{E}^{(2,a)}_X(\widehat{u}^{(2,a)})\quad\text{weakly in }\ L^2(\Omega\times Cyl^{(2,a)})^9
$$

with

$$
\begin{aligned}
\mathcal{E}^{(2,a)}_{22}(\mathbb{U})=&\frac{1}{\eta^{(2,a)}}\Bigg[\begin{pmatrix}e_{12}(\mathbb{U})\\ e_{22}(\mathbb{U})\\ 0\end{pmatrix}\cdot\mathbf{t}^{(2,a)}-\begin{pmatrix}\partial_{22}\mathbb{U}_3\\ -\partial_{12}\mathbb{U}_3\\ 0\end{pmatrix}\cdot\bigg(\Big(X_3+\frac{\Phi^{(2,a)}}{\gamma}\Big)\mathbf{e}_1-(X_1-a)\,\mathbf{n}^{(2,a)}\bigg)\Bigg],\\
\mathcal{E}^{(2,a)}_{12}(\mathbb{U})=&\frac{1}{2\eta^{(2,a)}}\Bigg[\begin{pmatrix}e_{12}(\mathbb{U})\\ e_{22}(\mathbb{U})\\ 0\end{pmatrix}\cdot\mathbf{e}_1+\begin{pmatrix}\partial_{22}\mathbb{U}_3\\ -\partial_{12}\mathbb{U}_3\\ 0\end{pmatrix}\cdot\Big(X_3\mathbf{t}^{(2,a)}+\Phi^{(2,a)}\mathbf{e}_2\Big)\Bigg],\\
\mathcal{E}^{(2,a)}_{23}(\mathbb{U})=&\frac{1}{2\eta^{(2,a)}}\Bigg[\begin{pmatrix}e_{12}(\mathbb{U})\\ e_{22}(\mathbb{U})\\ 0\end{pmatrix}\cdot\mathbf{n}^{(2,a)}-\begin{pmatrix}\partial_{22}\mathbb{U}_3\\ -\partial_{12}\mathbb{U}_3\\ 0\end{pmatrix}\cdot\Big((X_1-a)\,\mathbf{t}^{(2,a)}-\frac{\Phi^{(2,a)}d\Phi^{(2,a)}}{\gamma}\mathbf{e}_1\Big)\Bigg].
\end{aligned}
\tag{3.93}
$$

and local displacements $\mathcal{E}^{(2,a)}_X(\widehat{u}^{(2,a)})$ in the form (3.88).

3.6.5 The contact limit conditions

Recall the decomposition in the contact parts, see (3.49), and note that it reduces to

$$\begin{aligned}
u_\varepsilon^{(1,q)}(x) &= \mathbb{U}_\varepsilon^{(1,q)}(p\varepsilon + z_1) + \mathcal{R}_\varepsilon^{(1,q)}(p\varepsilon + z_1) \wedge z_2\mathbf{e}_2 + \overline{u}_\varepsilon^{(1,q)}(x),\\
u_\varepsilon^{(2,p)}(x) &= \mathbb{U}_\varepsilon^{(2,p)}(q\varepsilon + z_2) + \mathcal{R}_\varepsilon^{(2,p)}(q\varepsilon + z_2) \wedge z_1\mathbf{e}_1 + \overline{u}_\varepsilon^{(2,p)}(x).
\end{aligned}$$

for a.e. $x \in \mathbf{C}_{pq}$ or equivalently $z = (z_1, z_2) \in \omega_r$ and $|z_3| = r$. Using additionally the splitting (3.60), we obtain for the displacements in the contact parts

$$\begin{aligned}
u_\varepsilon^{(1,q)}(x) &= \mathbb{U}_\varepsilon(p\varepsilon + z_1, q\varepsilon) + \mathbb{U}_\varepsilon^{(g)}(p\varepsilon + z_1, q\varepsilon) + \widetilde{\mathbb{U}}_\varepsilon^{(1,q)}(p\varepsilon + z_1)\\
&\quad + \Big[\mathcal{R}_\varepsilon(p\varepsilon + z_1, q\varepsilon) + \mathcal{R}_\varepsilon^{(g)}(p\varepsilon + z_1, q\varepsilon) + \widetilde{\mathcal{R}}_\varepsilon^{(1,q)}(p\varepsilon + z_1)\Big] \wedge z_2\mathbf{e}_2 + \overline{u}_\varepsilon^{(1,q)}(x),\\
u_\varepsilon^{(2,p)}(x) &= \mathbb{U}_\varepsilon(p\varepsilon, q\varepsilon + z_2) - \mathbb{U}_\varepsilon^{(g)}(p\varepsilon, q\varepsilon + z_2) + \widetilde{\mathbb{U}}_\varepsilon^{(2,p)}(q\varepsilon + z_2)\\
&\quad + \Big[\mathcal{R}_\varepsilon(p\varepsilon, q\varepsilon + z_2) - \mathcal{R}_\varepsilon^{(g)}(p\varepsilon, q\varepsilon + z_2) + \widetilde{\mathcal{R}}_\varepsilon^{(2,p)}(q\varepsilon + z_2)\Big] \wedge z_1\mathbf{e}_1 + \overline{u}_\varepsilon^{(2,p)}(x).
\end{aligned}$$

From (3.36), one obtains (same equalities with $\mathcal{R}$)

$$\left.\begin{aligned}
\mathbb{U}_\varepsilon(p\varepsilon + z_1, q\varepsilon + z_2) &= \mathbb{U}_\varepsilon(p\varepsilon + z_1, q\varepsilon) + z_2\frac{\partial \mathbb{U}_\varepsilon}{\partial z_2}(p\varepsilon + z_1, q\varepsilon + z_2),\\
\mathbb{U}_\varepsilon(p\varepsilon + z_1, q\varepsilon + z_2) &= \mathbb{U}_\varepsilon(p\varepsilon, q\varepsilon + z_2) + z_1\frac{\partial \mathbb{U}_\varepsilon}{\partial z_1}(p\varepsilon + z_1, q\varepsilon + z_2),
\end{aligned}\right. \qquad \forall (z_1, z_2) \in \omega_r.$$

These identities yield for a.e. $x \in \mathbf{C}_{pq}$ that the difference between two beams in contact can be written as

$$\begin{aligned}
&u_\varepsilon^{(1,q)}(x) - u_\varepsilon^{(2,p)}(x)\\
&= -z_2\Big(\frac{\partial \mathbb{U}_\varepsilon}{\partial z_2} - \mathcal{R}_\varepsilon \wedge \mathbf{e}_2\Big)(p\varepsilon + z_1, q\varepsilon + z_2) + z_1\Big(\frac{\partial \mathbb{U}_\varepsilon}{\partial z_1} - \mathcal{R}_\varepsilon \wedge \mathbf{e}_1\Big)(p\varepsilon + z_1, q\varepsilon + z_2)\\
&+ \mathbb{U}^{(g)}(p\varepsilon + z_1, q\varepsilon) + \mathbb{U}^{(g)}(p\varepsilon, q\varepsilon + z_2) + \widetilde{\mathbb{U}}^{(1,q)}(p\varepsilon + z_1) - \widetilde{\mathbb{U}}^{(2,p)}(q\varepsilon + z_2)\\
&+ \Big[-z_2\frac{\partial \mathcal{R}_\varepsilon}{\partial z_2}(p\varepsilon + z_1, q\varepsilon + z_2) + \mathcal{R}_\varepsilon^{(g)}(p\varepsilon + z_1, q\varepsilon) + \widetilde{\mathcal{R}}_\varepsilon^{(1,q)}(p\varepsilon + z_1)\Big] \wedge z_2\mathbf{e}_2\\
&+ \Big[z_1\frac{\partial \mathcal{R}_\varepsilon}{\partial z_1}(p\varepsilon + z_1, q\varepsilon + z_2) + \mathcal{R}_\varepsilon^{(g)}(p\varepsilon, q\varepsilon + z_2) - \widetilde{\mathcal{R}}_\varepsilon^{(2,q)}(q\varepsilon + z_2)\Big] \wedge z_1\mathbf{e}_1\\
&+ \overline{u}_\varepsilon^{(1,q)}(x) - \overline{u}_\varepsilon^{(2,p)}(x).
\end{aligned} \tag{3.94}$$

This expansion allows to estimate the jump and obtain the correct convergences via the following Lemma.

Lemma 3.6.10. *The difference of $u_\varepsilon^{(1,q)}$ and $u_\varepsilon^{(2,p)}$ satisfies*

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \|u_\varepsilon^{(1,q)} - u_\varepsilon^{(2,p)}\|^2_{L^2(\mathbf{C}_{pq})} \le C\varepsilon^6.$$

Proof. The estimate of the Lemma is an immediate consequence of (3.94) and the Lemmas 3.4.2, 3.4.12 as well as the estimates (3.86). □

To obtain the limit it is necessary to introduce a third unfolding operator. Therefore, let

$$\mathbf{C} \doteq \bigcup_{a,b=0}^{1} \mathbf{C}_{ab}, \qquad \mathbf{C}_{ab} = \omega_\kappa + a\,\mathbf{e}_1 + b\,\mathbf{e}_2, \qquad (a,b) \in \{0,1\}^2,$$

denote the limit contact area. Then, the unfolding operator for the contact areas is defined for every $\varphi \in L^p\left(\bigcup_{pq} \mathbf{C}_{pq}\right)$ by

$$T_\varepsilon^C(\varphi)(z_1,z_2,X_1,X_2) = \varphi\left(2\varepsilon\left[\frac{z'}{2\varepsilon}\right] + \varepsilon\begin{pmatrix} a \\ b\end{pmatrix} + \varepsilon\left(X' - \begin{pmatrix} a \\ b\end{pmatrix}\right)\right),$$

with $T^C(\varphi) \in L^p(\Omega \times \mathbf{C})$. Note that this operator is related to the previous defined unfolding operators via the identities

$$T_\varepsilon^C(\varphi)(\cdot,X_1,X_2) = \mathcal{T}_\varepsilon(\varphi)_{|\Omega\times\mathbf{C}}(\cdot,X_1,X_2), \qquad \text{for } \varphi \in L^p(\Omega), \tag{3.95}$$

$$T_\varepsilon^C(\varphi)(\cdot,X_1,X_2) = \Pi_\varepsilon^{(\alpha)}(\varphi)_{|\Omega\times\mathbf{C}} = \Pi_\varepsilon^{(\alpha)}(\varphi)(\cdot,X_1,X_2,(-1)^{\alpha+a+b}\kappa), \quad \text{for } \varphi \in L^p(\mathbf{P}^{[1]}), \tag{3.96}$$

The following Lemma demonstrates the main property of T_ε^C

Lemma 3.6.11. *The unfolding operator T_ε^C satisfies*

$$\|T_\varepsilon^C(\varphi)\|_{L^p(\Omega\times\mathbf{C})} \le C\|\varphi\|_{L^p(\bigcup_{pq}\mathbf{C}_{pq})}, \qquad \textit{for every } \varphi \in L^p\Big(\bigcup_{pq}\mathbf{C}_{pq}\Big).$$

Proof. Follows directly from $(3.95)_1$ and Lemma 3.5.3. □

Due to Lemmas 3.5.5, 3.6.6, 3.6.7, 3.6.10 and 3.6.11 the following limit is obtained ($\cdot$ represents the macroscopic variable $z = (z_1, z_2)$):

$$\begin{aligned}
&\frac{1}{\varepsilon^3}\big[T_\varepsilon^C(u^{[1]}) - T_\varepsilon^C(u^{[2]})\big] \rightharpoonup \\
&\quad -(X_2-b)\begin{pmatrix} \partial_2\mathbb{U}_1 + \partial_{X_2}\widehat{\mathbb{U}}_1 + \widehat{\mathcal{R}}_3 \\ \partial_2\mathbb{U}_2 + \partial_{X_2}\widehat{\mathbb{U}}_2 \\ \mathcal{Z}_{23} + \partial_{X_2}\widehat{\mathbb{U}}_3 - \widehat{\mathcal{R}}_1 \end{pmatrix} + (X_1-a)\begin{pmatrix} \partial_1\mathbb{U}_1 + \partial_{X_1}\widehat{\mathbb{U}}_1 \\ \partial_1\mathbb{U}_2 + \partial_{X_1}\widehat{\mathbb{U}}_2 - \widehat{\mathcal{R}}_3 \\ \mathcal{Z}_{13} + \partial_{X_1}\widehat{\mathbb{U}}_3 + \widehat{\mathcal{R}}_2 \end{pmatrix} \\
&\quad + \widehat{\mathbb{U}}^{(g)}(\cdot,X_1,b) + \widehat{\mathbb{U}}^{(g)}(\cdot,a,X_2) + \widehat{\mathbb{U}}^{(1,b)}(\cdot,X_1) - \widehat{\mathbb{U}}^{(2,a)}(\cdot,X_2) \\
&\quad - (X_2-b)^2\big(\partial_2\mathcal{R} + \partial_{X_2}\widehat{\mathcal{R}}\big)(\cdot,X_1,X_2)\wedge\mathbf{e}_2 + \big[\widehat{\mathcal{R}}^{(g)}(\cdot,X_1,b) + \widehat{\mathcal{R}}^{(1,b)}(\cdot,X_1)\big]\wedge(X_2-b)\mathbf{e}_2 \\
&\quad + (X_1-a)^2\big(\partial_1\mathcal{R} + \partial_{X_1}\widehat{\mathcal{R}}\big)(\cdot,X_1,X_2)\wedge\mathbf{e}_1 + \big[\widehat{\mathcal{R}}^{(g)}(\cdot,a,X_2) - \widehat{\mathcal{R}}^{(2,a)}(\cdot,X_2)\big]\wedge(X_1-a)\mathbf{e}_1 \\
&\quad + \overline{u}^{(1,b)}(\cdot,X_1,X_2,(-1)^{a+b+1}\kappa) - \overline{u}^{(2,a)}(\cdot,X_1,X_2,(-1)^{a+b}\kappa) \quad \text{weakly in } L^2(\Omega\times\mathbf{C}_{ab})^3.
\end{aligned}$$

Moreover, since $\widehat{\mathbb{U}}_\alpha(\cdot,X_1,X_2)$ belongs to $\mathcal{Q}^1_{per}(\mathcal{Y})$, one has in $\omega \times \big([a,a+1]\times[b,b+1]\big)$

$$\begin{aligned}
&(X_1-a)\partial_{X_1}\widehat{\mathbb{U}}_\alpha(\cdot,X_1,X_2) = \widehat{\mathbb{U}}_\alpha(\cdot,X_1,X_2) - \widehat{\mathbb{U}}_\alpha(\cdot,a,X_2), \\
&(X_2-b)\partial_{X_2}\widehat{\mathbb{U}}_\alpha(\cdot,X_1,X_2) = \widehat{\mathbb{U}}_\alpha(\cdot,X_1,X_2) - \widehat{\mathbb{U}}_\alpha(\cdot,X_1,b), \\
\Longrightarrow\ &(X_1-a)\partial_{X_1}\widehat{\mathbb{U}}_\alpha(\cdot,X_1,X_2) - (X_2-b)\partial_{X_2}\widehat{\mathbb{U}}_\alpha(\cdot,X_1,X_2) = \widehat{\mathbb{U}}_\alpha(\cdot,X_1,b) - \widehat{\mathbb{U}}_\alpha(\cdot,a,X_2),
\end{aligned}$$

and similar identities for $\widehat{\mathcal{R}}_\alpha$ and $\widetilde{\mathbb{U}}_3$. Using (3.74), one obtains in $\omega \times \big([a,a+1]\times[b,b+1]\big)$

$$\begin{aligned}&(X_1-a)\partial_{X_1}\widehat{\mathbb{U}}_3(\cdot,X_1,X_2)-(X_2-b)\partial_{X_2}\widehat{\mathbb{U}}_3(\cdot,X_1,X_2)\\&\qquad=\widehat{\mathbb{U}}_3(\cdot,X_1,b)-\widehat{\mathbb{U}}_3(\cdot,a,X_2)-\frac{1}{2}(X_1-a)^2\partial_{11}\mathbb{U}_3+\frac{1}{2}(X_2-b)^2\partial_{22}\mathbb{U}_3.\end{aligned}\tag{3.97}$$

The above limit is equal to (taking into account the fact that $\mathcal{R}=\partial_2\mathbb{U}_3\mathbf{e}_1-\partial_1\mathbb{U}_3\mathbf{e}_2$, equalities (3.89)-(3.92) and the above ones)

$$\begin{aligned}&-(X_2-b)\begin{pmatrix}\partial_2\mathbb{U}_1\\\partial_2\mathbb{U}_2\\\mathcal{Z}_{23}\end{pmatrix}+(X_1-a)\begin{pmatrix}\partial_1\mathbb{U}_1\\\partial_1\mathbb{U}_2\\\mathcal{Z}_{13}\end{pmatrix}-\frac{1}{2}(X_2-b)^2\partial_{22}\mathbb{U}_3\mathbf{e}_3+\frac{1}{2}(X_1-a)^2\partial_{11}\mathbb{U}_3\mathbf{e}_3\\&+\widehat{\mathbb{U}}^{(1,b)}(\cdot,X_1)-\widehat{\mathbb{U}}^{(2,a)}(\cdot,X_2)+\widehat{\mathbb{U}}(\cdot,X_1,b)-\widehat{\mathbb{U}}(\cdot,a,X_2)+\widehat{\mathbb{U}}^{(g)}(\cdot,X_1,b)+\widehat{\mathbb{U}}^{(g)}(\cdot,a,X_2)\\&+\big[\widehat{\mathcal{R}}(\cdot,X_1,b)+\widehat{\mathcal{R}}^{(g)}(\cdot,X_1,b)+\widehat{\mathcal{R}}^{(1,b)}(\cdot,X_1)\big]\wedge(X_2-b)\mathbf{e}_2+\overline{u}^{(1,b)}(\cdot,X_1,X_2,(-1)^{a+b+1}\kappa)\\&-\big[\widehat{\mathcal{R}}(\cdot,a,X_2)-\widehat{\mathcal{R}}^{(g)}(\cdot,a,X_2)+\widehat{\mathcal{R}}^{(2,a)}(\cdot,X_2)\big]\wedge(X_1-a)\mathbf{e}_1-\overline{u}^{(2,a)}(\cdot,X_1,X_2,(-1)^{a+b}\kappa)\\&=\mathbf{M}_{ab}(\mathbb{U})(X_1,X_2)+\widehat{u}^{(1,b)}(\cdot,X_1,X_2,(-1)^{a+b+1}\kappa)-\widehat{u}^{(2,a)}(\cdot,X_1,X_2,(-1)^{a+b}\kappa).\end{aligned}$$

with the macroscopic part

$$\mathbf{M}_{ab}(\mathbb{U})(X_1,X_2)=\begin{pmatrix}(X_1-a)e_{11}(\mathbb{U})-(X_2-b)e_{12}(\mathbb{U})\\(X_1-a)e_{12}(\mathbb{U})-(X_2-b)e_{22}(\mathbb{U})\\\frac{1}{2}(X_1-a)^2\partial_{11}\mathbb{U}_3-\frac{1}{2}(X_2-b)^2\partial_{22}\mathbb{U}_3\end{pmatrix}.\tag{3.98}$$

Finally, remember that $g_\varepsilon=\varepsilon^3 g$ with $g\in\mathcal{C}(\overline{\Omega})^3$ (coming from the assumption (3.65)). Then, the unfolded limit contact condition for $(\widehat{u}^{(1)},\widehat{u}^{(2)})\in L^2(\Omega;\widehat{\mathbf{W}}^{(1)})\times L^2(\Omega;\widehat{\mathbf{W}}^{(2)})$ is defined by

$$|\mathbf{M}_{\alpha,ab}(\mathbb{U})+\widehat{u}_\alpha^{(1,b)}-\widehat{u}_\alpha^{(2,a)}|\le g_\alpha\quad\text{a.e. in }\ \Omega\times\mathbf{C}_{ab},\ (a,b)\in\{0,1\}^2,\tag{3.99}$$

$$0\le(-1)^{a+b}\big(\mathbf{M}_{3,ab}(\mathbb{U})+\widehat{u}_3^{(1,b)}-\widehat{u}_3^{(2,a)}\big)\le g_3\quad\text{a.e. in }\ \Omega\times\mathbf{C}_{ab},\ (a,b)\in\{0,1\}^2,\tag{3.100}$$

for the in-plane and outer-plane components respectively.

3.6.6 The limit space

Consequently, after investigating the limit displacements it is possible to define the limit space for the unfolded problem. Thus, first set

$$\begin{aligned}\mathcal{H}^1(\Omega)&\doteq\big\{V\in H^1(\Omega)\mid V=0\text{ on }z_2=0\big\},\\\mathcal{H}^2(\Omega)&\doteq\big\{V\in H^2(\Omega)\mid V=\partial_2V=0\text{ on }z_2=0\big\}.\end{aligned}$$

Then, the limit fields $\big(\mathbb{U}_1,\mathbb{U}_2,\mathbb{U}_3,\widehat{u}^{(1)},\widehat{u}^{(2)}\big)$ belong to the convex set

$$\mathbf{X}\doteq\mathcal{H}^1(\Omega)^2\times\mathcal{H}^2(\Omega)\times L^2(\Omega;\widehat{\mathbf{W}}^{(1)})\times L^2(\Omega;\widehat{\mathbf{W}}^{(2)}).$$

In fact, we restrict the space further

$$\begin{aligned}\mathcal{X} \doteq \Big\{ \big(\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \widehat{v}^{(1)}, \widehat{v}^{(2)}\big) \in \mathbf{X} \,\Big|\, \\ |\mathbf{M}_{ab,\alpha}(\mathbb{V}) + \widehat{v}^{(1,b)}_\alpha - \widehat{v}^{(2,a)}_\alpha| \leq g_\alpha \ \text{a.e. in } \ \Omega \times \mathbf{C}_{ab},\ (a,b) \in \{0,1\}^2, \\ 0 \leq (-1)^{a+b}\big(M_{ab,3}(\mathbb{V}) + \widehat{v}^{(1,b)}_3 - \widehat{v}^{(2,a)}_3\big) \leq g_3 \ \text{a.e. in } \ \Omega \times \mathbf{C}_{ab},\ (a,b) \in \{0,1\}^2 \Big\}, \end{aligned} \tag{3.101}$$

in order to satisfy the contact condition. The space $\mathcal{X}$ is a closed subset of the space

$$\mathcal{X} \subset \mathcal{H}^1(\Omega)^2 \times \mathcal{H}^2(\Omega) \times L^2(\Omega; H^1(Cyls^{(1)})^3) \times L^2(\Omega; H^1(Cyls^{(2)})^3)$$

endowed with the product norm.

3.7 The test-functions

In this section the used variables have to be split according to the splitting of the unfolding operator, i.e., the global and the local variable. Hence, note that for $z \in \mathbb{R}^2$ there exists a unique decomposition

$$z = [z] + \{z\}, \qquad [z] \in \mathbb{Z}^2, \quad \{z\} \in [0,1)^2. \tag{3.102}$$

The composition of the test-functions has to take the contact into account, i.e., the test-functions have to satisfy the contact condition in (3.32) and yield in the limit (3.99)-(3.100). To ensure this behavior, it is necessary to choose the test-functions in a special way. First, for illustration of the split cell-domain, see Figure 3.2.

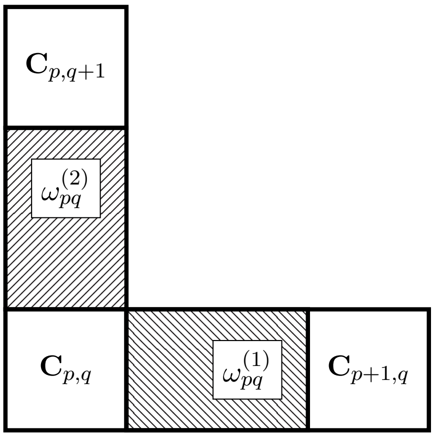


Figure 3.2: The 2D-cells for test-functions, with the different areas.

Let $\big(\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \widehat{v}^{(1)}, \widehat{v}^{(2)}\big)$ be in the space $\mathcal{X} \cap \mathcal{C}^2(\overline{\Omega})^2 \times \mathcal{C}^3(\overline{\Omega}) \times \mathcal{C}^1(\overline{\Omega}; \widehat{\mathbf{W}}^{(1)}) \times \mathcal{C}^1(\overline{\Omega}; \widehat{\mathbf{W}}^{(2)})$ such that $\widehat{v}^{(2)}(\cdot, 0) = 0$ vanishes at the boundary $z_2 = 0$. Now, replace $\widehat{v}$ by $\widehat{v}'$ defined by

$$\begin{aligned}\widehat{v}'^{(1,b)} &= \widehat{v}^{(1,b)} + \frac{1}{2}\big(\partial_1\mathbb{V}_2 - \partial_2\mathbb{V}_1\big)\mathbf{e}_3 \wedge \big(\Phi^{(1,b)}\mathbf{e}_3 + X_3\mathbf{n}^{(1,b)} + (X_2 - b)\mathbf{e}_2\big), \\ \widehat{v}'^{(2,a)} &= \widehat{v}^{(2,a)} + \frac{1}{2}\big(\partial_1\mathbb{V}_2 - \partial_2\mathbb{V}_1\big)\mathbf{e}_3 \wedge \big(\Phi^{(2,a)}\mathbf{e}_3 + X_3\mathbf{n}^{(2,a)} + (X_1 - a)\mathbf{e}_1\big).\end{aligned}$$

We easily check that $\big(\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \widehat{v}'^{(1)}, \widehat{v}'^{(2)}\big)$ satisfies the following contact conditions

$$
\begin{aligned}
&|M'_{ab,\alpha}(\mathbb{V}) + \widehat{v}_\alpha'^{(1,b)} - \widehat{v}_\alpha'^{(2,a)}| \leq g_\alpha \;\text{ a.e. in }\; \Omega \times \mathbf{C}_{ab},\; (a,b) \in \{0,1\}^2,\\
&0 \leq (-1)^{a+b}\big(M'_{ab,3}(\mathbb{V}) + \widehat{v}_3'^{(1,b)} - \widehat{v}_3'^{(2,a)}\big) \leq g_3 \;\text{ a.e. in }\; \Omega \times \mathbf{C}_{ab},\; (a,b) \in \{0,1\}^2,\\
&\mathbf{M}'_{ab}(\mathbb{V})(X_1, X_2) = \begin{pmatrix} (X_1 - a)e_{11}(\mathbb{V}) - (X_2 - b)\partial_2 \mathbb{V}_1 \\ (X_1 - a)\partial_1 \mathbb{V}_2 - (X_2 - b)e_{22}(\mathbb{V}) \\ \frac{1}{2}(X_1 - a)^2 \partial_{11}\mathbb{V}_3 - \frac{1}{2}(X_2 - b)^2 \partial_{22}\mathbb{V}_3 \end{pmatrix}.
\end{aligned}
$$

First, we define the functions $\mathbb{V}^{(1)}_{\varepsilon,\alpha}(\cdot, q\varepsilon) \in W^{1,\infty}(0,L)$ and $\mathbb{V}^{(1)}_{\varepsilon,3}(\cdot, q\varepsilon) \in W^{2,\infty}(0,L)$ for every $q \in \{0, \dots, 2N_\varepsilon\}$ as well as $\mathbb{V}^{(2)}_{\varepsilon,\alpha}(p\varepsilon, \cdot) \in W^{1,\infty}(0,L)$ and $\mathbb{V}^{(2)}_{\varepsilon,3}(p\varepsilon, \cdot) \in W^{2,\infty}(0,L)$ for every $p \in \{1, \dots, 2N_\varepsilon\}$.

Denote $z' = (z_1, z_2)$ and $p = \Big[\dfrac{z_1}{\varepsilon}\Big]$, $q = \Big[\dfrac{z_2}{\varepsilon} + \dfrac{1}{2}\Big]$. Then define

$$
\begin{aligned}
\mathbb{V}^{(1)}_{\varepsilon,\alpha}(z') &= \begin{cases} \mathbb{V}_\alpha(p\varepsilon, q\varepsilon) + (z_1 - p\varepsilon)\partial_1 \mathbb{V}_\alpha(p\varepsilon, q\varepsilon) \quad \text{in } C_{pq}, \\ \mathbb{V}^{(1)}_{\varepsilon,\alpha} \text{ linear interpolated in the stripe } \omega^{(1)}_{pq}, \end{cases}\\
\big(\partial_2 \mathbb{V}_3\big)^{(1)}_\varepsilon(z') &= \begin{cases} \partial_2 \mathbb{V}_3(p\varepsilon, q\varepsilon) + (z_1 - p\varepsilon)\partial_{12} \mathbb{V}_3(p\varepsilon, q\varepsilon) \quad \text{in } C_{pq}, \\ \big(\partial_2 \mathbb{V}_3\big)^{(1)}_\varepsilon \text{ linear interpolated in the stripe } \omega^{(1)}_{pq}, \end{cases}
\end{aligned}
$$

and

$$
\mathbb{V}^{(1)}_{\varepsilon,3}(z') = \begin{cases} \mathbb{V}_3(p\varepsilon, q\varepsilon) + (z_1 - p\varepsilon)\partial_1 \mathbb{V}_3(p\varepsilon, q\varepsilon) + \dfrac{1}{2}(z_1 - p\varepsilon)^2 \partial_{11} \mathbb{V}_3(p\varepsilon, q\varepsilon) \quad \text{in } C_{pq}, \\ \mathbb{V}^{(1)}_{\varepsilon,3} \text{ cubic interpolated in the stripe } \omega^{(1)}_{pq}. \end{cases}
$$

On the strips in direction $\mathbf{e}_2$ define $\mathbb{V}^{(2)}_{\varepsilon,\beta}$ accordingly by

$$
\begin{aligned}
\mathbb{V}^{(2)}_{\varepsilon,\alpha}(z') &= \begin{cases} \mathbb{V}_\alpha(p\varepsilon, q\varepsilon) + (z_2 - q\varepsilon)\partial_2 \mathbb{V}_\alpha(p\varepsilon, q\varepsilon) \quad \text{in } C_{pq}, \\ \text{linear interpolated in the stripe } \omega^{(2)}_{pq}, \end{cases}\\
\big(\partial_1 \mathbb{V}_3\big)^{(2)}_\varepsilon(z') &= \begin{cases} \partial_1 \mathbb{V}_3(p\varepsilon, q\varepsilon) + (z_2 - q\varepsilon)\partial_{12} \mathbb{V}_3(p\varepsilon, q\varepsilon) \quad \text{in } C_{pq}, \\ \big(\partial_1 \mathbb{V}_3\big)^{(2)}_\varepsilon \text{ linear interpolated in the stripe } \omega^{(2)}_{pq}, \end{cases}
\end{aligned}
$$

and

$$
\mathbb{V}^{(2)}_{\varepsilon,3}(z') = \begin{cases} \mathbb{V}_3(p\varepsilon, q\varepsilon) + (z_2 - q\varepsilon)\partial_2 \mathbb{V}_3(p\varepsilon, q\varepsilon) + \dfrac{1}{2}(z_2 - q\varepsilon)^2 \partial_{22} \mathbb{V}_3(p\varepsilon, q\varepsilon) \quad \text{in } C_{pq}, \\ \mathbb{V}^{(2)}_{\varepsilon,3} \text{ cubic interpolated in the stripe } \omega^{(2)}_{pq}. \end{cases}
$$

At last, the remaining displacements $\widehat{v}^{(1)}$ and $\widehat{v}^{(2)}$ are subjected to an analogous transformation. Hence, define

$$\widehat{v}'^{(1,b)}_{\varepsilon}(z',X) = \begin{cases} \widehat{v}'^{(1,b)}(p\varepsilon, q\varepsilon, X) & \text{a.e. in } C_{pq}\times(-\kappa,\kappa), \\ \text{linear interpolated with respect to } z_1 \text{ in the stripe } \omega^{(1)}_{pq}, \end{cases}$$
$$\widehat{v}'^{(2,a)}_{\varepsilon}(z',X) = \begin{cases} \widehat{v}'^{(2,a)}(p\varepsilon, q\varepsilon, X) & \text{in } C_{pq}\times(-\kappa,\kappa), \\ \text{linear interpolated with respect to } z_2 \text{ in the stripe } \omega^{(2)}_{pq}. \end{cases}$$

Lemma 3.7.1. *One has the following strong convergences:*

$$\begin{aligned}
&\Pi^{[\alpha]}_{\varepsilon}(\mathbb{V}^{(\alpha)}_{\varepsilon}) \to \mathbb{V} \quad \textit{strongly in } L^2(\Omega\times Cyls^{(\alpha)})^3,\\
&\Pi^{[\alpha]}_{\varepsilon}(\partial_\alpha\mathbb{V}^{(\alpha)}_{\varepsilon}) \to \partial_\alpha\mathbb{V} \quad \textit{strongly in } L^2(\Omega\times Cyls^{(\alpha)})^3,\\
&\Pi^{[\alpha]}_{\varepsilon}(\partial_{\alpha\alpha}\mathbb{V}^{(\alpha)}_{\varepsilon,3}) \to \partial_{\alpha\alpha}\mathbb{V}_3 \quad \textit{strongly in } L^2(\Omega\times Cyls^{(\alpha)}),\\
&\Pi^{[1]}_{\varepsilon}\big((\partial_2\mathbb{V}_3)^{(1)}_{\varepsilon}\big) \to \partial_2\mathbb{V}_3 \quad \textit{strongly in } L^2(\Omega\times Cyls^{(1)}),\\
&\Pi^{[1]}_{\varepsilon}\big(\partial_1(\partial_2\mathbb{V}_3)^{(1)}_{\varepsilon}\big) \to \partial_{12}\mathbb{V}_3 \quad \textit{strongly in } L^2(\Omega\times Cyls^{(1)}),\\
&\Pi^{[2]}_{\varepsilon}\big((\partial_1\mathbb{V}_3)^{(2)}_{\varepsilon}\big) \to \partial_1\mathbb{V}_3 \quad \textit{strongly in } L^2(\Omega\times Cyls^{(2)}),\\
&\Pi^{[2]}_{\varepsilon}\big(\partial_2(\partial_1\mathbb{V}_3)^{(2)}_{\varepsilon}\big) \to \partial_{21}\mathbb{V}_3 \quad \textit{strongly in } L^2(\Omega\times Cyls^{(2)}),\\
&\Pi^{[\alpha]}_{\varepsilon}(\widehat{v}'^{(\alpha,c)}_{\varepsilon}) \to \widehat{v}'^{(\alpha,c)} \quad \textit{strongly in } L^2(\Omega; H^1(Cyls^{(\alpha)}))^3.
\end{aligned}$$

Proof. The proof of this Lemma is an easy consequence of the unfolding properties and the regularity of the test-functions. □

The beam test displacements in the directions $\mathbf{e}_1$ and $\mathbf{e}_2$ are composed by (recall that $z' = (z_1,z_2)$ and $z=(z_1,z_2,z_3)$)

$$V^{(1,b)}_{\varepsilon}(z) = V^{e(1,b)}_{\varepsilon}(z) + \widehat{v}'^{(1,b)}_{\varepsilon}\Big(z', 2\Big\{\frac{z_1}{\varepsilon}\Big\}, 2\Big\{\frac{z_2}{2\varepsilon}\Big\} - b, \frac{z_3}{\varepsilon}\Big), \tag{3.103}$$
$$V^{(2,a)}_{\varepsilon}(z) = V^{e(2,a)}_{\varepsilon}(z) + \widehat{v}'^{(2,a)}_{\varepsilon}\Big(z', 2\Big\{\frac{z_1}{\varepsilon}\Big\} - a, 2\Big\{\frac{z_2}{2\varepsilon}\Big\}, \frac{z_3}{\varepsilon}\Big). \tag{3.104}$$

Observe that the above couple $(V^{(1,b)}_{\varepsilon}, V^{(2,a)}_{\varepsilon})$ belongs to $\mathcal{V}_\varepsilon$.
The elementary displacements for the $\mathbf{e}_1$-directed and the $\mathbf{e}_2$-directed beams are defined respectively by

$$V^{e(1,b)}_{\varepsilon} = \begin{pmatrix} \varepsilon^2\mathbb{V}^{(1)}_{\varepsilon,1} \\ \varepsilon^2\mathbb{V}^{(1)}_{\varepsilon,2} \\ \varepsilon\mathbb{V}^{(1)}_{\varepsilon,3}\end{pmatrix} + \begin{pmatrix} \varepsilon^2\big(\partial_2\mathbb{V}_3\big)^{(1)}_{\varepsilon} \\ -\varepsilon^2\partial_1\mathbb{V}^{(1)}_{\varepsilon,3} \\ 0 \end{pmatrix} \wedge \Big(\Phi^{(1,b)}\Big(2\Big\{\frac{z_1}{2\varepsilon}\Big\}\Big)\mathbf{e}_3 + \Big(2\Big\{\frac{z_2}{2\varepsilon}\Big\} - b\Big)\mathbf{e}_2 + \frac{z_3}{\varepsilon}\mathbf{n}^{(1,b)}\Big(2\Big\{\frac{z_1}{2\varepsilon}\Big\}\Big)\Big),$$

$$V_\varepsilon^{e(2,a)} = \begin{pmatrix} \varepsilon^2 \mathbb{V}_{\varepsilon,1}^{(2)} \\ \varepsilon^2 \mathbb{V}_{\varepsilon,2}^{(2)} \\ \varepsilon \mathbb{V}_{\varepsilon,3}^{(2)} \end{pmatrix} + \begin{pmatrix} \varepsilon^2 \partial_2 \mathbb{V}_{\varepsilon,3}^{(1)} \\ -\varepsilon^2 \big(\partial_1 \mathbb{V}_3\big)_\varepsilon^{(2)} \\ 0 \end{pmatrix} \wedge \left(\Phi^{(2,a)} \left(2\left\{\frac{z_2}{2\varepsilon}\right\}\right) \mathbf{e}_3 + \left(2\left\{\frac{z_1}{2\varepsilon}\right\} - a\right) \mathbf{e}_1 \right.$$
$$\left. + \frac{z_3}{\varepsilon} \mathbf{n}^{(2,a)} \left(2\left\{\frac{z_2}{2\varepsilon}\right\}\right) \right).$$

The test-functions are build to satisfy the contact-conditions before and after the limit as well as to yield the same strain tensor in the limit. In the following, we show the strain-tensor and the contact condition and their limit for both types of test-functions.

The unfolded limiting strain tensor of the test-functions is then an immediate consequence of their definition and the convergences in Lemma 3.7.1 and the limit is written in the exact same way as in section 3.6.4.

Corollary 3.7.2. *The unfolded strain tensor of the test-functions 3.103 satisfies*

$$\frac{1}{\varepsilon^2}\Pi^{[1]}\left(e_z(V_\varepsilon^{(1,b)})\right) \to \mathcal{E}^{(1,b)}(\mathbb{V}) + \mathcal{E}_X^{(1,b)}(\widehat{v}^{(1,b)}), \qquad \textit{strongly in } L^2(\Omega \times Cyls^{(1)})^{3\times 3},$$
$$\frac{1}{\varepsilon^2}\Pi^{[2]}\left(e_z(V_\varepsilon^{(2,a)})\right) \to \mathcal{E}^{(2,a)}(\mathbb{V}) + \mathcal{E}_X^{(2,a)}(\widehat{v}^{(2,a)}), \qquad \textit{strongly in } L^2(\Omega \times Cyls^{(2)})^{3\times 3},$$

where $\mathcal{E}^{(1,b)}$ *and* $\mathcal{E}_X^{(1,b)}$, *respectively* $\mathcal{E}^{(2,a)}$ *and* $\mathcal{E}_X^{(2,a)}$ *are the same as in* (3.91), (3.87), (3.93) *and* (3.88).

Proof. Easy consequence of Lemma 3.7.1 and the properties of the unfolding operator. □

3.7.1 The contact condition of the test-functions

It is necessary to check the contact condition for the test-functions, as they must satisfy this cone-condition to be in the correct space. Due to the special choice of test-function in the section before, this is an easy task. Indeed, the functions on the contact parts $\mathbf{C}_{pq}$ are chosen such that a Taylor expansion of the macroscopic fields is exact and does not admit any remainder terms. To check this, observe that, on the contact area $\mathbf{C}_{pq}$ the elementary test-functions reduce to

$$\widetilde{\mathbb{V}}_\varepsilon^{e(1,b)}(z', (-1)^{a+b+1} r) = \begin{pmatrix} \varepsilon^2 \mathbb{V}_{\varepsilon,1}^{(1)}(z') \\ \varepsilon^2 \mathbb{V}_{\varepsilon,2}^{(1)}(z') \\ \varepsilon \mathbb{V}_{\varepsilon,3}^{(1)}(z') \end{pmatrix} + \begin{pmatrix} \varepsilon^2 \big(\partial_2 \mathbb{V}_3\big)_\varepsilon^{(1)}(z') \\ -\varepsilon^2 \partial_1 \mathbb{V}_{\varepsilon,3}^{(1)}(z') \\ 0 \end{pmatrix} \wedge \left(2\left\{\frac{z_2}{2\varepsilon}\right\} - b\right) \mathbf{e}_2,$$
$$\widetilde{\mathbb{V}}_\varepsilon^{e(2,a)}(z', (-1)^{a+b} r) = \begin{pmatrix} \varepsilon^2 \mathbb{V}_{\varepsilon,1}^{(2)}(z') \\ \varepsilon^2 \mathbb{V}_{\varepsilon,2}^{(2)}(z') \\ \varepsilon \mathbb{V}_{\varepsilon,3}^{(2)}(z') \end{pmatrix} + \begin{pmatrix} \varepsilon^2 \partial_2 \mathbb{V}_{\varepsilon,3}^{(1)}(z') \\ -\varepsilon^2 \big(\partial_1 \mathbb{V}_3\big)_\varepsilon^{(2)}(z') \\ 0 \end{pmatrix} \wedge \left(2\left\{\frac{z_1}{2\varepsilon}\right\} - a\right) \mathbf{e}_1.$$

Consider now the difference of the two test displacements in $\mathbf{C_{pq}}$. One has

$$\begin{aligned}
&\widetilde{\mathbb{V}}_\varepsilon^{(1,b)}(z',(-1)^{a+b+1}r) - \widetilde{\mathbb{V}}^{(2,a)}(z',(-1)^{a+b}r) = \\
&\qquad \varepsilon^3 \begin{pmatrix} \frac{z_1-p\varepsilon}{\varepsilon}\partial_1\mathbb{V}_1(p\varepsilon,q\varepsilon) - \frac{z_2-q\varepsilon}{\varepsilon}\partial_2\mathbb{V}_1(p\varepsilon,q\varepsilon) \\ \frac{z_1-p\varepsilon}{\varepsilon}\partial_1\mathbb{V}_2(p\varepsilon,q\varepsilon) - \frac{z_2-q\varepsilon}{\varepsilon}\partial_2\mathbb{V}_2(p\varepsilon,q\varepsilon) \\ \frac{(z_1-p\varepsilon)^2}{2\varepsilon^2}\partial_{11}\mathbb{V}_3(p\varepsilon,q\varepsilon) - \frac{(z_2-q\varepsilon)^2}{2\varepsilon^2}\partial_{22}\mathbb{V}_3(p\varepsilon,q\varepsilon) \end{pmatrix} \\
&\qquad + \varepsilon^3\widehat{v}'^{(1,b)}\Big(p\varepsilon,q\varepsilon,\frac{z_1-p\varepsilon}{\varepsilon},\frac{z_2-q\varepsilon}{\varepsilon},(-1)^{a+b+1}\kappa\Big) \\
&\qquad\qquad - \varepsilon^3\widehat{v}'^{(2,a)}\Big(p\varepsilon,q\varepsilon,\frac{z_1-p\varepsilon}{\varepsilon},\frac{z_2-q\varepsilon}{\varepsilon},(-1)^{a+b}\kappa\Big). \qquad (3.105)
\end{aligned}$$

Hence, the contact conditions are satisfied for every ε by the definition of the test-functions. Indeed, note that by the conditions on the test functions (3.101) the microscopic contact in (3.32) as well as in the unfolded limit (3.98)-(3.100) follows immediately and even includes the case of a rigid contact where $\mathbf{g} \equiv 0$.

Now, that a set of test-functions is prepared and verified, it is necessary to span the full space of displacements. Finally, we conclude by density of the spaces

$$\mathcal{C}^1(\overline{\Omega}) \cap \mathcal{H}^1(\Omega) \subset \mathcal{H}^1(\Omega), \quad \mathcal{C}^2(\overline{\Omega}) \cap \mathcal{H}^2(\Omega) \subset \mathcal{H}^2(\Omega), \quad \mathcal{C}^1(\overline{\Omega};\widehat{\mathbf{W}}^{(\alpha)}) \subset L^2(\Omega;\widehat{\mathbf{W}}^{(\alpha)}),$$

that the convergences of the unfolded strain tensor and the contact condition hold not only for these specific test-functions, but for all functions in $\mathcal{X}$.

3.8 The limit problem

In this section, all tools and results developed in this thesis lead to the homogenization of the textile elasticity problem. Thus, recall the initial variational inequality in the vectorial notation:

$$\begin{aligned}
&\text{Find } u_\varepsilon \in V_\varepsilon \text{ such that:} \\
&\int_{\mathcal{S}_\varepsilon} A_\varepsilon\, E_x(u_\varepsilon)\cdot E_x(u_\varepsilon-\varphi)\,dx - \int_{\mathcal{S}_\varepsilon} f_\varepsilon\cdot(u_\varepsilon-\varphi)\,dx \le 0, \qquad \forall\varphi\in V_\varepsilon.
\end{aligned} \qquad (3.106)$$

Let us denote by $\widetilde{\mathbf{C}}_\varepsilon^{(\alpha,c)}$ the orthogonal matrices as in (3.26) for the different beam directions, such that $E_z^{(\alpha,c)}(u^{(\alpha,c)}) = \widetilde{\mathbf{C}}_\varepsilon^{(\alpha,c)} E_x(u^{(\alpha,c)})$ and define the matrices

$$\widetilde{A}_\varepsilon^{(1,q)} = (\widetilde{\mathbf{C}}^{(1,q)})^{-1} A_\varepsilon^{(1,q)} \widetilde{\mathbf{C}}^{(1,q)} \qquad \text{and} \qquad \widetilde{A}_\varepsilon^{(2,p)} = (\widetilde{\mathbf{C}}^{(2,p)})^{-1} A_\varepsilon^{(2,p)} \widetilde{\mathbf{C}}^{(2,p)}$$

respectively. Then, unfolding the problem (3.106) we obtain (for every $\varphi \in V_\varepsilon$)

$$\begin{aligned}
&\sum_{q=1}^{N_\varepsilon}\Big(\int_{P_r^{(1,q)}}\Big[\widetilde{A}_\varepsilon^{(1,q)}E_z(u_\varepsilon^{(1,q)})\cdot E_z(u_\varepsilon^{(1,q)}-\varphi^{(1,q)})\ -f_\varepsilon^{(1)}\cdot(u_\varepsilon^{(1,q)}-\varphi^{(1,q)})\Big]\ |\eta_\varepsilon^{(1,q)}|dz\Big)\\
+&\sum_{p=0}^{N_\varepsilon}\Big(\int_{P_r^{(2,p)}}\Big[\widetilde{A}_\varepsilon^{(2,p)}E_z(u_\varepsilon^{(2,p)})\cdot E_z(u_\varepsilon^{(2,p)}-\varphi^{(2,p)})\ -f_\varepsilon^{(2)}\cdot\big(u_\varepsilon^{(2,p)}-\varphi^{(2,p)}\big)\Big]\ |\eta_\varepsilon^{(2,p)}|dz\Big)\leq 0.
\end{aligned}\tag{3.107}$$

For the following analysis we introduce a new notation in order to simplify the expressions of the different microscopic and macroscopic problems.

Notation 3.8.1. *Set*

$$\mathbf{M}_{ab}(\zeta)(X_1,X_2)=\begin{pmatrix}(X_1-a)\zeta_1-(X_2-b)\zeta_2\\(X_1-a)\zeta_2-(X_2-b)\zeta_3\\\frac{1}{2}(X_1-a)^2\zeta_4-\frac{1}{2}(X_2-b)^2\zeta_5\end{pmatrix}.$$

Moreover, define the displacements

$$\begin{aligned}
\widehat{W}^{(1,b)}(\zeta)(X)&=\theta_1(X_1)\begin{pmatrix}\zeta_1\\\zeta_2\\0\end{pmatrix}+\begin{pmatrix}\zeta_6\\-\zeta_4\\0\end{pmatrix}\wedge(\theta_2(X_1)\mathbf{e}_1+\theta_1(X_1)(X_2-b)\mathbf{e}_2)\\
&\qquad+\begin{pmatrix}\zeta_6\\-\zeta_4\\0\end{pmatrix}\wedge\Big(\Phi^{(1,b)}(X_1)\mathbf{e}_3+X_3\mathbf{n}^{(1,b)}(X_1)\Big),\\
\widehat{W}^{(2,a)}(\zeta)(X)&=\theta_1(X_2)\begin{pmatrix}\zeta_2\\\zeta_3\\0\end{pmatrix}+\begin{pmatrix}\zeta_5\\-\zeta_6\\0\end{pmatrix}\wedge(\theta_2(X_2)\mathbf{e}_2+(X_1-a)\theta_1(X_2)\mathbf{e}_1)\\
&\qquad+\begin{pmatrix}\zeta_5\\-\zeta_6\\0\end{pmatrix}\wedge\Big(\Phi^{(2,a)}(X_2)\mathbf{e}_3+X_3\mathbf{n}^{(2,a)}(X_2)\Big).
\end{aligned}$$

where $\theta_1\in\mathcal{C}^1_{per}(0,2)$ (resp. $\theta_2\in\mathcal{C}^1_{per}(0,2)$) is 2-periodic and satisfies

$$\theta_1(t)=t-c\quad(\textit{resp.}\ \ \theta_2(t)=\frac{1}{2}(t-c)^2)\ \ \textit{a.e. in}\ [c-\kappa,c+\kappa],\quad c\in\{0,1\}.$$

Then the difference on the contact area can be expressed by

$$\widehat{W}^{(1,b)}(\zeta)(X)-\widehat{W}^{(2,a)}(\zeta)(X)=\mathbf{M}_{ab}(\zeta)(X_1,X_2)\qquad \textit{a.e. on}\quad \mathbf{C}_{ab}\tag{3.108}$$

and hence resembles the original contact condition.
Similarly, define the strain tensor in vectorial notation in the according form

$$\mathcal{E}(\zeta)=\mathcal{E}^{(1,b)}(\zeta)\mathbb{1}_{Cyls^{(1,b)}}+\mathcal{E}^{(2,a)}(\zeta)\mathbb{1}_{Cyls^{(2,a)}}=\sum_{i=1}^{6}\zeta_n\mathcal{E}(\mathbf{e}_n),$$

with

$$\mathcal{E}^{(\alpha,c)}(\zeta) = \left(\mathcal{E}_{11}^{(\alpha,c)}, \mathcal{E}_{22}^{(\alpha,c)}, \mathcal{E}_{33}^{(\alpha,c)}, \sqrt{2}\mathcal{E}_{12}^{(\alpha,c)}, \sqrt{2}\mathcal{E}_{13}^{(\alpha,c)}, \sqrt{2}\mathcal{E}_{23}^{(\alpha,c)}\right)^T \tag{3.109}$$

and

$$\begin{aligned}
\mathcal{E}_{11}^{(1,b)}(\zeta) =& \frac{1}{\eta^{(1,b)}}\left[\begin{pmatrix}\zeta_1\\ \zeta_2\\ 0\end{pmatrix}\cdot \mathbf{t}^{(1,b)} + \begin{pmatrix}\zeta_6\\ -\zeta_4\\ 0\end{pmatrix}\cdot\left(\left(X_3 + \frac{\Phi^{(1,b)}}{\gamma}\right)\mathbf{e}_2 - (X_2 - b)\,\mathbf{n}^{(1,b)}\right)\right],\\
\mathcal{E}_{12}^{(1,b)}(\zeta) =& \frac{1}{2\eta^{(1,b)}}\left[\begin{pmatrix}\zeta_1\\ \zeta_2\\ 0\end{pmatrix}\cdot \mathbf{e}_2 - \begin{pmatrix}\zeta_6\\ -\zeta_4\\ 0\end{pmatrix}\cdot\left(X_3\mathbf{t}^{(1,b)} + \Phi^{(1,b)}\mathbf{e}_1\right)\right],\\
\mathcal{E}_{13}^{(1,b)}(\zeta) =& \frac{1}{2\eta^{(1,b)}}\left[\begin{pmatrix}\zeta_1\\ \zeta_2\\ 0\end{pmatrix}\cdot \mathbf{n}^{(1,b)} + \begin{pmatrix}\zeta_6\\ -\zeta_4\\ 0\end{pmatrix}\cdot\left((X_2 - b)\,\mathbf{t}^{(1,b)} - \frac{\Phi^{(1,b)}d_1\Phi^{(1,b)}}{\gamma}\mathbf{e}_2\right)\right],
\end{aligned}$$

and $\mathcal{E}_{22}^{(1,b)}(\zeta) = \mathcal{E}_{33}^{(1,b)}(\zeta) = \mathcal{E}_{23}^{(1,b)}(\zeta) = 0$. *Accordingly, the tensor* $\mathcal{E}^{(2,a)}(\zeta)$ *is defined by*

$$\begin{aligned}
\mathcal{E}_{22}^{(2,a)}(\zeta) =& \frac{1}{\eta^{(2,a)}}\left[\begin{pmatrix}\zeta_2\\ \zeta_3\\ 0\end{pmatrix}\cdot \mathbf{t}^{(2,a)} - \begin{pmatrix}\zeta_5\\ -\zeta_6\\ 0\end{pmatrix}\cdot\left(\left(X_3 + \frac{\Phi^{(2,a)}}{\gamma}\right)\mathbf{e}_1 - (X_1 - a)\,\mathbf{n}^{(2,a)}\right)\right],\\
\mathcal{E}_{12}^{(2,a)}(\zeta) =& \frac{1}{2\eta^{(2,a)}}\left[\begin{pmatrix}\zeta_2\\ \zeta_3\\ 0\end{pmatrix}\cdot \mathbf{e}_1 + \begin{pmatrix}\zeta_5\\ -\zeta_6\\ 0\end{pmatrix}\cdot\left(X_3\mathbf{t}^{(2,a)} + \Phi^{(2,a)}\mathbf{e}_2\right)\right],\\
\mathcal{E}_{23}^{(2,a)}(\zeta) =& \frac{1}{2\eta^{(2,a)}}\left[\begin{pmatrix}\zeta_2\\ \zeta_3\\ 0\end{pmatrix}\cdot \mathbf{n}^{(2,a)} - \begin{pmatrix}\zeta_5\\ -\zeta_6\\ 0\end{pmatrix}\cdot\left((X_1 - a)\,\mathbf{t}^{(2,a)} - \frac{\Phi^{(2,a)}d\Phi^{(2,a)}}{\gamma}\mathbf{e}_1\right)\right].
\end{aligned}$$

and $\mathcal{E}_{11}^{(2,a)}(\zeta) = \mathcal{E}_{33}^{(2,a)}(\zeta) = \mathcal{E}_{13}^{(2,a)}(\zeta) = 0$.
Additionally, without renaming rewrite the local strain tensor in vectorial form, i.e.

$$\mathcal{E}_X^{(\alpha,c)} = \left(\mathcal{E}_{X,11}^{(\alpha,c)}, \mathcal{E}_{X,22}^{(\alpha,c)}, \mathcal{E}_{X,33}^{(\alpha,c)}, \sqrt{2}\mathcal{E}_{X,12}^{(\alpha,c)}, \sqrt{2}\mathcal{E}_{X,13}^{(\alpha,c)}, \sqrt{2}\mathcal{E}_{X,23}^{(\alpha,c)}\right)^T. \tag{3.110}$$

Furthermore, for the sake of comprehensibility and readability define

$$\begin{aligned}
\widetilde{\mathbf{A}}(X) &= \sum_{\alpha=1}^{2}\sum_{c=0}^{1}\widetilde{A}^{(\alpha,c)}(X)\mathbb{1}_{Cyls^{(\alpha,c)}}(X), & \mathcal{E}_X(\varphi) &= \sum_{\alpha=1}^{2}\sum_{c=0}^{1}\mathcal{E}_X^{(\alpha,c)}(\varphi)\mathbb{1}_{Cyls^{(\alpha,c)}}(X),\\
\boldsymbol{\eta}(X) &= \sum_{\alpha=1}^{2}\sum_{c=0}^{1}\boldsymbol{\eta}^{(\alpha,c)}(X)\mathbb{1}_{Cyls^{(\alpha,c)}}(X), & \rho(X) &= \sum_{\alpha=1}^{2}\sum_{c=0}^{1}\frac{1}{|Cyls^{(\alpha,c)}|}\mathbb{1}_{Cyls^{(\alpha,c)}}(X).
\end{aligned} \tag{3.111}$$

Finally, define different function spaces accounting for different contact conditions. First, for every $\widehat{w} = (\widehat{w}^{(1)}, \widehat{w}^{(2)}) \in \widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)}$ *set*

$$\sum_{(\alpha,c)\in\{1,2\}\times\{0,1\}} \int_{Cyls^{(\alpha,c)}} \widehat{w}^{(\alpha,c)}\, dX = \int_{Cyls} \widehat{w}\, dX.$$

Then, define

$$\widehat{\mathbf{W}}_{lin} \doteq \Big\{ \widehat{w} = (\widehat{w}^{(1)}, \widehat{w}^{(2)}) \in \widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)} \;\Big|\; \widehat{w}^{(1)} = \widehat{w}^{(2)} \text{ a.e. on } \mathbf{C}_{ab},\ (a,b)\in\{0,1\}^2,\ \int_{Cyls} \widehat{w}\, dX = 0 \Big\},$$

and $K_{\zeta,z}$ *and* K_z *the convex subsets of* $\widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)}$, *(*$(\zeta, z) \in \mathbb{R}^6 \times \overline{\Omega}$*)*

$$\begin{aligned} K_{\zeta,z} \doteq \Big\{ \widehat{v} = (\widehat{v}^{(1)}, \widehat{v}^{(2)}) \in \widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)} \mid |\mathbf{M}_{ab,\alpha}(\zeta) + \widehat{v}^{(1,b)}_\alpha - \widehat{v}^{(2,a)}_\alpha| \le g_\alpha(z) \text{ a.e. on } \mathbf{C}_{ab},\\ 0 \le (-1)^{a+b}\big(\mathbf{M}_{ab,3}(\zeta) + \widehat{v}^{(1,b)}_3 - \widehat{v}^{(2,a)}_3\big) \le g_3(z) \text{ a.e. on } \mathbf{C}_{ab}, \text{ and } \int_{Cyls} \widehat{w}\, dX = 0 \Big\}, \end{aligned}$$

$$\begin{aligned} K_{z} \doteq \Big\{ \widehat{v} = (\widehat{v}^{(1)}, \widehat{v}^{(2)}) \in \widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)} \mid |\widehat{v}^{(1,b)}_\alpha - \widehat{v}^{(2,a)}_\alpha| \le g_\alpha(z) \text{ a.e. on } \mathbf{C}_{ab},\\ 0 \le (-1)^{a+b}\big(\widehat{v}^{(1,b)}_3 - \widehat{v}^{(2,a)}_3\big) \le g_3(z) \text{ a.e. on } \mathbf{C}_{ab}, \text{ and } \int_{Cyls} \widehat{w}\, dX = 0 \Big\}. \end{aligned}$$

3.8.1 The unfolded limit problem

Lemma 3.8.2. *Let* $(\widehat{w}, \widehat{v})$ *be in* $\widehat{\mathbf{W}}_{lin} \times \widehat{\mathbf{W}}_{lin}$ *(resp. in* $K_z \times K_z$, $K_{\zeta,z} \times K_{\zeta,z}$*) satisfying*

$$\mathcal{E}_X(\widehat{w}) = \mathcal{E}_X(\widehat{v}) \quad \text{a.e. in} \quad Cyls, \tag{3.112}$$

then

$$\widehat{v} = \widehat{w} \quad \text{a.e. in} \quad Cyls.$$

We also have

$$\begin{aligned} &\forall\, \widehat{w} \in \widehat{\mathbf{W}}_{lin}, && \|\widehat{w}\|_{H^1(Cyls)} \le C\|\mathcal{E}_X(\widehat{w})\|_{L^2(Cyls)},\\ &\forall\, \widehat{w} \in K_z, && \|\widehat{w}\|_{H^1(Cyls)} \le C\big(\|\mathcal{E}_X(\widehat{w})\|_{L^2(Cyls)} + \|g\|_{L^\infty(\Omega)}\big),\\ &\forall\, \widehat{w} \in K_{\zeta,z}, && \|\widehat{w}\|_{H^1(Cyls)} \le C\big(\|\mathcal{E}_X(\widehat{w})\|_{L^2(Cyls)} + \|g\|_{L^\infty(\Omega)} + |\zeta|\big). \end{aligned} \tag{3.113}$$

Proof. Let $(\widehat{w}, \widehat{v})$ be in $\widehat{\mathbf{W}}_{lin} \times \widehat{\mathbf{W}}_{lin}$ satisfying (3.112). Set $r^{(1,b)} = \widehat{w}^{(1,b)} - \widehat{v}^{(1,b)}$ and similarly $r^{(2,a)} = \widehat{w}^{(2,a)} - \widehat{v}^{(2,a)}$ with $(a,b) \in \{0,1\}^2$. These displacements are rigid motions, so we write

$$\begin{aligned} r^{(1,b)}(X) &= A^{(1,b)} + B^{(1,b)} \wedge (X_1\mathbf{e}_1 + \Phi^{(1,b)}(X_1)\mathbf{e}_3) + B^{(1,b)} \wedge \big((X_2 - b)\mathbf{e}_2 + X_3\mathbf{n}^{(1,b)}(X_1)\big),\\ r^{(2,a)}(X) &= A^{(2,a)} + B^{(2,a)} \wedge (X_2\mathbf{e}_2 + \Phi^{(2,a)}(X_2)\mathbf{e}_3) + B^{(2,a)} \wedge \big((X_1 - a)\mathbf{e}_1 + X_3\mathbf{n}^{(2,a)}(X_2)\big). \end{aligned} \tag{3.114}$$

Furthermore, they are periodic in the respective direction. Hence

$$r^{(1,b)}(0, X_2, X_3) = r^{(1,b)}(2, X_2, X_3)$$

yields $B^{(1,b)} \wedge \mathbf{e}_1 = 0$ and $r^{(2,a)}(X_1, 0, X_3) = r^{(2,a)}(X_1, 2, X_3)$ analogously $B^{(2,a)} \wedge \mathbf{e}_2 = 0$. They also satisfy the contact conditions, hence

$$r^{(1,\alpha)}(X_1, X_2, (-1)^{\alpha+\beta}\kappa) = r^{(2,\beta)}(X_1, X_2, (-1)^{\alpha+\beta+1}\kappa), \qquad \forall\, (X_1, X_2) \in \mathbf{C}_{\alpha\beta}.$$

The first condition yields

$$A^{(1,0)} + B^{(1,0)} \wedge X_2\mathbf{e}_2 = A^{(2,0)} + B^{(2,0)} \wedge X_1\mathbf{e}_1 \qquad \forall (X_1, X_2) \in (-\kappa, \kappa)^2.$$

That gives $B^{(1,0)} \wedge \mathbf{e}_2 = B^{(2,0)} \wedge \mathbf{e}_1 = 0$ and then taking into account the preceding equalities one has $B^{(1,0)} = B^{(2,0)} = 0$ and also $A^{(1,0)} = A^{(2,0)}$. In the same way, we obtain

$$B^{(1,b)} = B^{(2,b)} = 0, \qquad A^{(1,b)} = A^{(2,a)} \qquad \forall (a, b) \in \{0, 1\}^2.$$

Thus, the difference of $\widehat{w}$ and $\widehat{v}$ is constant and there exits $A \in \mathbb{R}^3$ such that $\widehat{w} - \widehat{v} = A$. The last condition in the definition of $\widehat{\mathbf{W}}_{lin}$ implies $A = 0$. The Korn inequality gives $(3.113)_1$.
Now, if $(\widehat{w}, \widehat{v})$ belongs to $K_z \times K_z$ the periodicity yields again $B^{(1,b)} \wedge \mathbf{e}_1 = 0$ and $B^{(2,a)} \wedge \mathbf{e}_2 = 0$. The difference between the two cases lies in the contact conditions and we obtain on $\mathbf{C}_{00}$ that

$$\begin{aligned} &|A_1^{(1,0)} - A_1^{(2,0)}| \le g_1(z), \quad |A_2^{(1,0)} - A_2^{(2,0)}| \le g_2(z), \\ &0 \le A_3^{(1,0)} + X_2 B_1^{(1,0)} - A_3^{(2,0)} + X_1 B_2^{(2,0)} \le g_3(z), \end{aligned} \qquad \forall\, (X_1, X_2) \in \mathbf{C}_{00}. \tag{3.115}$$

Since $(X_1, X_2) \in (-\kappa, \kappa)^2$, the third condition in (3.115) gives $0 \le A_3^{(1,0)} - A_3^{(2,0)} \le g_3(z)$ as well as $2\kappa\big(|B_1^{(1,0)}| + |B_2^{(2,0)}|\big) \le g_3(z)$. In the same way, we get similar conditions for the other contact parts:

$$|A^{(1,a)} - A^{(2,b)}| + |B^{(1,a)}| + |B^{(2,b)}| \le C\|g\|_{L^\infty(\Omega)} \qquad \forall (a, b) \in \{0, 1\}^2.$$

Set

$$A = \frac{1}{4}\big(A^{(1,0)} + A^{(1,1)} + A^{(2,0)} + A^{(2,1)}\big).$$

One has

$$|A^{(1,a)} - A| \le C\|g\|_{L^\infty(\Omega)}, \quad |A^{(2,b)} - A| \le C\|g\|_{L^\infty(\Omega)} \qquad \forall (a, b) \in \{0, 1\}^2.$$

This leads to

$$\|r^{(1,a)} - A\|_{L^2(Cyls^{(1,a)})} + \|r^{(2,0)} - A\|_{L^2(Cyls^{(2,b)})} \le C\|g\|_{L^\infty(\Omega)} \qquad \forall (a, b) \in \{0, 1\}^2. \tag{3.116}$$

Finally, the above inequalities and the Korn inequality yield

$$\|(\widehat{w} - \widehat{v})^{(1,a)} - A\|_{L^2(Cyls^{(1,a)})} + \|(\widehat{w} - \widehat{v})^{(2,b)} - A\|_{L^2(Cyls^{(2,b)})} \le C\|g\|_{L^\infty(\Omega)}, \quad \forall (a, b) \in \{0, 1\}^2.$$

The last condition in the definition of K_z implies $A = 0$. The Korn inequality gives $(3.113)_2$.

In the last case $(\widehat{w}, \widehat{v})$ in $K_{\zeta,z} \times K_{\zeta,z}$ we replace (3.116) by

$$\|r^{(1,a)} - A\|_{L^2(Cyls^{(1,a)})} + \|r^{(2,0)} - A\|_{L^2(Cyls^{(2,b)})} \leq C(|\zeta| + \|g\|_{L^\infty(\Omega)}) \qquad \forall (a,b) \in \{0,1\}^2.$$

The conclusion is analogous. □

Theorem 3.8.3. *Suppose that $f_\varepsilon^{(\alpha)}$ is defined as in* (3.63) *and that*

$$g_\varepsilon = \varepsilon^3 g, \qquad g \in \mathcal{C}(\overline{\Omega})^3. \tag{3.117}$$

Moreover, assume that $A_\varepsilon^{(\alpha)} = A^{(\alpha)}\left(\frac{\cdot}{\varepsilon}\right)$ with $A^{(\alpha)} \in [L^\infty(Cyls^{(\alpha)})]^{6\times 6}$ satisfies the assumptions of section3.3.4.

Let $u_\varepsilon = (u^{(1,1)}, \dots, u^{(1,2N_\varepsilon)}, u^{(2,0)}, \dots, u^{(2,2N_\varepsilon)}) \in \mathcal{V}_\varepsilon$ be a solution to problem (3.106). *Then there exists a subsequence, still denoted by ε, and $(\mathbb{U}, \widehat{u}) \in \mathcal{X}$ such that the fields satisfy the unfolded limit problem:*

Find $(\mathbb{U}, \widehat{u}) \in \mathcal{X}$ such that for every $(\mathbb{V}, \widehat{v}) \in \mathcal{X}$:

$$\int_{\Omega\times Cyls} \rho\widetilde{\mathbf{A}}\Big[\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{u})\Big] \cdot \Big[\mathcal{E}(\zeta - \xi) + \mathcal{E}_X(\widehat{u} - \widehat{v})\Big] |\boldsymbol{\eta}| \, dz dX \leq \int_\Omega F \cdot (\mathbb{U} - \mathbb{V}) \, dz, \tag{3.118}$$

where

$$\zeta = (e_{11}(\mathbb{U}), e_{12}(\mathbb{U}), e_{22}(\mathbb{U}), \partial_{11}\mathbb{U}_3, \partial_{22}\mathbb{U}_3, \partial_{12}\mathbb{U}_3),$$
$$\xi = (e_{11}(\mathbb{V}), e_{12}(\mathbb{V}), e_{22}(\mathbb{V}), \partial_{11}\mathbb{V}_3, \partial_{22}\mathbb{V}_3, \partial_{12}\mathbb{V}_3).$$

The tensor-fields $\mathcal{E}$ and $\mathcal{E}_X$ are defined in (3.111) *(and in section3.6.4) and $F = f^{(1)} + f^{(2)}$.*

Proof. Choose the test-functions according to section3.7. Then the limit (3.118) is a consequence of the unfolding for integrals, the assumptions and the convergences in Sections 3.6 and 3.7. The form of the right-hand side follows by the integration over the other parts of the displacement, which vanish due to symmetry.

Note that until now the test-functions are in the space

$$(\mathbb{V}, \widehat{v}^{(\alpha)}) \in \mathcal{X} \cap \big[\mathcal{C}^1(\overline{\Omega})^2 \times \mathcal{C}^2(\overline{\Omega}) \times \mathcal{C}^1(\overline{\Omega}, \widehat{\mathbf{W}}^{(\alpha)})\big].$$

The density-argument of this space in $\mathcal{X}$ is a bit more involved due to the cone-condition coming from the contact. This issue is resolved by truncation and regularization of the functions, which then allow together with the typical density argument to conclude the claim. □

Before investigating the existence and uniqueness, it is necessary to describe the homogenized problem completely. Hence introduce the correctors and their respective problems. In fact, since the problem (3.118) is nonlinear, it is split into multiple problems, of which most are linear but one remaining problem captures the nonlinearity.

The corrector-problem for the field $\widehat{u}$ is obtained by choosing $\mathbb{V} = \mathbb{U}$ in (3.118) leading to the following microscopic problem:

$$\begin{aligned}&\text{For } (\zeta, z) \text{ in } \mathbb{R}^6 \times \overline{\Omega}, \text{ find } \widehat{v}_{\zeta,z} \in K_{\zeta,z},\\ &\int_{Cyls} \rho\widetilde{\mathbf{A}}\Big[\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v}_{\zeta,z})\Big] \cdot \mathcal{E}_X(\widehat{v}_{\zeta,z} - \widehat{w})|\boldsymbol{\eta}|\, dX \le 0, \qquad \forall \widehat{w} \in K_{\zeta,z}.\end{aligned} \tag{3.119}$$

This variational inequality admits solutions by the Stampacchia-Theorem (see [33]). Two solutions $\widehat{v}_{\zeta,z}$ and $\widehat{w}_{\zeta,z}$ of this problem satisfy

$$\mathcal{E}_X(\widehat{v}_{\zeta,z}) = \mathcal{E}_X(\widehat{w}_{\zeta,z}).$$

Indeed, consider the problems of the two solutions with specific test-functions

$$\begin{aligned}&\int_{Cyls} \rho\widetilde{\mathbf{A}}\Big[\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v}_{\zeta,z})\Big] \cdot \mathcal{E}_X(\widehat{v}_{\zeta,z} - \widehat{w}_{\zeta,z})|\boldsymbol{\eta}|\, dX \le 0,\\ &\int_{Cyls} \rho\widetilde{\mathbf{A}}\Big[\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{w}_{\zeta,z})\Big] \cdot \mathcal{E}_X(\widehat{w}_{\zeta,z} - \widehat{v}_{\zeta,z})|\boldsymbol{\eta}|\, dX \le 0.\end{aligned}$$

Adding these two inequalities yields

$$\int_{Cyls} \rho\widetilde{\mathbf{A}}\mathcal{E}_X(\widehat{w}_{\zeta,z} - \widehat{v}_{\zeta,z}) \cdot \mathcal{E}_X(\widehat{w}_{\zeta,z} - \widehat{v}_{\zeta,z})|\boldsymbol{\eta}|\, dX \le 0. \tag{3.120}$$

Hence, it follows that $\mathcal{E}_X(\widehat{v}_{\zeta,z}) = \mathcal{E}_X(\widehat{w}_{\zeta,z})$ since by coercivity (3.120) is also non-negative and solutions of (3.119) differ only from rigid motions, see Lemma 3.8.2. Thus, there exist rigid displacements $r_{\zeta,z}^{(\alpha,c)}$, $(\alpha, c) \in \{1, 2\} \times \{0, 1\}$ such that

$$\widehat{w}_{\zeta,z}^{(\alpha,c)} - \widehat{v}_{\zeta,z}^{(\alpha,c)} = r_{\zeta,z}^{(\alpha,c)}, \qquad (\alpha, c) \in \{1, 2\} \times \{0, 1\}.$$

One has

$$\|r_{\zeta,z}^{(\alpha,c)}\|_{L^2(Cyls^{(\alpha,c)})} \le C\big(|\zeta| + \|g\|_{L^\infty(\Omega)}\big), \qquad (\alpha, c) \in \{1, 2\} \times \{0, 1\}.$$

Now, we introduce the six typical linear corrector problems as the solution of the following variational problems:

$$\begin{aligned}&\text{Find } \widehat{\chi}_n \in \widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)} \text{ such that } \widehat{\chi}_n^{(1,b)} + \widehat{W}^{(1,b)}(\mathbf{e}_n) = \widehat{\chi}_n^{(2,a)} + \widehat{W}^{(2,a)}(\mathbf{e}_n) \text{ a.e. on } \mathbf{C}_{ab},\\ &\int_{Cyls} \rho\widetilde{\mathbf{A}}\Big[\mathcal{E}(\mathbf{e}_n) + \mathcal{E}_X(\widehat{\chi}_n)\Big] \cdot \mathcal{E}_X(\widehat{w})|\boldsymbol{\eta}|\, dX = 0, \qquad \forall\, \widehat{w} \in \widehat{\mathbf{W}}_{lin}, \quad n \in \{1, \dots, 6\},\end{aligned} \tag{3.121}$$

and the unit-vectors $\mathbf{e}_n \in \mathbb{R}^6$. Due to the definition of $\widehat{\mathbf{W}}_{lin}$ the problems (3.121) admit unique solutions.

Denote $\mathbf{S}$ the vector space generated by $\{\widehat{\chi}_1,\dots,\widehat{\chi}_6\}$. Every function $\widehat{v}\in K_{\zeta,z}$ is uniquely written as

$$\widehat{v}=\sum_{i=1}^{6}\zeta_i\widehat{\chi}_i+\widehat{w},\qquad \sum_{i=1}^{6}\zeta_i\widehat{\chi}_i\in\mathbf{S},\ \ \widehat{w}\in K_z.$$

Hence, the solution of (3.119) is uniquely decomposed as

$$\widehat{v}_{\zeta,z}=\widehat{v}_{\zeta,\,lin}+\widehat{\chi}_{\zeta,z},\qquad \widehat{v}_{\zeta,\,lin}=\sum_{i=1}^{6}\zeta_i\widehat{\chi}_i\in\mathbf{S},\quad \widehat{\chi}_{\zeta,z}\in K_z. \tag{3.122}$$

The additional corrector $\widehat{\chi}_{\zeta,z}$ takes into account the nonlinearity and is the solution of the variational problem

For (ζ,z) in $\mathbb{R}^6\times\overline{\Omega}$, find $\widehat{\chi}_{\zeta,z}\in K_z$,

$$\int_{Cyls}\rho\widetilde{\mathbf{A}}\Big[\mathcal{E}(\zeta)+\mathcal{E}_X(\widehat{v}_{\zeta,lin})+\mathcal{E}_X(\widehat{\chi}_{\zeta,z})\Big]\cdot\mathcal{E}_{X,kl}(\widehat{\chi}_{\zeta,z}-\widehat{w})|\boldsymbol{\eta}|\,dX\le 0,\qquad \forall\widehat{w}\in K_z. \tag{3.123}$$

This variational inequality admits solutions by the Stampacchia-Theorem [33]. Here, two solutions $\widehat{\chi}_{\zeta,z}$ and $\widetilde{\chi}_{\zeta,z}$ of (3.123) differ only by rigid motions (see Lemma 3.8.2). Hence, there exist rigid displacements $r^{(\alpha,c)}_{\zeta,z}$, $(\alpha,c)\in\{1,2\}\times\{0,1\}$ such that

$$\widehat{\chi}^{(\alpha,c)}_{\zeta,z}-\widetilde{\chi}^{(\alpha,c)}_{\zeta,z}=r^{(\alpha,c)}_{\zeta,z},\qquad (\alpha,c)\in\{1,2\}\times\{0,1\}$$

and one has

$$\|r^{(\alpha,c)}_{\zeta,z}\|_{L^2(Cyls^{(\alpha,c)})}\le C\|g\|_{L^\infty(\Omega)},\qquad (\alpha,c)\in\{1,2\}\times\{0,1\}.$$

Lemma 3.8.4. *The map $(\zeta,z)\in\mathbb{R}^6\times\overline{\Omega}\longmapsto\mathcal{E}_X(\widehat{\chi}_{\zeta,z})$ is continuous. Moreover, one has*

$$\forall(\zeta,z)\in\mathbb{R}^6\times\overline{\Omega},\qquad \begin{aligned}&\big\|\mathcal{E}_X(\widehat{\chi}_{\zeta,z})\big\|_{L^2(Cyls)}\le C|\zeta|,\\ &\big\|\widehat{\chi}_{\zeta,z}\big\|_{H^1(Cyls)}\le C\big(|\zeta|+\|g\|_{L^\infty(\Omega)}\big).\end{aligned} \tag{3.124}$$

The constants do not depend on (ζ,z).

Proof. First, choose $\widehat{w}=0$ in (3.123), that leads to the estimate $(3.124)_1$. Then $(3.124)_2$ is a consequence of Lemma 3.8.2.

Now we prove that the map is continuous. Let (ζ,z) be in $\mathbb{R}^6\times\overline{\Omega}$ and $\{(\zeta_n,z_n)\}_{n\in\mathbb{N}^*}$ a sequence satisfying

$$(\zeta_n,z_n)\in\mathbb{R}^6\times\overline{\Omega},\qquad \zeta_n\longrightarrow\zeta,\quad\text{and}\quad z_n\longrightarrow z.$$

Due to (3.124) and Lemma 3.8.2, the sequence $\{\widehat{\chi}_{\zeta_n,z_n}\}_{n\in\mathbb{N}^*}\subset H^1(Cyls)^3$ is uniformly bounded in $H^1(Cyls)^3$. Hence, there exist a subsequence $\{n'\}$ and $\widehat{\chi}_0\in H^1(Cyls)^3$ such that

$$\begin{aligned}&\widehat{\chi}_{\zeta_{n'},z_{n'}}\rightharpoonup\widehat{\chi}_0\quad\text{weakly in } H^1(Cyls)^3,\\ &\widehat{\chi}_{\zeta_{n'},z_{n'}}\longrightarrow\widehat{\chi}_0\quad\text{strongly in } L^2(Cyls)^3\end{aligned}\qquad\text{and } \widehat{\chi}_{\zeta_{n'},z_{n'}}(X)\longrightarrow\widehat{\chi}_0(X)\quad\text{for a.e. } X\in Cyls. \tag{3.125}$$

First, using the definition of K_z and passing to the limit gives $((a,b)\in\{0,1\}^2)$

$$\left.\begin{aligned} |\widehat{\chi}^{(1,b)}_{0,\alpha} - \widehat{\chi}^{(2,a)}_{0,\alpha}| \le g_\alpha(z)\\ 0 \le (-1)^{a+b}\big(\widehat{\chi}^{(1,b)}_{0,3} - \widehat{\chi}^{(2,a)}_{0,3}\big) \le g_3(z)\end{aligned}\right\}\quad \text{a.e. on } \mathbf{C}_{ab}, \quad \text{and} \quad \int_{Cyls}\widehat{\chi}_0\, dX = 0,$$

which implies that $\widehat{\chi}_0 \in K_z$. Then, from (3.123) one has for all $\widehat{w}_{n'} \in K_{z_{n'}}$ that

$$\begin{aligned}\int_{Cyls}\rho\widetilde{\mathbf{A}}\mathcal{E}_X(\widehat{\chi}_{\zeta_{n'},z_{n'}})\cdot\mathcal{E}_X(\widehat{\chi}_{\zeta_{n'},z_{n'}})|\boldsymbol{\eta}|\,dX \le -\int_{Cyls}\rho\widetilde{\mathbf{A}}\mathcal{E}(\zeta_{n'})\cdot\mathcal{E}_X(\widehat{\chi}_{\zeta_{n'},z_{n'}} - \widehat{w}_{n'})|\boldsymbol{\eta}|\,dX\\ -\int_{Cyls}\rho\widetilde{\mathbf{A}}\mathcal{E}_X(\widehat{v}_{\zeta_{n'},lin})\cdot\mathcal{E}_X(\widehat{\chi}_{\zeta_{n'},z_{n'}} - \widehat{w}_{n'})|\boldsymbol{\eta}|\,dX + \int_{Cyls}\rho\widetilde{\mathbf{A}}\mathcal{E}_X(\widehat{\chi}_{\zeta_{n'},z_{n'}})\cdot\mathcal{E}_X(\widehat{w}_{n'})|\boldsymbol{\eta}|\,dX.\end{aligned}\tag{3.126}$$

Now, for every $\widehat{w} \in K_z$, we build a sequence $\widehat{w}_{n'}$ of admissible test-displacements strongly converging to $\widehat{w}$ in $\widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)}$. Set $(i \in \{1,2,3\},\ \alpha \in \{1,2\})$

$$\Big(\widehat{w}^{(\alpha)}_{n'}\Big)_i = \frac{g_i(z_{n'})}{g_i(z)}\Big(\widehat{w}^{(\alpha)}\Big)_i \quad \text{if } g_i(z) \neq 0, \qquad \Big(\widehat{w}^{(\alpha)}_{n'}\Big)_i = \Big(\widehat{w}^{(\alpha)}\Big)_i \quad \text{if } g_i(z) = 0.$$

Clearly, due to the continuity of g, the sequence $\{(\widehat{w}^{(1)}_{n'}, \widehat{w}^{(2)}_{n'})\}_{n'\in\mathbb{N}}$ strongly converges to $(\widehat{w}^{(1)}, \widehat{w}^{(2)})$ in $\widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)}$. Then, observe that the left-hand side of (3.126) is converging by weak lower semi-continuity of the integral and the weak convergence of $\mathcal{E}_X(\widehat{\chi}_{\zeta_{n'},z_{n'}})$. In the right-hand side we have a sum of integrals with a product of a weakly L^2-convergent term with another one, which converges strongly. Hence, for all $\widehat{w} \in K_z$ one has

$$\begin{aligned}\int_{Cyls}\rho\widetilde{\mathbf{A}}\mathcal{E}_X(\widehat{\chi}_0)\cdot\mathcal{E}_X(\widehat{\chi}_0)|\boldsymbol{\eta}|\,dX &\le \liminf_{n'\to 0}\int_{Cyls}\rho\widetilde{\mathbf{A}}\mathcal{E}_X(\widehat{\chi}_{\zeta_{n'},z_{n'}})\cdot\mathcal{E}_X(\widehat{\chi}_{\zeta_{n'},z_{n'}})|\boldsymbol{\eta}|\,dX\\ &\le \limsup_{n'\to 0}\int_{Cyls}\rho\widetilde{\mathbf{A}}\mathcal{E}_X(\widehat{\chi}_{\zeta_{n'},z_{n'}})\cdot\mathcal{E}_X(\widehat{\chi}_{\zeta_{n'},z_{n'}})|\boldsymbol{\eta}|\,dX\\ &\le -\int_{Cyls}\rho\widetilde{\mathbf{A}}\mathcal{E}(\zeta)\cdot\mathcal{E}_X(\widehat{\chi}_0 - \widehat{w})|\boldsymbol{\eta}|\,dX\\ &\quad -\int_{Cyls}\rho\widetilde{\mathbf{A}}\mathcal{E}_X(\widehat{v}_{\zeta,lin})\cdot\mathcal{E}_X(\widehat{\chi}_0 - \widehat{w})|\boldsymbol{\eta}|\,dX + \int_{Cyls}\rho\widetilde{\mathbf{A}}\mathcal{E}_X(\widehat{\chi}_0)\cdot\mathcal{E}_X(\widehat{w})|\boldsymbol{\eta}|\,dX.\end{aligned}$$

Therefore, the field $\widehat{\chi}_0$ solves the problem (3.123). Recall that

$$\mathcal{E}_X(\widehat{\chi}_{\zeta_{n'},z_{n'}}) \rightharpoonup \mathcal{E}_X(\widehat{\chi}_0) \quad \text{weakly in } L^2(Cyls)^6.$$

Due to the uniqueness of the strain tensor of the solution to problem (3.123), one has $\mathcal{E}_X(\widehat{\chi}_0) = \mathcal{E}_X(\widehat{\chi}_{\zeta,z})$. As a consequence the whole sequence $\{\mathcal{E}_X(\widehat{\chi}_{\zeta_n,z_n})\}_{n\in\mathbb{N}^*}$ converges to $\mathcal{E}_X(\widehat{\chi}_0) = \mathcal{E}_X(\widehat{\chi}_{\zeta,z})$. That gives the continuity of the map $(\zeta,z) \in \mathbb{R}^6 \times \overline{\Omega} \longmapsto \mathcal{E}_X(\widehat{\chi}_{\zeta,z})$. $\square$

Remark 3.8.5. *Denote* $\widehat{v}_\zeta$ *the solution of* (3.119) *with* $g_i = 1$, $i = 1,2,3$. *Then consider the variational inequality* (3.119) *with* $g_1 = g_2 = g_3 = G(z)$. *In this case one has*

$$\mathcal{E}_X(\widehat{\chi}_{\zeta,z}) = G(z)\mathcal{E}_X(\widehat{\chi}_{\zeta/G(z)}).$$

Proposition 3.8.6. *Under the assumptions of Theorem 3.8.3, the function A^{hom} defined by ($n \in \{1, \dots, 6\}$)*

$$A^{hom}_n(z,\zeta) = \int_{Cyls} \rho \widetilde{\mathbf{A}}\, [\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v}_{\zeta,z})] \cdot \big(\mathcal{E}(\mathbf{e}_n) + \mathcal{E}_X(\widehat{\chi}_n)\big)|\boldsymbol{\eta}|\, dX, \tag{3.127}$$

with $\widehat{v}_{\zeta,z}$ the solution of problem (3.119) *is of Caratheodory type and monotone.*

Proof. First note that from Lemma 3.8.4 the map

$$(\zeta, z) \in \mathbb{R}^6 \times \overline{\Omega} \longmapsto \mathcal{E}_X(\widehat{v}_{\zeta,z}) \in \mathcal{C}(\mathbb{R}^6 \times \overline{\Omega}; L^2(Cyls))^6$$

is continuous. Hence, the map $(\zeta, z) \in \mathbb{R}^6 \times \overline{\Omega} \longmapsto A^{hom}(\zeta, z) \in \mathcal{C}(\mathbb{R}^6 \times \overline{\Omega}; \mathbb{R}^6)$ is continuous. Moreover, due to (3.122) and $(3.124)_1$, it satisfies

$$|A^{hom}(z,\zeta)| \le C|\zeta| \quad \text{for every } (z,\zeta) \in \overline{\Omega} \times \mathbb{R}^6. \tag{3.128}$$

Monotonicity is easily shown by

$$\begin{aligned}
&\big(A^{hom}(z,\zeta) - A^{hom}(z,\xi)\big) \cdot (\zeta - \xi) \\
&= \int_{Cyls} \rho \widetilde{\mathbf{A}}\, [\mathcal{E}(\zeta - \xi) + \mathcal{E}_X(\widehat{v}_{\zeta,z} - \widehat{v}_{\xi,z})] \ \cdot\ [\mathcal{E}(\zeta - \xi) + \mathcal{E}_X(\widehat{v}_{\zeta,z} - \widehat{v}_{\xi,z})]\, |\boldsymbol{\eta}|\, dX \\
&\quad + \int_{Cyls} \rho \widetilde{\mathbf{A}}\, [\mathcal{E}(\zeta - \xi) + \mathcal{E}_X(\widehat{v}_{\zeta,z} - \widehat{v}_{\xi,z})] \ \cdot\ \mathcal{E}_X(\widehat{\chi}_{\xi,z} - \widehat{\chi}_{\zeta,z})|\boldsymbol{\eta}|\, dX.
\end{aligned}$$

The last integral is non-negative by problem (3.123) and the first one by coercivity of the matrix A. Hence, using the above Lemma 3.8.7 we arrive at

$$\big(A^{hom}(z,\zeta) - A^{hom}(z,\xi)\big) \cdot (\zeta - \xi) \ge C \int_{Cyls} |\mathcal{E}(\zeta - \xi) + \mathcal{E}_X(\widehat{v}_{\zeta,z} - \widehat{v}_{\xi,z})|^2 \ dX \ge 0 \tag{3.129}$$

with constants independent of ζ, ξ, z and $C > 0$. □

Lemma 3.8.7. *There exist two constant C_1, $C' > 0$ such that*

$$\forall (z,\zeta) \in \Omega \times \mathbb{R}^9, \qquad |\zeta| \ge C_1 \|g\|_{L^\infty(\Omega)} \implies A^{hom}(z,\zeta) \cdot \zeta \ge C'|\zeta|^2.$$

Proof. Step 1. In this step we show that there exists a constant $C_0 > 0$ such that, if the equation

$$\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v}) = 0$$

admits a solution in $K_{\zeta,z}$, $(\zeta, z) \in \mathbb{R}^6 \times \overline{\Omega}$, then $|\zeta| \le C_0 \|g\|_{L^\infty(\Omega)}$.
The solution of the above equation is given by

$$\begin{aligned}
\widehat{v}^{(1,b)} &= \mathcal{A}^{(1,b)} + \mathcal{B}^{(1,b)} \wedge \big((X_2 - b)\mathbf{e}_2 + X_3 \mathbf{n}(X_1)\big), \\
\widehat{v}^{(2,a)} &= \mathcal{A}^{(2,a)} + \mathcal{B}^{(2,a)} \wedge \big((X_1 - a)\mathbf{e}_1 + X_3 \mathbf{n}(X_2)\big)
\end{aligned}$$

with

$$\mathcal{B}^{(1,b)}(X_1) = \mathbf{b}^{(1,b)} - (X_1 - 1)\begin{pmatrix}\zeta_6\\ -\zeta_4\\ 0\end{pmatrix}, \qquad \mathcal{B}^{(2,a)}(X_1) = \mathbf{b}^{(2,a)} - (X_2 - 1)\begin{pmatrix}\zeta_5\\ -\zeta_6\\ 0\end{pmatrix},$$

$$\mathcal{A}^{(1,b)}(X_1) = \mathbf{a}^{(1,b)} + (X_1 - 1)\left[\mathbf{b}^{(1,b)} \wedge \mathbf{e}_1 - \begin{pmatrix}\zeta_1\\ \zeta_2\\ 0\end{pmatrix}\right] - \frac{1}{2}(X_1 - 1)^2\begin{pmatrix}\zeta_6\\ -\zeta_4\\ 0\end{pmatrix} \wedge \mathbf{e}_1 - \Phi^{(1,b)}(X_1)\mathcal{B}^{(1,b)}(X_1) \wedge \mathbf{e}_3,$$

$$\mathcal{A}^{(2,a)}(X_2) = \mathbf{a}^{(2,a)} + (X_2 - 1)\left[\mathbf{b}^{(2,a)} \wedge \mathbf{e}_2 - \begin{pmatrix}\zeta_2\\ \zeta_3\\ 0\end{pmatrix}\right] - \frac{1}{2}(X_2 - 1)^2\begin{pmatrix}\zeta_5\\ -\zeta_6\\ 0\end{pmatrix} \wedge \mathbf{e}_2 - \Phi^{(2,a)}(X_2)\mathcal{B}^{(2,a)}(X_2) \wedge \mathbf{e}_3,$$

where $\mathbf{b}^{(1,b)}$, $\mathbf{a}^{(1,b)}$, $\mathbf{b}^{(2,a)}$, $\mathbf{a}^{(2,a)}$ belong to $\mathbb{R}^3$.
First, note that the functions $X_1 \longmapsto (X_1 - 1)^2$ and $X_2 \longmapsto (X_2 - 1)^2$ can be extended in 2-periodic functions. Then, the periodicity of $\mathcal{A}^{(1,b)}$ and $\mathcal{B}^{(1,b)}$ (resp. $\mathcal{A}^{(2,a)}$ and $\mathcal{B}^{(2,a)}$) with respect to X_1 (resp. X_2) yields $\zeta_1 = \zeta_3 = \zeta_4 = \zeta_5 = \zeta_6 = 0$ and

$$\mathbf{b}^{(1,b)} = \begin{pmatrix}\mathbf{b}_1^{(1,b)}\\ 0\\ \zeta_2\end{pmatrix}, \quad \mathbf{b}^{(2,a)} = \begin{pmatrix}0\\ \mathbf{b}_2^{(2,a)}\\ -\zeta_2\end{pmatrix}.$$

This reduces the displacements tremendously to

$$\widehat{v}^{(1,b)}(X) = \mathbf{a}^{(1,b)} - \begin{pmatrix}\mathbf{b}_1^{(1,b)}\\ 0\\ \zeta_2\end{pmatrix} \wedge \left[\Phi^{(1,b)}(X_1)\mathbf{e}_3 + (X_2 - b)\mathbf{e}_2 + X_3\mathbf{n}^{(1,b)}(X_1)\right],$$

$$\widehat{v}^{(2,a)}(X) = \mathbf{a}^{(2,a)} - \begin{pmatrix}0\\ \mathbf{b}_2^{(2,a)}\\ -\zeta_2\end{pmatrix} \wedge \left[\Phi^{(2,a)}(X_2)\mathbf{e}_3 + (X_1 - a)\mathbf{e}_1 + X_3\mathbf{n}^{(2,a)}(X_2)\right].$$

Then the displacements on the contact parts read as

$$\widehat{v}^{(1,b)}(X) = \mathbf{a}^{(1,b)} - \begin{pmatrix}\mathbf{b}_1^{(1,b)}\\ 0\\ \zeta_2\end{pmatrix} \wedge (X_2 - b)\mathbf{e}_2, \qquad \widehat{v}^{(2,a)}(X) = \mathbf{a}^{(2,a)} - \begin{pmatrix}0\\ \mathbf{b}_2^{(2,a)}\\ -\zeta_2\end{pmatrix} \wedge (X_1 - a)\mathbf{e}_1.$$

Hence,

$$\mathbf{M}_{ab}(\zeta) + \widehat{v}^{(1,b)} - \widehat{v}^{(2,a)} = \mathbf{a}^{(1,b)} - \mathbf{a}^{(2,a)} + \begin{pmatrix}-2(X_2 - b)\zeta_2\\ 2(X_1 - a)\zeta_2\\ (X_2 - b)\mathbf{b}_1^{(1,b)} + (X_1 - a)\mathbf{b}_2^{(2,a)}\end{pmatrix}$$

and thereby $2\kappa|\zeta_2| \leq \|g\|_{L^\infty(\Omega)}$.

Step 2. In this step, we prove by contradiction that there exists a constant $C_1\|g\|_{L^\infty(\Omega)} > 0$ such that for all $(z,\zeta) \in \overline{\Omega} \times \mathbb{R}^9$ and all $\widehat{v} \in K_{\zeta,z}$ it holds that

$$|\zeta| \geq C_1\|g\|_{L^\infty(\Omega)} = 2C_0\|g\|_{L^\infty(\Omega)} \implies \int_{Cyls} \big(\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v})\big)^2\, dX \geq C_1\|g\|_{L^\infty(\Omega)}|\zeta|^2. \tag{3.130}$$

Suppose (3.130) is not satisfied. Then, for every $n \in \mathbb{N}^*$ there exists $(z_n,\zeta_n) \in \overline{\Omega} \times \mathbb{R}^9$ with $|\zeta_n| \geq C_1\|g\|_{L^\infty(\Omega)}$ and $\widehat{v}_n \in K_{\zeta_n,z_n}$ such that

$$\int_{Cyls} \big(\mathcal{E}(\zeta_n) + \mathcal{E}_X(\widehat{v}_n)\big)^2\, dX \leq \frac{1}{n}|\zeta_n|^2, \qquad n \in \mathbb{N}^*. \tag{3.131}$$

• Case 1: a subsequence of $\{|\zeta_n|\}_n$ is bounded. From (3.131) and $(3.113)_3$ the sequence $\{\widehat{v}_n\}_n$ is bounded in $\widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)}$. Then, there exists a subsequence of $\{n\}$ (still denoted $\{n\}$) such that

$$\zeta_n \longrightarrow \zeta, \qquad \widehat{v}_n \rightharpoonup \widehat{v} \quad \text{weakly in} \quad \widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)}.$$

One has $|\zeta| \geq C_1\|g\|_{L^\infty(\Omega)}$. Now, passing to the limit in (3.131) gives

$$\int_{Cyls} \big(\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v})\big)^2\, dX \leq \liminf_{n\to+\infty} \int_{Cyls} \big(\mathcal{E}(\zeta_n) + \mathcal{E}_X(\widehat{v}_n)\big)^2\, dX \leq 0.$$

Hence,

$$\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v}) = 0.$$

Using Step 1, this yields that $|\zeta| \leq \dfrac{C_1\|g\|_{L^\infty(\Omega)}}{2} = C_0\|\mathbf{g}\|_{L^\infty(\Omega)}$, which obviously contradicts the fact that $|\zeta| \geq C_1\|g\|_{L^\infty(\Omega)}$. Hence, $\lim\limits_{n\to+\infty} |\zeta_n| = +\infty$.

• Case 2: $\lim\limits_{n\to+\infty} |\zeta_n| = +\infty$. Set $\zeta'_n = \frac{C_1\|g\|_{L^\infty(\Omega)}\zeta_n}{|\zeta_n|}$ and $\widehat{v}'_n = \frac{C_1\|g\|_{L^\infty(\Omega)}}{|\zeta_n|}\widehat{v}_n$. Then one has $|\zeta'_n| = C_1\|g\|_{L^\infty(\Omega)}$ and from (3.131)

$$\int_{Cyls} \big(\mathcal{E}(\zeta'_n) + \mathcal{E}_X(\widehat{v}'_n)\big)^2\, dX \leq \frac{1}{n}|\zeta'_n|^2 \leq \frac{C_1^2}{n}\|g\|^2_{L^\infty(\Omega)}.$$

Then, proceeding as in the first case one obtains a contradiction and (3.130) is proved.

Step 3. In this step we show

$$\forall (z,\zeta) \in \Omega \times \mathbb{R}^9, \qquad |\zeta| \geq C_1\|g\|_{L^\infty(\Omega)} \implies A^{hom}(z,\zeta)\cdot\zeta \geq C'|\zeta|^2.$$

For every $(z,\zeta) \in \Omega \times \mathbb{R}^9$ such that $|\zeta| \geq C_1\|g\|_{L^\infty(\Omega)}$ one has

$$\begin{aligned}
A^{hom}(z,\zeta)\cdot\zeta &= \int_{Cyls} \rho\widetilde{\mathbf{A}}\left[\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v}_{\zeta,\,lin}) + \mathcal{E}_X(\widehat{\chi}_{\zeta,z})\right]\cdot\big(\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v}_{\zeta,\,lin})\big)|\boldsymbol{\eta}|\, dX \\
&= \int_{Cyls} \rho\widetilde{\mathbf{A}}\left[\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v}_{\zeta,z})\right]\cdot\left[\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v}_{\zeta,z})\right]|\boldsymbol{\eta}|\, dX \\
&\qquad - \int_{Cyls} \rho\widetilde{\mathbf{A}}\left[\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v}_{\zeta,z})\right]\cdot\mathcal{E}_X(\widehat{\chi}_{\xi,z})|\boldsymbol{\eta}|\, dX.
\end{aligned}$$

The last integral is non-negative by problem (3.123). For the first term we apply Step 2 and the coercivity of the matrix A. We conclude thereby that

$$A^{hom}(z,\zeta)\cdot\zeta \geq C'|\zeta|^2.$$

□

Theorem 3.8.8. *Under the assumptions of Theorem 3.8.3 the homogenized problem:*

$$\begin{aligned}&\textit{Find } \mathbb{U}\in[\mathcal{H}^1(\Omega)]^2\times\mathcal{H}^2(\Omega) \textit{ such that:}\\ &\int_\Omega A^{hom}\big(z,\zeta(z)\big)\cdot\xi(z)\,dz = \int_\Omega F(z)\,\mathbb{V}(z)\,dz, \quad \forall\,\mathbb{V}\in[\mathcal{H}^1(\Omega)]^2\times\mathcal{H}^2(\Omega)\end{aligned} \tag{3.132}$$

with

$$\begin{aligned}\zeta &= (e_{11}(\mathbb{U}), e_{12}(\mathbb{U}), e_{22}(\mathbb{U}), \partial_{11}\mathbb{U}_3, \partial_{22}\mathbb{U}_3, \partial_{12}\mathbb{U}_3),\\ \xi &= (e_{11}(\mathbb{V}), e_{12}(\mathbb{V}), e_{22}(\mathbb{V}), \partial_{11}\mathbb{V}_3, \partial_{22}\mathbb{V}_3, \partial_{12}\mathbb{V}_3),\end{aligned}$$

and the nonlinear differential operator ($m\in\{1,\dots,6\}$)

$$A^{hom}_m(\cdot,\zeta) = \int_{Cyls}\rho\widetilde{\mathbf{A}}\left[\mathcal{E}(\zeta)+\mathcal{E}_X(\widehat{v}_{\zeta,z})\right]\,\cdot\,\left[\mathcal{E}(\mathbf{e}_m)+\mathcal{E}_X(\widehat{\chi}_m)\right]|\boldsymbol{\eta}|\,dX, \tag{3.133}$$

admits solutions.

Proof. The solvability of the problem (3.132) is a direct consequence of the Caratheodory-type, monotonicity, coercivity and boundedness (3.128) of the function A^{hom}. □

The operator-structure of the homogenized problem is known as *Leray-Lions*-operator.

3.8.2 The linear case

As seen in the previous section the limit-problem is an overall nonlinear problem due to the contact. In particular, this corresponds to the contact $\mathbf{g}_\varepsilon\sim\varepsilon^3$ but in the case where $\mathbf{g}_\varepsilon=0$ or at least $\mathbf{g}_\varepsilon\sim\varepsilon^{3+\delta}$ with $\delta>0$ the problem reduces to a linear problem in both the microscopic and the macroscopic level. Indeed, in this case the limiting contact-condition degenerates to an equation

$$\mathbf{M}_{ab}(\zeta)+\widehat{u}^{(1,b)}-\widehat{u}^{(1,b)}=0, \tag{3.134}$$

with $\zeta=(e_{11}(\mathbb{U}), e_{12}(\mathbb{U}), e_{22}(\mathbb{U}), \partial_{11}\mathbb{U}_3, \partial_{22}\mathbb{U}_3, \partial_{12}\mathbb{U}_3)$ as above. Thus, we find that the corrector problem (3.119) reduces to (3.121). Hence, all necessary information is already captured by the linear correctors and the nonlinear corrector vanishes, i.e., $\widehat{\chi}_{\zeta,z}=0$. This reduces the homogenized operator to a matrix with the entries

$$A^{hom,lin}_{nm} = \int_{Cyls}\rho\widetilde{\mathbf{A}}\left[\mathcal{E}(\mathbf{e}_n)+\mathcal{E}_X(\widehat{\chi}_n)\right]\cdot\left[\mathcal{E}(\mathbf{e}_m)+\mathcal{E}_X(\widehat{\chi}_m)\right]|\boldsymbol{\eta}|\,dX, \quad m,n\in\{1,\dots,6\} \tag{3.135}$$

and leads to the homogenized problem:

$$\begin{aligned}&\text{Find } \mathbb{U} \in [\mathcal{H}^1(\Omega)]^2 \times \mathcal{H}^2(\Omega) \text{ such that:}\\ &\int_\Omega A^{hom,lin}\zeta \cdot \xi \, dz = \int_\Omega F\,\mathbb{V} \, dz, \quad \forall\, \mathbb{V} \in [\mathcal{H}^1(\Omega)]^2 \times \mathcal{H}^2(\Omega)\end{aligned} \tag{3.136}$$

with

$$\begin{aligned}\zeta &= (e_{11}(\mathbb{U}), e_{12}(\mathbb{U}), e_{22}(\mathbb{U}), \partial_{11}\mathbb{U}_3, \partial_{22}\mathbb{U}_3, \partial_{12}\mathbb{U}_3),\\ \xi &= (e_{11}(\mathbb{V}), e_{12}(\mathbb{V}), e_{22}(\mathbb{V}), \partial_{11}\mathbb{V}_3, \partial_{22}\mathbb{V}_3, \partial_{12}\mathbb{V}_3).\end{aligned}$$

Theorem 3.8.9. *Under the assumptions of 3.8.8 and additionally that the contact satisfies* $\|\mathbf{g}_\varepsilon\| \leq \varepsilon^{3+\delta}$ *with* $\delta > 0$ *the problem* (3.136) *is uniquely solvable.*

Proof. The existence is a direct consequence of Theorem 3.8.8. The uniqueness is a consequence of the coerciveness and the Lax-Milgram-Lemma. □

Chapter 4

Homogenization of the textile in the von-Kármán regime

While the first part considered a geometrical linear setting and yielded a macroscopically linear limit plate, we investigate below the textile in the von-Kármán regime yielding a nonlinear plate model. The von-Kármán plate is widely used by mathematicians and engineers, yet for long time not entirely accepted due to inert assumptions. The reasons are discussed and partially resolved in multiple publications and we want to refer particularly to [12, 14, 20, 21].

For the homogenization of the textile with a von-Kármán plate in the limit a different approach as in chapter 3 is needed. Although nonlinear elasticity models are usually stated with respect to deformations, the von-Kármán plate is still stated with respect to displacements. However, it is necessary to care about additional nonlinear terms arising from the Green-St.Venant strain tensor. To achieve the von-Kármán model in the limit we consider an elastic energy of order $\|e(u)\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{5/2}$, which is in consensus with [4, 20, 21].

For the von-Kármán limit for textile structures we restrict the contact condition introduced in Section 3.3.2 to $g_\varepsilon = 0$. This corresponds to a textile with glued fibers. This assumption is necessary to use the extension operator for the displacements of the structure.

The homogenization starts with the definition of the structure. Similar to Section 3.3.2 a decomposition of plate-displacements is used and provides the basis for the estimates on the different displacement fields. Due to the nonlinear setting some additional estimations with respect to the geometric energy $\|dist(\nabla v, SO(3))\|_{L^2(\Omega_\varepsilon)}$ are introduced. The asymptotic behavior of the displacements are derived with the help of the unfolding-rescaling operator. For the asymptotic behavior of the problem it is important to consider the nonlinear term in the Green-Lagrange strain tensor. With the established convergences the limiting von-Kármán energy is deduced with a Γ-convergence argument. The existence of minimizers for the limiting von-Kármán plate is shown. The uniqueness of solution, though, is not provable. The end of the section is dedicated to the derivation of the homogenized problem and the associated cell problems. Although the initial and the homogenized problem are nonlinear, the cell-problems for the von-Kármán plate are linear and in fact the same as obtained for a linear elastic plate. At the end we show that for isotropic homogeneous fibers the textile is orthotropic.

4.1 The structure

4.1.1 Parametrization of the yarns

Bearing in mind that we want to transfer partial results of chapter 3 we choose the same structure as before, see Section 3.3 for further information. For convenience, the basic definitions are recalled hereafter.

The middle line of a beam is parametrized by rescaled function $\Phi_\varepsilon = \varepsilon\Phi(\frac{z}{\varepsilon})$ of

$$\Phi(z) = \begin{cases} -\kappa, & \text{if } z \in [0,\kappa], \\ \kappa\Big(6\frac{(z-\kappa)^2}{(1-2\kappa)^2} - 4\frac{(z-\kappa)^3}{(1-2\kappa)^3} - 1\Big) & \text{if } z \in [\kappa, 1-\kappa], \\ \kappa & \text{if } z \in [1-\kappa, 1], \\ \Phi(2-z) & \text{if } z \in [1,2], \end{cases} \tag{4.1}$$

with $\kappa \in (0, \frac{1}{3})$. The reference beams in the directions $\mathbf{e}_1$ and $\mathbf{e}_2$ are defined by

$$P_r^{(1)} \doteq \big\{z \in \mathbb{R}^3 \mid z_1 \in (0,L),\ (z_2,z_3) \in \Omega_r\big\}, \quad P_r^{(2)} \doteq \big\{z \in \mathbb{R}^3 \mid z_2 \in (0,L),\ (z_1,z_3) \in \Omega_r\big\}.$$

The curved beams for the textile structure are defined by

$$\mathcal{P}_\varepsilon^{(1,q)} \doteq \Big\{x \in \mathbb{R}^3 \mid x = \psi_\varepsilon^{(1,q)}(z),\ \ z \in P_r^{(1)}\Big\}, \quad \mathcal{P}_\varepsilon^{(2,p)} \doteq \Big\{x \in \mathbb{R}^3 \mid x = \psi_\varepsilon^{(2,p)}(z),\ \ z \in P_r^{(2)}\Big\},$$

with the diffeomorphisms

$$\psi_\varepsilon^{(1,q)}(z) \doteq M_\varepsilon^{(1,q)}(z_1) + z_2\mathbf{e}_2 + z_3\mathbf{n}_\varepsilon^{(1,q)}(z_1), \qquad \psi_\varepsilon^{(2,p)}(z) \doteq M_\varepsilon^{(2,p)}(z_2) + z_1\mathbf{e}_1 + z_3\mathbf{n}_\varepsilon^{(2,p)}(z_2),$$

and the corresponding middle lines

$$M_\varepsilon^{(1,q)}(z_1) \doteq z_1\mathbf{e}_1 + q\varepsilon\mathbf{e}_2 + (-1)^{q+1}\Phi_\varepsilon(z_1)\mathbf{e}_3, \qquad M_\varepsilon^{(2,p)}(z_2) \doteq p\varepsilon\mathbf{e}_1 + z_2\mathbf{e}_2 + (-1)^{p}\Phi_\varepsilon(z_2)\mathbf{e}_3.$$

The periodicity cell of the structure $\mathcal{Y}^* \subset Y = (0,1)^2 \times (-2\kappa, 2\kappa)$ consist of the curved beam-parts within Y, as depicted in figure 4.1.

4.1.2 The complete structure

Denote the whole structure (see chapter 3 for more details) by

$$\Omega_\varepsilon^* \doteq \Omega_\varepsilon \cap \Big(\bigcup_{p=0}^{2N_\varepsilon} \mathcal{P}_\varepsilon^{(1,q)} \cup \bigcup_{q=0}^{2N_\varepsilon} \mathcal{P}_\varepsilon^{(2,p)}\Big), \qquad \Omega_\varepsilon \doteq \omega \times (-2\kappa\varepsilon, 2\kappa\varepsilon), \qquad \omega = (0,L)^2, \tag{4.2}$$

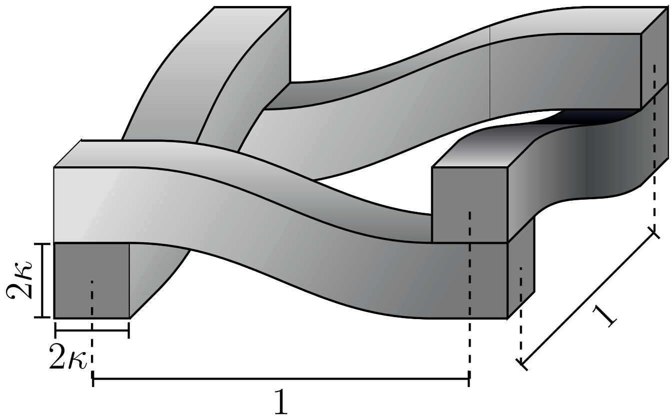


Figure 4.1: The Periodicity cell $\mathcal{Y}^* \subset Y = (0,1)^2 \times (-2\kappa, 2\kappa)$ of the full structure.

4.1.3 An extension result

The extension (see Appendix C) heavily depends on the fact that the beams are glued. For a more general contact condition as in Section 3.3 it is necessary to treat the two directions separately and obtain two deformations, which give the same limit for $g_\varepsilon \sim \varepsilon^4$. Nevertheless, the more general case would exceed the bounds of this work.

Proposition 4.1.1. *For every deformation v in $H^1(\Omega_\varepsilon^*)^3$ there exists a deformation $\widetilde{v}$ in $H^1(\Omega_\varepsilon)^3$ satisfying*

$$\begin{aligned} &\widetilde{v}_{|\Omega_\varepsilon^*} = v, \\ &\big\| dist\big(\widetilde{v}, SO(3)\big) \big\|_{L^2(\Omega_\varepsilon)} \le C \big\| dist(v, SO(3)) \big\|_{L^2(\Omega_\varepsilon^*)}. \end{aligned} \tag{4.3}$$

The constant does not depend on ε.

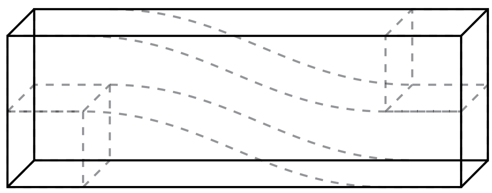


Figure 4.2: First extension domain for one beam segment.

Proof. Now that the general extension for Lipschitz domains in nonlinear elasticity is recalled in Appendix C, we specify the extension procedure for the domain Ω_ε^*.

First, divide the domain Ω_ε^* into portions included in domains isometric to the parallelotope $(0, \varepsilon + 2\kappa\varepsilon) \times (0, 2\kappa\varepsilon) \times (0, 4\kappa\varepsilon)$[1] as depicted in Figure 4.2. These portions include a curved beam and parts of the beams in the perpendicular direction with which the beam is in contact. Besides a rotation and/or a reflection, all the portions are of the same form and itself Lipschitz-domains. Furthermore, note that these portions intersect each other and

[1] We reduce the parallelotopes that are in contact with the boundary of Ω.

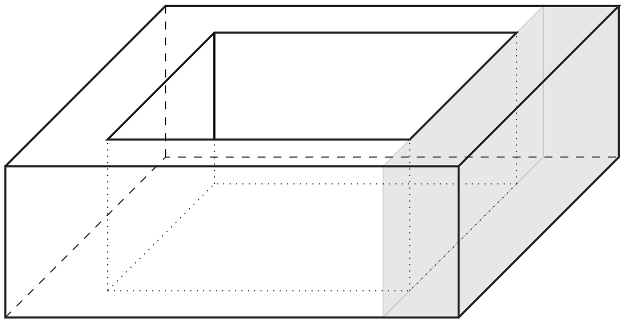


Figure 4.3: Periodicity cell of the periodic plate with holes after the first extension.

every contact cylinder $C_{pq} \times (-2\kappa\varepsilon, 2\kappa\varepsilon)$ with $C_{pq} = (p\varepsilon - \kappa\varepsilon, p\varepsilon + \kappa\varepsilon) \times (q\varepsilon - \kappa\varepsilon, q\varepsilon + \kappa\varepsilon)$ is used in four of such domains.

Since every portion is a Lipschitz domain the extension procedure given in Lemma C.1 is applicable for every $v \in H^1(\Omega_\varepsilon^*)$ and yields an extension to the paling parallelotope, e.g., $(p\varepsilon - \kappa\varepsilon, (p+1)\varepsilon + \kappa\varepsilon) \times (q\varepsilon - \kappa\varepsilon, q\varepsilon + \kappa\varepsilon) \times (-2\kappa\varepsilon, 2\kappa\varepsilon)$. For the second step define new domains included in $(p\varepsilon - \kappa\varepsilon, (p+1)\varepsilon + \kappa\varepsilon) \times (q\varepsilon - \kappa\varepsilon, (q+1)\varepsilon + \kappa\varepsilon) \times (-2\kappa\varepsilon, 2\kappa\varepsilon)$ by collecting four of the above portions as depicted in Figure Figure 4.3. Note that the contact cylinders in every corner of the new domain is used by two portions. Obviously, this domain is again a Lipschitz domain and hence we extend all the fields into the holes using again Lemma C.1.

To obtain the full extension we reassemble the structure. Keep in mind, that the domains $(p\varepsilon - \kappa\varepsilon, (p+1)\varepsilon + \kappa\varepsilon) \times (q\varepsilon - \kappa\varepsilon, (q+1)\varepsilon + \kappa\varepsilon) \times (-2\kappa\varepsilon, 2\kappa\varepsilon)$ have an overlap. This overlap includes every beam twice and the contact cylinders again fourfold, i.e., the overlap consists exactly of the domains before the last extension. Together with the intersections from the step before we obtain that the contact cylinders are the most used domains for the extension, namely eight times. This influences the estimate and finally we give the final extension $\widetilde{v}$ which satisfies

$$\left\|\operatorname{dist}\left(\widetilde{v}, SO(3)\right)\right\|_{L^2(\Omega_\varepsilon)} \leq C\left\|\operatorname{dist}(v, SO(3))\right\|_{L^2(\Omega_\varepsilon^*)},$$

where the constant does not depend on ε. By construction we have $\widetilde{v}_{|\Omega_\varepsilon^*} = v$. □

Henceforth, we use the extended deformation $v \in H^1(\Omega_\varepsilon)$, which is a deformation of a periodic plate without holes. This allows to use the results in the papers [4] and general results of [5, 16, 18] and of chapter 3.

4.1.4 Space of deformations

The structure is clamped on its lateral boundary. Moreover, here we assume a glued contact, which corresponds to the case $g_\varepsilon \equiv 0$ in chapter 3. This allows to obtain one deformation field for the whole structure Ω_ε^* instead of one for each beam.

Denote

$$\gamma = \partial\Omega \cap \{x_2 = 0\} = (0, L) \times \{0\}, \qquad \Gamma_\varepsilon = \gamma \times (-2\kappa\varepsilon, 2\kappa\varepsilon).$$

The set of the admissible deformations are

$$\begin{aligned} \mathbf{V}_\varepsilon &\doteq \Big\{ v \in H^1(\Omega_\varepsilon^*)^3 \ \mid \ \text{such that} \ \ v = I_d \quad \text{a.e. on } \partial\Omega_\varepsilon^* \cap \Gamma_\varepsilon \Big\}, \\ \mathbf{D}_\varepsilon &\doteq \Big\{ v \in H^1(\Omega_\varepsilon)^3 \ \mid \ \text{such that} \ \ v = I_d \quad \text{a.e. on } \Gamma_\varepsilon \Big\}. \end{aligned} \tag{4.4}$$

Remark 4.1.2. *Every deformation belonging to* $\mathbf{V}_\varepsilon$ *is extended into the domain* $(0, L) \times (-\kappa\varepsilon, 0) \times (-2\kappa\varepsilon, 2\kappa\varepsilon)$ *by setting* $v = I_d$ *in this open set. Then, Proposition 4.1.1 gives an extension of* v *whose restriction to* Ω_ε *belongs to* $\mathbf{D}_\varepsilon$ *and satisfies* (4.3).

4.2 The non-linear elasticity problem

The deformations and the terms of their decompositions are estimated in terms of the geometric energy $\|\text{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\varepsilon^*)}$.

Denote $\widehat{W}$ the local elastic energy density, then the total elastic energy is given by

$$\mathcal{J}_\varepsilon(v) = \int_{\Omega_\varepsilon^*} \widehat{W}_\varepsilon\big(\cdot, \nabla v\big)\, dx - \int_{\Omega_\varepsilon^*} f_\varepsilon \cdot (v - I_d)\, dx, \qquad \text{if } \det(\nabla v) > 0,$$

where I_d is the identity map. The local density energy $\widehat{W} \ : \ \mathcal{Y}^* \times \mathbf{S}_3 \longrightarrow \mathbb{R}^+ \cup \{+\infty\}$ is assumed to be

$$\widehat{W}_\varepsilon(\cdot, F) = \begin{cases} Q\Big(\dfrac{\cdot}{\varepsilon}, \dfrac{1}{2}(F^T F - \mathbf{I}_3)\Big) & \text{if} \quad \det(F) > 0, \\ +\infty & \text{if} \quad \det(F) \le 0, \end{cases}$$

where $\mathbf{S}_3$ is the space of symmetric real-valued 3×3-matrices. The quadratic form Q is defined by

$$Q(y, S) = a_{ijkl}(y) S_{ij} S_{kl} \quad \text{for a.e. } y \in \mathcal{Y}^* \text{ and for all } S \in \mathbf{S}_3,$$

where for $(i, j, k, l) \in \{1, 2, 3\}^4$ the a_{ijkl}'s belong to $L^\infty(\mathcal{Y}^*)$ and are periodic with respect to $\mathbf{e}_1$ and $\mathbf{e}_2$.
Moreover, the tensor a is symmetric, i.e., $a_{ijkl} = a_{jikl} = a_{klji}$. Also it is positive definite and satisfies

$$\exists c_0 > 0, \text{ such that} \quad c_0\, S_{ij} S_{ij} \le a_{ijkl}(y) S_{ij} S_{kl} \quad \text{for a.e. } y \in \mathcal{Y}^* \text{ and for all } S \in \mathbf{S}_3. \tag{4.5}$$

Note that the energy density

$$\widehat{W}_\varepsilon(x, \nabla v(x)) = \begin{cases} Q\Big(\dfrac{x}{\varepsilon}, E(v)(x)\Big) & \text{if} \quad \det(\nabla v(x)) > 0, \\ +\infty & \text{if} \quad \det(\nabla v(x)) \le 0, \end{cases} \qquad \text{for a.e. } x \in \Omega_\varepsilon^*$$

depends on the strain tensor $E(v) = \dfrac{1}{2}\big((\nabla v)^T \nabla v - \mathbf{I}_3\big)$ with $\mathbf{I}_3$ the unit 3×3 matrix.

Remark 4.2.1. *As a classical example of a local elastic energy satisfying the above assumptions, we mention the following St Venant-Kirchhoff's law with Lamé constants* λ *and* μ *for* *which*

$$\widehat{W}(F) = \begin{cases} \dfrac{\lambda}{8}\big(tr(F^TF - \mathbf{I}_3)\big)^2 + \dfrac{\mu}{4}tr\big((F^TF - \mathbf{I}_3)^2\big) & \textit{if} \quad \det(F) > 0, \\ +\infty & \textit{if} \quad \det(F) \le 0. \end{cases}$$

Now we are in the position to state the problem and we set

$$m_\varepsilon = \inf_{v \in \mathbf{V}_\varepsilon} J_\varepsilon(v)^2.$$

4.3 Preliminary estimates

4.3.1 Recalls about the plate deformations

Denote the in-plane variables by $x' = (x_1, x_2) \in \mathbb{R}^2$ and the space of displacements by

$$\mathbf{U}_\varepsilon \doteq \Big\{ u \in H^1(\Omega_\varepsilon)^3 \;\; | \;\; u = 0 \;\text{ a.e. on } \Gamma_\varepsilon \Big\}.$$

Lemma 4.3.1. *Let $v \in \mathbf{V}_\varepsilon$ be a deformation and $\widetilde{v} \in \mathbf{D}_\varepsilon$ the extended deformation given by Proposition 4.1.1 and Remark 4.1.2. The associated displacement $u = \widetilde{v} - I_d$ belongs to $\mathbf{U}_\varepsilon$ and satisfies*

$$\|e(u)\|_{L^2(\Omega_\varepsilon)} \le C_0 \|dist(\nabla v, SO(3))\|_{L^2(\Omega^*_\varepsilon)} + \frac{C_1}{\varepsilon^{5/2}} \|dist(\nabla v, SO(3))\|^2_{L^2(\Omega^*_\varepsilon)} \tag{4.6}$$

The constants do not depend on ε and v (they depend only on Ω, $\mathcal{Y}^*$ and κ).

Proof. In [4, Lemma 4.3] it is proved that there exists a constant which does not depend on ε and $\widetilde{v}$ such that

$$\|e(u)\|_{L^2(\Omega_\varepsilon)} \le C\|dist(\nabla\widetilde{v}, SO(3))\|_{L^2(\Omega_\varepsilon)}\Big(1 + \frac{1}{\varepsilon^{5/2}}\|dist(\nabla\widetilde{v}, SO(3))\|_{L^2(\Omega_\varepsilon)}\Big).$$

Then, Proposition 4.1.1 gives a constant which does not depend on ε and v such that

$$\big\|\text{dist}\big(\widetilde{v}, SO(3)\big)\big\|_{L^2(\Omega_\varepsilon)} \le C\big\|\text{dist}(v, SO(3))\big\|_{L^2(\Omega^*_\varepsilon)}.$$

This ends the proof of the lemma. □

Remark 4.3.2. *In fact, it is possible to estimate the structure on the level of beams in contact as in chapter 3 but in the context of geometric nonlinear beams, i.e., with the decomposition*

[2] It is well known that the existence of a minimizer for $\mathcal{J}_\varepsilon$ is still an open problem.

of deformations. This different approach yields the estimation

$$\|e(u)\|^2_{L^2(\Omega^*_\varepsilon)} \le C\Big(1+\frac{\varepsilon^2}{r^2}\Big)\|dist(\nabla v, SO(3))\|^2_{L^2(\Omega_\varepsilon)} + C\frac{\varepsilon}{r^6}\|dist(\nabla v, SO(3))\|^4_{L^2(\Omega_\varepsilon)} + C\frac{1}{r^2\varepsilon^5}\|g_\varepsilon\|^4_{L^2(\Omega)}, \quad (4.7)$$

where $u = v - I_d$ is the associated displacement. However, the proof is more involved, since it relies on deformations and their decomposition and the distinction between every beam.

Note that from this it is easily deduced that there is a restriction on the contact condition $\|g_\varepsilon\|_{L^2(\Omega)} \le C\varepsilon^3$ to obtain an energy in the von-Kármán regime $\|e(u)\|^2_{L^2(\Omega_\varepsilon)} \le C\varepsilon^5$. The considered glued contact with $g_\varepsilon \equiv 0$ obviously satisfies this constraint and yields for $r = \kappa\varepsilon$ the same estimation as (4.6).

4.3.2 Recalls about the plate displacements

Set

$$H^1_\gamma(\Omega) \doteq \big\{\phi \in H^1(\Omega) \mid \phi = 0 \ \text{ a.e. on } \ \gamma\big\},$$
$$H^2_\gamma(\Omega) \doteq \big\{\phi \in H^2(\Omega) \mid \phi = 0, \ \ \nabla\phi = 0 \text{ a.e. on } \ \gamma\big\}.$$

Below we recall a definition from [16, chapter 11] (see also [24, 28]).

Definition 4.3.3. *Elementary displacements are elements u_e of $H^1(\Omega_\varepsilon)^3$ satisfying for a.e. $x = (x', x_3) \in \Omega_\varepsilon$ (where $x' \in \Omega$)*

$$\begin{aligned} u_{e,1}(x) &= \mathcal{U}_1(x') + x_3\mathcal{R}_1(x'),\\ u_{e,2}(x) &= \mathcal{U}_2(x') + x_3\mathcal{R}_2(x'),\\ u_{e,3}(x) &= \mathcal{U}_3(x'). \end{aligned}$$

Here

$$\mathcal{U} = (\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3) \in H^1(\Omega)^3 \quad \text{and} \quad \mathcal{R} = \mathcal{R}_1\mathbf{e}_1 + \mathcal{R}_2\mathbf{e}_2 \in H^1(\Omega)^2.$$

The following Lemma is proved in [16, Theorem 11.4 and Proposition 11.6].

Lemma 4.3.4. *Let u be in $\mathbf{U}_\varepsilon$. The displacement u can be decomposed as the sum*

$$u = u_e + \overline{u} \quad (4.8)$$

of an elementary displacement u_e and a residual displacement $\overline{u}$, both belonging to $\mathbf{U}_\varepsilon$ and satisfying

$$\mathcal{U} \in H^1_\gamma(\Omega)^3, \quad \mathcal{R} \in H^1_\gamma(\Omega)^2, \quad \|\overline{u}\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|\nabla\overline{u}\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon\|e(u)\|_{L^2(\Omega_\varepsilon)}. \quad (4.9)$$

Moreover, one has

$$
\begin{aligned}
&\|\mathcal{U}_\alpha\|_{H^1(\Omega)} + \varepsilon\big(\|\mathcal{U}_3\|_{H^1(\Omega)} + \|\mathcal{R}\|_{H^1(\Omega)}\big) \le \frac{C}{\varepsilon^{1/2}}\|e(u)\|_{L^2(\Omega_\varepsilon)},\\
&\big\|\partial_\alpha \mathcal{U}_3 + \mathcal{R}_\alpha\big\|_{L^2(\Omega)} \le \frac{C}{\varepsilon^{1/2}}\|e(u)\|_{L^2(\Omega_\varepsilon)},\\
&\|u_\alpha\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|u_3\|_{L^2(\Omega_\varepsilon)} \le C\|e(u)\|_{L^2(\Omega_\varepsilon)},\\
&\sum_{\alpha,\beta=1}^{2}\Big\|\frac{\partial u_\alpha}{\partial x_\beta}\Big\|_{L^2(\Omega_\varepsilon)} + \Big\|\frac{\partial u_3}{\partial x_3}\Big\|_{L^2(\Omega_\varepsilon)} \le C\|e(u)\|_{L^2(\Omega_\varepsilon)},\\
&\sum_{\alpha=1}^{2}\Big(\Big\|\frac{\partial u_\alpha}{\partial x_3}\Big\|_{L^2(\Omega_\varepsilon)} + \Big\|\frac{\partial u_3}{\partial x_\alpha}\Big\|_{L^2(\Omega_\varepsilon)}\Big) \le \frac{C}{\varepsilon}\|e(u)\|_{L^2(\Omega_\varepsilon)}.
\end{aligned}
\tag{4.10}
$$

The constants do not depend on ε.

4.3.3 Assumptions on the forces

The forces have to admit a certain scaling with respect to the ε-scaling of the domain. For the textile we require forces of the type

$$
\begin{aligned}
&f_{\varepsilon,1} = \varepsilon^2 f_1,\\
&f_{\varepsilon,2} = \varepsilon^2 f_2, \qquad \text{a.e. in } \Omega_\varepsilon^*,\\
&f_{\varepsilon,3} = \varepsilon^3 f_3,
\end{aligned}
\tag{4.11}
$$

with $f \in L^2(\Omega)^3$. In order to obtain a von-Kármán model at the limit, the applied forces must satisfy the condition

$$
\|f\|_{L^2(\Omega)} \le C^*. \tag{4.12}
$$

This constant depends on the reference cell $\mathcal{Y}^*$, the mid-surface Ω of the structure and the local elastic energy W (see Lemma 4.3.5).

The scaling of the force gives rise to the order of the energy in the elasticity problem. This is proven in the Lemma below.

Lemma 4.3.5. *Let $v \in \mathbf{V}_\varepsilon$ be a deformation such that $J_\varepsilon(v) \le 0$. Assume (4.11) on the forces. There exists a constant C^* independent of ε and the applied forces such that, if $\|f\|_{L^2(\Omega)} < C^*$, one has*

$$
\|dist(\nabla v, SO(3))\|_{L^2(\Omega_\varepsilon^*)} \le C\varepsilon^{5/2}\|f\|_{L^2(\Omega)}.
$$

The constant C does not depend on ε.

Proof. Using (4.5) gives rise to the estimation

$$
c_0\|dist(\nabla v, SO(3))\|^2_{L^2(\Omega_\varepsilon^*)} \le \Big|\int_{\Omega_\varepsilon^*} f_\varepsilon \cdot (v - I_d)\, dx\Big|. \tag{4.13}
$$

Introduce $u = \widetilde{v} - I_d \in \mathbf{U}_\varepsilon$ the associated displacement to the extended deformation (see Lemma 4.3.1). Then, with (4.11) and the estimates $(4.10)_3$ we obtain

$$\begin{aligned}\Big|\int_{\Omega_\varepsilon^*} f_\varepsilon \cdot (v - I_d)\, dx\Big| &\le \varepsilon^{5/2}\|f_\alpha\|_{L^2(\Omega)}\|u_\alpha\|_{L^2(\Omega_\varepsilon^*)} + \varepsilon^{7/2}\|f_3\|_{L^2(\Omega)}\|u_3\|_{L^2(\Omega_\varepsilon^*)} \\ &\le \varepsilon^{5/2}\|f_\alpha\|_{L^2(\Omega)}\|u_\alpha\|_{L^2(\Omega_\varepsilon)} + \varepsilon^{7/2}\|f_3\|_{L^2(\Omega)}\|u_3\|_{L^2(\Omega_\varepsilon)} \\ &\le C_2\varepsilon^{5/2}\|f\|_{L^2(\Omega)}\|e(u)\|_{L^2(\Omega_\varepsilon)}.\end{aligned} \tag{4.14}$$

Eventually, the above inequality with (4.13) and Lemma 4.3.1 give

$$\begin{aligned}c_0\|dist(\nabla v, SO(3))\|^2_{L^2(\Omega_\varepsilon^*)} \le C_2C_0\varepsilon^{5/2}\|f\|_{L^2(\Omega)}\|dist(\nabla v, SO(3))\|_{L^2(\Omega_\varepsilon^*)} \\ + C_2C_1\|f\|_{L^2(\Omega)}\|dist(\nabla v, SO(3))\|^2_{L^2(\Omega_\varepsilon^*)}.\end{aligned}$$

If $C_2C_1\|f\|_{L^2(\Omega)} < c_0$, then

$$\|dist(\nabla v, SO(3))\|_{L^2(\Omega_\varepsilon^*)} \le \frac{C_2\varepsilon^{5/2}}{c_0 - C_2C_1\|f\|_{L^2(\Omega)}}\|f\|_{L^2(\Omega)}.$$

□

Now, if the deformation $v \in \mathbf{V}_\varepsilon$ satisfies $\mathcal{J}_\varepsilon(v) \le 0$, it is possible to give a lower bound for the infimum of the functional $\mathcal{J}_\varepsilon$. With the assumptions (4.5) on the problem and (4.11)-(4.12) on the forces together with the Lemmas 4.3.1-4.3.5 and inequality (4.14) the energy is estimated by

$$c_0\|(\nabla v)^T\nabla v - \mathbf{I}_3\|^2_{L^2(\Omega_\varepsilon^*)} \le \int_{\Omega_\varepsilon^*} \widehat{W}(\nabla v)dx \le \int_{\Omega_\varepsilon^*} f_\varepsilon \cdot (v - I_d)\, dx \le C\varepsilon^5\|f\|^2_{L^2(\Omega)}. \tag{4.15}$$

As a consequence, there exists a constant c independent of ε such that

$$-c\varepsilon^5 \le \mathcal{J}_\varepsilon(v) \le 0 \tag{4.16}$$

Recalling that $m_\varepsilon = \inf_{v\in\mathbf{V}_\varepsilon} J_\varepsilon(v)$ yields

$$-c \le \frac{m_\varepsilon}{\varepsilon^5} \le 0. \tag{4.17}$$

Our aim is to give the asymptotic behavior of the rescaled sequence $\left\{\frac{m_\varepsilon}{\varepsilon^5}\right\}_\varepsilon$ and to characterize its limit as the minimum of a functional.

4.4 Asymptotic behavior

In this section, we consider a sequence $\{v_\varepsilon\}_\varepsilon$ of deformations satisfying

$$\|dist(\nabla v_\varepsilon, SO(3))\|_{L^2(\Omega_\varepsilon^*)} \le C\varepsilon^{5/2}. \tag{4.18}$$

Hereafter, we are interested in the asymptotic behavior of the sequence of displacements $\{u_\varepsilon\}_\varepsilon = \{v_\varepsilon - I_d\}_\varepsilon$, where u_ε is the associated displacement to the extended deformation v_ε.

From Lemma 4.3.1, one has

$$\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{5/2}. \tag{4.19}$$

Recalling the estimates of Lemma 4.3.4 with this assumption we have

$$\begin{aligned}
&\|\overline{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|\nabla\overline{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{7/2},\\
&\left\|\mathcal{U}_{\varepsilon,3}\right\|_{H^1(\Omega)} + \left\|\mathcal{R}_\varepsilon\right\|_{H^1(\Omega)} \leq C\varepsilon,\\
&\left\|\mathcal{U}_{\varepsilon,\alpha}\right\|_{H^1(\Omega)} \leq C\varepsilon^2,\\
&\left\|\partial_\alpha\mathcal{U}_{\varepsilon,3} + \mathcal{R}_{\varepsilon,\alpha}\right\|_{L^2(\Omega)} \leq C\varepsilon^2.
\end{aligned} \tag{4.20}$$

The constants do not depend on ε.

Lemma 4.4.1 (See [4, Section 7])**.** *Under the assumptions of Lemma 4.3.4, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, $\mathcal{U}_3 \in H^2_\gamma(\Omega)$ and $\mathcal{U}_\alpha,\ \mathcal{R}_\alpha \in H^1_\gamma(\Omega)$ $(\alpha \in \{1,2\})$ such that*

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathcal{U}_{\varepsilon,3} \to \mathcal{U}_3 \quad \textit{strongly in } H^1_\gamma(\Omega),\\
&\frac{1}{\varepsilon}\mathcal{R}_{\varepsilon,\alpha} \rightharpoonup \mathcal{R}_\alpha \quad \textit{weakly in } H^1_\gamma(\Omega), \ \ \textit{and strongly in } L^4(\Omega)\\
&\frac{1}{\varepsilon^2}\mathcal{U}_{\varepsilon,\alpha} \rightharpoonup \mathcal{U}_\alpha \quad \textit{weakly in } H^1_\gamma(\Omega),\\
&\frac{1}{\varepsilon^2}\big(\partial_\alpha\mathcal{U}_{\varepsilon,3} + \mathcal{R}_{\varepsilon,\alpha}\big) \rightharpoonup \mathcal{Z}_\alpha \quad \textit{weakly in } L^2(\Omega).
\end{aligned} \tag{4.21}$$

Moreover, one has

$$\partial_\alpha\mathcal{U}_3 + \mathcal{R}_\alpha = 0. \tag{4.22}$$

Proof. The convergences and equalities are easy consequences of the above estimates (4.20). The identity (4.22) comes from $(4.21)_{2,4}$ and the weak convergence of $(4.21)_1$. To see the strong convergence of $(4.21)_1$ note that $(4.10)_{1,2}$ and the strong convergence of $\frac{1}{\varepsilon}\mathcal{R}_{\varepsilon,\alpha}$ in $L^2(\Omega)$ imply

$$\frac{1}{\varepsilon}\partial_\alpha\mathcal{U}_{\varepsilon,3} = \frac{1}{\varepsilon}(\partial_\alpha\mathcal{U}_{\varepsilon,3} + \mathcal{R}_{\varepsilon,\alpha}) - \frac{1}{\varepsilon}\mathcal{R}_{\varepsilon,\alpha} \to 0 - \mathcal{R}_\alpha = \partial_\alpha\mathcal{U}_3, \qquad \text{strongly in } L^2(\Omega),$$

where the last equality in the comes from (4.22). □

4.4.1 The unfolding and rescaling operators

For the asymptotic behavior we introduce two operators: $\mathcal{T}_\varepsilon$ for the homogenization in Ω and $\mathfrak{T}_\varepsilon$ for the homogenization and dimension reduction in Ω^*. Both operators can be found in [16, 28] thus we recall here only the important properties for this thesis. Denote

$$\mathcal{Y}' := (0,2)^2, \qquad \mathcal{Y} := (0,2)^2 \times (-2\kappa, 2\kappa).$$

Let $\mathcal{Y}^* \subset \mathcal{Y}$ be the reference cell. The cell $\mathcal{Y}^*$ is deduced from Y^* (see Figure 4.1) after two symmetries with respect to the planes $y_1 = 1$ and $y_2 = 1$ the cell consisting of the beam structure.

Definition 4.4.2. *For every measurable function $\phi \in L^1(\Omega)$ we recall the definition of the unfolded function $\mathcal{T}_\varepsilon(\phi) \in L^1(\Omega \times \mathcal{Y}')$*

$$\mathcal{T}_\varepsilon(\phi)(x', y') = \phi\Big(\varepsilon\Big[\frac{x'}{\varepsilon}\Big] + \varepsilon y'\Big) \qquad \textit{for a.e. } (x', y') \in \Omega \times \mathcal{Y}'.$$

For every measurable function $\psi \in L^1(\Omega^*)$ the unfolding and the rescaling operator $\mathfrak{T}_\varepsilon$ is defined by

$$\mathfrak{T}_\varepsilon(\psi)(x', y', y_3) = \psi\Big(\varepsilon\Big[\frac{x'}{\varepsilon}\Big] + \varepsilon y', \varepsilon y_3\Big) \qquad \textit{for a.e. } (x', y) \in \Omega \times \mathcal{Y}^*.$$

Lemma 4.4.3. *There exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, $\widehat{\mathcal{U}}_\alpha, \widehat{\mathcal{R}}_\alpha \in L^2(\Omega; H^1_{per}(\mathcal{Y}'))$ and $\mathfrak{u} \in L^2(\Omega; H^1_{per}(\mathcal{Y}'))$ such that*

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon(\nabla \mathcal{U}_{\varepsilon,3}) \longrightarrow \nabla \mathcal{U}_3 \qquad \textit{strongly in } L^2(\Omega \times \mathcal{Y}')^2,\\
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon(\mathcal{R}_\varepsilon) \longrightarrow \mathcal{R} \qquad \textit{strongly in } L^2(\Omega \times \mathcal{Y}')^2,\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon(\nabla \mathcal{R}_{\varepsilon,\alpha}) \rightharpoonup \nabla \mathcal{R}_\alpha + \nabla_y \widehat{\mathcal{R}}_\alpha \qquad \textit{weakly in } L^2(\Omega \times \mathcal{Y}')^2,\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon(\nabla \mathcal{U}_{\varepsilon,\alpha}) \rightharpoonup \nabla \mathcal{U}_\alpha + \nabla_y \widehat{\mathcal{U}}_\alpha \qquad \textit{weakly in } L^2(\Omega \times \mathcal{Y}')^2,\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon(\partial_\alpha \mathcal{U}_{\varepsilon,3} + \mathcal{R}_{\varepsilon,\alpha}) \rightharpoonup \mathcal{Z}_\alpha + \nabla_{y_\alpha} \mathfrak{u} + \widehat{\mathcal{R}} \qquad \textit{weakly in } L^2(\Omega \times \mathcal{Y}').
\end{aligned} \tag{4.23}$$

Moreover, there exists $\overline{u} \in L^2(\Omega; H^1_{per}(\mathcal{Y}^*))^3$ such that

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathfrak{T}_\varepsilon(\overline{u}_\varepsilon) \rightharpoonup \overline{u} \qquad \textit{weakly in } L^2(\Omega; H^1(\mathcal{Y}^*)),\\
&\frac{1}{\varepsilon^2}\mathfrak{T}_\varepsilon(\nabla \overline{u}_\varepsilon) \rightharpoonup \nabla_y \overline{u} \qquad \textit{weakly in } L^2(\Omega \times \mathcal{Y}^*)^9.
\end{aligned} \tag{4.24}$$

Furthermore, one has

$$\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\nabla u_\varepsilon (\nabla u_\varepsilon)^T\big) \longrightarrow \begin{pmatrix} \partial_1\mathcal{U}_3\partial_1\mathcal{U}_3 & \partial_1\mathcal{U}_3\partial_2\mathcal{U}_3 & 0\\ \partial_1\mathcal{U}_3\partial_2\mathcal{U}_3 & \partial_2\mathcal{U}_3\partial_2\mathcal{U}_3 & 0\\ 0 & 0 & \nabla\mathcal{U}_3\cdot\nabla\mathcal{U}_3 \end{pmatrix} \quad \textit{strongly in } L^1(\Omega \times \mathcal{Y}^*)^9. \tag{4.25}$$

The above convergence is weak in $L^2(\Omega \times \mathcal{Y}^*)^9$.

Proof. The first convergence $(4.23)_1$ is a consequence of (4.21) and the classical resuts of the periodic unfolding method, see [16]. Convergences $(4.23)_{2,3,4}$ come from the convergences in Lemma 4.4.1 and again of the classical results of the periodic unfolding method [16]. The last convergence $(4.23)_5$ is a consequence of [16, Lemma 11.11], together with the convergences $(4.21)_4$ and $(4.23)_3$.

For (4.25) recall

$$\nabla u_\varepsilon = \begin{pmatrix} \partial_1\mathcal{U}_{\varepsilon,1} + x_3\partial_1\mathcal{R}_{\varepsilon,1} & \partial_2\mathcal{U}_{\varepsilon,1} + x_3\partial_2\mathcal{R}_{\varepsilon,1} & \mathcal{R}_{\varepsilon,1} \\ \partial_1\mathcal{U}_{\varepsilon,2} + x_3\partial_1\mathcal{R}_{\varepsilon,2} & \partial_2\mathcal{U}_{\varepsilon,2} + x_3\partial_2\mathcal{R}_{\varepsilon,2} & \mathcal{R}_{\varepsilon,2} \\ \partial_1\mathcal{U}_{\varepsilon,3} & \partial_2\mathcal{U}_{\varepsilon,3} & 0 \end{pmatrix} + \nabla\overline{u}_\varepsilon.$$

Since $(4.23)_{1,2}$ are strong convergences and the other fields vanish due to $(4.23)_{3,2}$ and (4.24) we obtain

$$\frac{1}{\varepsilon}\mathcal{T}_\varepsilon(\nabla u_\varepsilon) \longrightarrow \begin{pmatrix} 0 & 0 & \mathcal{R}_1 \\ 0 & 0 & \mathcal{R}_2 \\ \partial_1\mathcal{U}_3 & \partial_2\mathcal{U}_3 & 0 \end{pmatrix} \quad \text{strongly in } L^2(\Omega\times\mathcal{Y}^*)^9. \tag{4.26}$$

Hence, using (4.22) the product converges:

$$\begin{aligned}\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon(\nabla u_\varepsilon(\nabla u_\varepsilon)^T) \longrightarrow & \begin{pmatrix} 0 & 0 & \mathcal{R}_1 \\ 0 & 0 & \mathcal{R}_2 \\ -\mathcal{R}_1 & -\mathcal{R}_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -\mathcal{R}_1 \\ 0 & 0 & -\mathcal{R}_2 \\ \mathcal{R}_1 & \mathcal{R}_2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{R}_1\mathcal{R}_1 & \mathcal{R}_1\mathcal{R}_2 & 0 \\ \mathcal{R}_1\mathcal{R}_2 & \mathcal{R}_2\mathcal{R}_2 & 0 \\ 0 & 0 & \mathcal{R}_1^2+\mathcal{R}_2^2 \end{pmatrix} \quad \text{strongly in } L^1(\Omega\times\mathcal{Y}^*)^9.\end{aligned}$$

Now note that the sequence $\left\{\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon(\nabla u_\varepsilon(\nabla u_\varepsilon)^T)\right\}_\varepsilon$ is bounded in $L^2(\Omega\times\mathcal{Y}^*)^9$ by

$$\begin{aligned}\|\nabla u_\varepsilon(\nabla u_\varepsilon)^T\|_{L^2(\Omega_\varepsilon)} &\le \|\nabla u_\varepsilon(\nabla u_\varepsilon)^T + 2e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon)} + 2\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon)} \\ &= \|\nabla v_\varepsilon(\nabla v_\varepsilon)^T - \mathbf{I}_3\|_{L^2(\Omega_\varepsilon)} + 2\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon^{5/2},\end{aligned}$$

ensuring that (4.25) is also weakly converging in $L^2(\Omega\times\mathcal{Y}^*)$. □

The convergence (4.26) implies

$$\mathfrak{T}_\varepsilon(\nabla v_\varepsilon) \to \mathbf{I}_3 \qquad \text{strongly in } L^2(\Omega\times\mathcal{Y}^*)^9. \tag{4.27}$$

Additionally, the displacements converge as follows

$$\begin{aligned}&\frac{1}{\varepsilon^2}\mathfrak{T}_\varepsilon(u_{\varepsilon,\alpha}) \to \mathcal{U}_\alpha - y_3\partial_\alpha\mathcal{U}_3 \qquad \text{strongly in } L^2(\Omega\times Y), \\ &\frac{1}{\varepsilon^1}\mathfrak{T}_\varepsilon(u_{\varepsilon,3}) \to \mathcal{U}_3 \qquad \text{strongly in } L^2(\Omega\times Y).\end{aligned} \tag{4.28}$$

The above convergences show that the limit displacement is of Kirchhoff-Love type.

The next Lemma presents the limit of the Green-Lagrange strain tensor, used in the energy.

Lemma 4.4.4. *For a subsequence we have*

$$\frac{1}{2\varepsilon^2}\mathfrak{T}_\varepsilon\big((\nabla v_\varepsilon)^T\nabla v_\varepsilon - \mathbf{I}_3\big) \rightharpoonup \mathbf{E}(\mathcal{U}) + e_y(\widehat{u}) \qquad \textit{weakly in} \qquad L^2(\Omega\times\mathcal{Y}^*)^9, \tag{4.29}$$

where the symmetric matrix $\mathbf{E}(\mathcal{U})$ *is defined by*

$$\mathbf{E}(\mathcal{U}) = \begin{pmatrix} -y_3 \frac{\partial^2 \mathcal{U}_3}{\partial x_1^2} + \mathcal{Z}_{11} & -y_3 \frac{\partial^2 \mathcal{U}}{\partial x_1 \partial x_2} + \mathcal{Z}_{12} & 0 \\ * & -y_3 \frac{\partial^2 \mathcal{U}_3}{\partial x_2^2} + \mathcal{Z}_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the new warping field by

$$\begin{aligned} \widehat{u}(x', y) = \overline{u}(x', y) &+ \frac{y_3}{2}(\mathcal{Z}_1(x') \cdot \mathbf{e}_3)\mathbf{e}_1 + \frac{y_3}{2}(\mathcal{Z}_2(x') \cdot \mathbf{e}_3)\mathbf{e}_2 \\ &+ y_3 \widehat{\mathcal{R}}(x', y') \wedge \mathbf{e}_3 + \mathfrak{u}(x', y') + y_3 |\nabla \mathcal{U}_3(x')|^2 \mathbf{e}_3 \quad \text{for a.e. } (x', y) \in \Omega \times \mathcal{Y}^*, \end{aligned}$$

with

$$\mathcal{Z}_{\alpha\beta} = e_{\alpha\beta}(\mathcal{U}) + \frac{1}{2} \frac{\partial \mathcal{U}_3}{\partial x_\alpha} \frac{\partial \mathcal{U}_3}{\partial x_\beta}, \qquad (\alpha, \beta) \in \{1, 2\}^2.$$

Proof. First, in the strain tensor $\nabla v (\nabla v)^T - \mathbf{I}_3$ replace the deformation by its associated displacement $u = v - I_d$. This yields

$$\nabla v (\nabla v)^T - \mathbf{I}_3 = \nabla u (\nabla u)^T + \nabla u + (\nabla u)^T = \nabla u (\nabla u)^T + 2e(u). \tag{4.30}$$

The first term on the right-hand side is already covered in (4.25). Hence, consider now $\frac{1}{\varepsilon^2} \mathfrak{T}_\varepsilon(e(u))$. This is already proved in [16] and yields

$$\frac{1}{\varepsilon^2} \mathfrak{T}_\varepsilon(e(u)) \rightharpoonup \begin{pmatrix} e_{11}(\mathcal{U}) - y_3 \frac{\partial^2 \mathcal{U}_3}{\partial x_1^2} & e_{12}(\mathcal{U}) - y_3 \frac{\partial^2 \mathcal{U}_3}{\partial x_1 \partial_2} & 0 \\ * & e_{22}(\mathcal{U}) - y_3 \frac{\partial^2 \mathcal{U}_3}{\partial x_2^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} + e_y(\widehat{u}), \quad \text{weakly in } L^2(\Omega \times \mathcal{Y}^*), \tag{4.31}$$

where we define

$$\begin{aligned} \widehat{u}(x', y) = \overline{u}(x', y) &+ \frac{y_3}{2}(\mathcal{Z}_1(x') \cdot \mathbf{e}_3)\mathbf{e}_1 + \frac{y_3}{2}(\mathcal{Z}_2(x') \cdot \mathbf{e}_3)\mathbf{e}_2 \\ &+ y_3 \widehat{\mathcal{R}}(x', y') \wedge \mathbf{e}_3 + \mathfrak{u}(x', y') + y_3 |\nabla \mathcal{U}_3(x')|^2 \mathbf{e}_3, \end{aligned}$$

for a.e. $(x', y) \in \Omega \times \mathcal{Y}^*$. Upon rewriting the result this yields the claim. □

Note that the antisymmetric part is responsible for the nonlinearity of the problem. Finally we prove that in the limit problems and in the case of glued yarns, one can replace the $e_{\alpha\beta}(\mathcal{U})$'s with the $\mathcal{Z}_{\alpha\beta}(\mathcal{U})$'s.

4.4.2 The limit problem

The limits of the previous section allow to investigate the limit of the elastic problem. Therefore, recall the energy of the elasticity problem in the limit

$$\mathcal{J}(\mathcal{U},\widehat{u}) = \frac{1}{|\mathcal{Y}^*|}\int_\Omega\int_{\mathcal{Y}^*} \widehat{W}\big(y,\mathbf{E}(\mathcal{U})+e_y(\widehat{u})\big)\,dydx' - \int_\Omega f\cdot\mathcal{U}\,dx'. \tag{4.32}$$

Define the limit displacement space

$$\mathbb{U} = \big\{\mathcal{U} = \big(\mathcal{U}_1,\mathcal{U}_2,\mathcal{U}_3\big)\in H^1(\Omega)^2\times H^2(\Omega)\ \mid\ \mathcal{U} = \partial_\alpha\mathcal{U}_3 = 0\quad \text{a.e. on } \gamma\big\}.$$

Furthermore, set

$$J(\mathcal{U},\widehat{u}) = \frac{1}{|\mathcal{Y}^*|}\int_\Omega\int_{\mathcal{Y}^*} \widehat{W}\big(y,\mathbf{E}(\mathcal{U})+e_y(\widehat{u})\big)\,dydx', \tag{4.33}$$

the part of the energy without the external force. Thus we can write

$$\mathcal{J}(\mathcal{U},\widehat{u}) = J(\mathcal{U},\widehat{u}) - \int_\Omega f\cdot\mathcal{U}\,dx'.$$

Before showing the convergence of the problem with a kind of Γ-convergence, we first prove that the limit-functional $\mathcal{J}$ attains a minimum on $\mathbb{U}\times L^2(\Omega;H^1_{per}(\mathcal{Y}^*))^3$.

Hereafter denote for every $(\xi,\zeta,\widehat{w})\in\mathbb{S}\doteq\mathbb{R}^3\times\mathbb{R}^3\times H^1_{per,0}(\mathcal{Y}^*)^3$ and denote $\widetilde{\mathcal{E}}$ the symmetric matrix

$$\widetilde{\mathcal{E}}(\xi,\zeta,\widehat{w}) = \begin{pmatrix} \xi_1 - y_3\zeta_1 + e_{11,y}(\widehat{w}) & \xi_3 - y_3\zeta_3 + e_{12,y}(\widehat{w}) & e_{13,y}(\widehat{w})\\ * & \xi_2 - y_3\zeta_2 + e_{22,y}(\widehat{w}) & e_{23,y}(\widehat{w})\\ * & * & e_{33,y}(\widehat{w})\end{pmatrix}.$$

Lemma 4.4.5. *We equip the space $\mathbb{S}\doteq\mathbb{R}^3\times\mathbb{R}^3\times H^1_{per,0}(\mathcal{Y}^*)^3$ with the semi-norm*

$$\|(\xi,\zeta,\widehat{w})\|_{\mathbb{S}} = \sqrt{\sum_{i,j=1}^3 \|\widetilde{\mathcal{E}}_{ij}(\xi,\zeta,\widehat{w})\|^2_{L^2(\mathcal{Y}^*)}}.$$

Then, this expression actually defines a norm on $\mathbb{S}$ equivalent to the product-norm.

Proof. To show that the semi-norm is actually a norm it is necessary to show the positive definiteness, i.e., that $\|(\xi,\zeta,\widehat{w})\|_{\mathbb{S}} = 0$ implies $(\xi,\zeta,\widehat{w}) = 0$.

Let $(\xi,\zeta,\widehat{w})\in\mathbb{S}$ satisfy $\|(\xi,\zeta,\widehat{w})\|_{\mathbb{S}} = 0$ and define the map

$$\tau(y) = \begin{pmatrix} y_1\left(\xi_1 - y_3\zeta_1\right) + y_2\left(\xi_3 - y_3\zeta_3\right)\\ y_1\left(\xi_3 - y_3\zeta_3\right) + y_2\left(\xi_2 - y_3\zeta_2\right)\\ -\frac{y_1^2}{2}\zeta_1 - \frac{y_2^2}{2}\zeta_2 - y_1y_2\zeta_3\end{pmatrix}.$$

Then rewrite

$$\widetilde{\mathcal{E}}(\xi,\zeta,\widehat{w}) = e_y(\tau+\widehat{w}). \tag{4.34}$$

Hence, $\tau(y) + \widehat{w}(y) = a + b \wedge y$ is a rigid motion. Then the properties of $\widehat{w} \in H^1_{per,0}(\mathcal{Y}^*)$ (periodicity in the directions $\mathbf{e}_1$, $\mathbf{e}_2$ and vanishing mean) imply that $a = b = \xi = \zeta = 0$ and thus also $\widehat{w} = 0$.

Finally, by a contradiction argument it is easy to prove that there exists a constant C such that

$$\|\xi\|_2 + \|\zeta\|_2 + \|\widehat{w}\|_{H^1_{per,0}(\mathcal{Y}^*)} \leq C\|(\xi, \zeta, \widehat{w})\|_{\mathbb{S}} \tag{4.35}$$

holds for all $(\xi, \zeta, \widehat{w}) \in \mathbb{S}$. $\square$

Lemma 4.4.6. *The functional $\mathcal{J}$ admits a minimum on $\mathbb{U} \times L^2(\Omega; H^1_{per,0}(\mathcal{Y}^*))^3$.*

Proof. First, from (4.5) and Lemma 4.4.5, there exists a constant $C > 0$ such that

$$\sum_{\alpha,\beta=1}^{2} \left[\left\| e_{\alpha\beta}(\mathcal{W}) + \frac{1}{2}\frac{\partial \mathcal{W}_3}{\partial x_\alpha}\frac{\partial \mathcal{W}_3}{\partial x_\beta} \right\|^2_{L^2(\Omega)} + \left\| \frac{\partial^2 \mathcal{W}_3}{\partial x_\alpha \partial x_\beta} \right\|^2_{L^2(\Omega)} \right] + \|\widehat{w}\|^2_{L^2(\Omega; H^1(\mathcal{Y}^*))} \leq CJ(\mathcal{W}, \widehat{w}),$$
$$\text{for all } (\mathcal{W}, \widehat{w}) \in \mathbb{U} \times L^2(\Omega; H^1_{per,0}(\mathcal{Y}^*))^3. \tag{4.36}$$

Set

$$m = \inf_{(\mathcal{U},\widehat{u}) \in \mathbb{U} \times L^2(\Omega; H^1_{per,0}(\mathcal{Y}^*))^3} \mathcal{J}(\mathcal{U}, \widehat{u})$$

where $m \in [-\infty, 0]$.

Step 1. We show that $m \in (-\infty, 0]$.

That m is actually finite is a consequence of the existence of a weak convergent subsequence and the weak sequential continuity of $\mathcal{J}$.

The boundedness of $\mathcal{U}_i$ are shown with the help of the functional J. First, consider $\mathcal{U}_3$ and note that by (4.36) together with the boundary conditions this field satisfies

$$\|\widehat{u}\|^2_{L^2(\Omega; H^1(\mathcal{Y}^*))} \leq J(\mathcal{U}, \widehat{u}), \qquad \|\mathcal{U}_3\|^2_{H^2(\Omega)} \leq C \sum_{\alpha,\beta=1}^{2} \left\| \frac{\partial^2 \mathcal{U}_3}{\partial x_\alpha \partial x_\beta} \right\|^2_{L^2(\Omega)} \leq C_2 J(\mathcal{U}, \widehat{u}). \tag{4.37}$$

Similarly, the estimate for $\mathcal{U}_\alpha$ is obtained. For this keep in mind that in the energy only $\mathcal{Z}_{\alpha\beta} = e_{\alpha\beta(\mathcal{U})} + \frac{1}{2}\partial_\alpha \mathcal{U}_3 \partial_\beta \mathcal{U}_3$ arises and we arrive at

$$\sum_{\alpha,\beta=1}^{2} \|e_{\alpha\beta}(\mathcal{U})\|^2_{L^2(\Omega)} \leq cJ(\mathcal{U}, \widehat{u}) + \|\nabla \mathcal{U}_3\|^4_{L^4(\Omega)} \leq cJ(\mathcal{U}, \widehat{u}) + C_1 \|\mathcal{U}_3\|^4_{H^2(\Omega)}$$
$$\leq cJ(\mathcal{U}, \widehat{u}) + [C_1 C_2 J(\mathcal{U}, \widehat{u})]^2 .$$

Note that we used here the embedding $H^2(\Omega) \hookrightarrow W^{1,4}(\Omega)$. The 2D-Korn inequality then yields

$$\|\mathcal{U}_1\|^2_{H^1(\Omega)} + \|\mathcal{U}_2\|^2_{H^1(\Omega)} \leq cJ(\mathcal{U}, \widehat{u}) + [C_1 C_2 J(\mathcal{U}, \widehat{u})]^2 . \tag{4.38}$$

With (4.37) and (4.38) the functional satisfies

$$
\begin{aligned}
J(\mathcal{U},\widehat{u}) &\le \|f_3\|_{L^2(\Omega)}\|\mathcal{U}_3\|_{L^2(\Omega)} + \sqrt{\|f_1\|^2_{L^2(\Omega)} + \|f_2\|^2_{L^2(\Omega)}}\left[\|\mathcal{U}_1\|_{L^2(\Omega)} + \|\mathcal{U}_2\|_{L^2(\Omega)}\right] \\
&\le \|f_3\|_{L^2(\Omega)}\sqrt{J(\mathcal{U},\widehat{u})} + \sqrt{\|f_1\|^2_{L^2(\Omega)} + \|f_2\|^2_{L^2(\Omega)}}\left[c\sqrt{J(\mathcal{U},\widehat{u})} + C_1C_2J(\mathcal{U},\widehat{u})\right].
\end{aligned} \tag{4.39}
$$

Thus, we also have

$$
J(\mathcal{U},\widehat{u}) \le c\|f\|_{L^2(\Omega)}\sqrt{J(\mathcal{U},\widehat{u})} + C_1C_2\|f\|_{L^2(\Omega)}J(\mathcal{U},\widehat{u}), \tag{4.40}
$$

which shows that $J(\mathcal{U},\widehat{u})$ is bounded if and only if $C_1C_2\|f\|_{L^2(\Omega)} \le 1$. In fact it is the same constraint as found in Lemma 4.3.5.

Finally, we have

$$
\begin{aligned}
&\forall(\mathcal{U},\widehat{u}) \in \mathbb{U}\times L^2(\Omega;H^1_{per,0}(\mathcal{Y}^*))^3, \\
&\qquad \mathcal{J}(\mathcal{U},\widehat{u}) \le 0 \quad \Longrightarrow \quad \|\mathcal{U}_1\|_{H^1(\Omega)} + \|\mathcal{U}_2\|_{H^1(\Omega)} + \|\mathcal{U}_3\|_{H^2(\Omega)} + \|\widehat{u}\|_{L^2(\Omega;H^1(\mathcal{Y}^*))} \le C.
\end{aligned}
$$

Then we easily show that $m \in (-\infty, 0]$, since $\mathcal{J}$ is weak lower semi-continuous.

Step 2. We show that m is a minimum.

Consider a minimizing sequence $\{(\mathcal{U}^n,\widehat{u}^n)\}_n \subset \mathbb{U}\times L^2(\Omega;H^1_{per,0}(\mathcal{Y}^*))^3$, i.e., the sequence satisfies $\mathcal{J}(\mathcal{U}^n,\widehat{u}^n) \le \mathcal{J}(0,0) = 0$ and

$$
m = \inf_{(\mathcal{U},\widehat{u})\in\mathbb{U}} \mathcal{J}(\mathcal{U},\widehat{u}) = \lim_{n\to+\infty} \mathcal{J}(\mathcal{U}^n,\widehat{u}^n).
$$

From step 1, one has

$$
\|\mathcal{U}^n_1\|_{H^1(\Omega)} + \|\mathcal{U}^n_2\|_{H^1(\Omega)} + \|\mathcal{U}^n_3\|_{H^2(\Omega)} + \|\widehat{u}^n\|_{L^2(\Omega;H^1(\mathcal{Y}^*))} \le C,
$$

where the constant does not depend on n.
Hence, there exists a subsequence of $\{(\mathcal{U}^n,\widehat{u}^n)\}_n$, still denoted $\{(\mathcal{U}^n,\widehat{u}^n)\}_n$, such that

$$
(\mathcal{U}^n,\widehat{u}^n) \rightharpoonup (\mathcal{U}',\widehat{u}') \quad \text{weakly in } \mathbb{U}\times L^2(\Omega;H^1_{per,0}(\mathcal{Y}^*))^3.
$$

Then the lower semi-continuity of $\mathcal{J}$ implies that

$$
\mathcal{J}(\mathcal{U}',\widehat{u}') = \liminf_{n\to+\infty} \mathcal{J}(\mathcal{U}^n,\widehat{u}^n) \le \lim_{n\to+\infty} \mathcal{J}(\mathcal{U}^n,\widehat{u}^n) \le m \tag{4.41}
$$

Since $m = \inf_{(\mathcal{U},\widehat{u})\in\mathbb{U}} \mathcal{J}(\mathcal{U},\widehat{u})$ we conclude that

$$
\mathcal{J}(\mathcal{U}',\widehat{u}') \le m \le \mathcal{J}(\mathcal{U},\widehat{u}),
$$

holds for every $(\mathcal{U},\widehat{u}) \in \mathbb{U}\times L^2(\Omega;H^1_{per,0}(\mathcal{Y}^*))^3$. Choosng $(\mathcal{U},\widehat{u}) = (\mathcal{U}',\widehat{u}')$ finishes the proof. $\square$

Theorem 4.4.7. *Under the assumptions on the forces* (4.11)-(4.12) *we have*

$$m = \lim_{\varepsilon\to 0} \frac{m_\varepsilon}{\varepsilon^5} = \min_{(\mathcal{U},\widehat{u})\in\mathbb{U}\times L^2(\Omega;H^1_{per}(\mathcal{Y}^*))^3} \mathcal{J}(\mathcal{U},\widehat{u}). \tag{4.42}$$

Proof. The following proof uses a kind of Γ-convergence technique. For more Literature about Gamma-Convergence see for instance [8, 20, 21, 36].

Step 1. In this step we show that

$$\min_{(\mathcal{U},\widehat{u})\in\mathbb{U}\times L^2(\Omega;H^1_{per}(\mathcal{Y}^*))^3} \mathcal{J}(\mathcal{U},\widehat{u}) \le \liminf_{\varepsilon\to 0} \frac{m_\varepsilon}{\varepsilon^5}.$$

Let $\{v_\varepsilon\}_\varepsilon \subset V_\varepsilon$, be a minimizing sequence of deformations, i.e., it satisfies

$$\lim_{\varepsilon\to 0} \frac{\mathcal{J}_\varepsilon(v_\varepsilon)}{\varepsilon^5} = \liminf_{\varepsilon\to 0} \frac{m_\varepsilon}{\varepsilon^5}.$$

Without loss of generality, we can assume that the sequence satisfies $\mathcal{J}_\varepsilon(v_\varepsilon) \le 0$ and hence the estimates of the previous sections yield

$$\|dist(\nabla v_\varepsilon, SO(3))\|^2_{L^2(\Omega^*_\varepsilon)} \le C\varepsilon^5 \qquad \text{and} \qquad \|(\nabla v_\varepsilon)^T \nabla v_\varepsilon - \mathbf{I}_3\|^2_{L^2(\Omega^*_\varepsilon)} \le C\varepsilon^5. \tag{4.43}$$

Therefore, we are allowed to use the decomposition defined in Definition 4.3.3 and obtain the estimates (4.20) and convergences as in Lemma 4.4.4 and 4.4.3. Then the assumptions on the force lead to

$$\begin{aligned}
\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^5} \int_{\Omega\times\mathcal{Y}^*} \mathfrak{T}_\varepsilon(f_\varepsilon \cdot (v_\varepsilon - I_d))dx'dy &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^5} \int_{\Omega\times\mathcal{Y}^*} \mathfrak{T}_\varepsilon(f_\varepsilon \cdot u_\varepsilon)dx'dy \\
&= |\mathcal{Y}^*| \int_\Omega f \cdot \mathcal{U}\, dx',
\end{aligned}$$

converging as a product of a weak and a strong convergence. As consequence, we have with the weak convergence of the strain tensor (4.29) together with the weak lower semi-continuity of $\mathcal{J}$ that

$$\liminf_{\varepsilon\to 0} \frac{\mathcal{J}_\varepsilon(v_\varepsilon)}{\varepsilon^5} \ge J(\mathcal{U},\widehat{u}) - |\mathcal{Y}^*| \int_\Omega f \cdot \mathcal{U}\, dx. \tag{4.44}$$

Step 2. We show that for every $(\mathcal{U}',\widehat{u}') \in \mathbb{U} \times L^2(\Omega; H^1_{per}(\mathcal{Y}^*))^3$, one has

$$\limsup_{\varepsilon\to 0} \frac{m_\varepsilon}{\varepsilon^5} \le \mathcal{J}(\mathcal{U}',\widehat{u}'). \tag{4.45}$$

To do that, let $(\mathcal{U}',\widehat{u}')$ be in $\mathbb{U} \times L^2(\Omega; H^1_{per}(\mathcal{Y}^*))^3$. We will build a sequence $\{v_\varepsilon\}_\varepsilon$ of admissible deformations such that

$$\limsup_{\varepsilon\to 0} \frac{m_\varepsilon}{\varepsilon^5} \le \lim_{n\to+\infty} \lim_{\varepsilon\to 0} \frac{\mathcal{J}_\varepsilon(V^{(n)}_\varepsilon)}{\varepsilon^5} = \mathcal{J}(\mathcal{U}',\widehat{u}').$$

Now consider a sequence of displacements $\{\mathcal{U}^{(n)}\}_n$ in $\mathbb{V} \cap \big(\mathcal{C}^1(\overline{\Omega})^2 \times \mathcal{C}^2(\overline{\Omega})\big)$ and additionally $\{\widehat{u}^{(n)}\}_n$ in $L^2(\Omega; H^1_{per}(\mathcal{Y}^*))^3 \cap \mathcal{C}^1(\overline{\Omega} \times \overline{\mathcal{Y}^*})^3$ with $\widehat{u}'_{n|x_2=0} = 0$, such that

$$\begin{aligned}
&\mathcal{U}^{(n)}_\alpha \to \mathcal{U}'_\alpha \qquad \text{strongly in } H^1(\Omega),\\
&\mathcal{U}^{(n)}_3 \to \mathcal{U}'_3 \qquad \text{strongly in } H^2(\Omega),\\
&\widehat{u}^{(n)} \to \widehat{u}' \qquad \text{strongly in } L^2(\Omega; H^1_{per}(\mathcal{Y}^*)).
\end{aligned} \tag{4.46}$$

Now, we show that there exists a sequence $\{v_\varepsilon\}_\varepsilon$ such that

$$\limsup_{\varepsilon\to 0} \frac{m_\varepsilon}{\varepsilon^5} \leq \mathcal{J}(\mathcal{U}^{(n)}, \widehat{u}^{(n)})$$

Therefore, define the associated sequence of deformations

$$\begin{aligned}
V^{(n)}_{\varepsilon,1}(x) &= x_1 + \varepsilon^2\Big(\mathcal{U}^{(n)}_1(x_1,x_2) - \frac{x_3}{\varepsilon}\partial_1\mathcal{U}^{(n)}_3(x_1,x_2) + \varepsilon\widehat{u}^{(n)}_1(x_1,x_2,\frac{x_3}{\varepsilon})\Big),\\
V^{(n)}_{\varepsilon,2}(x) &= x_2 + \varepsilon^2\Big(\mathcal{U}^{(n)}_2(x_1,x_2) - \frac{x_3}{\varepsilon}\partial_2\mathcal{U}^{(n)}_3(x_1,x_2) + \varepsilon\widehat{u}^{(n)}_2(x_1,x_2,\frac{x_3}{\varepsilon})\Big),\\
V^{(n)}_{\varepsilon,1}(x) &= x_3 + \varepsilon\Big(\mathcal{U}^{(n)}_3(x_1,x_2) + \varepsilon^2\widehat{u}^{(n)}_3(x_1,x_2,\frac{x_3}{\varepsilon})\Big).
\end{aligned}$$

By construction we have $V \in \mathbf{V}_\varepsilon$. Obviously, the deformation can be further restricted to the original structure Ω^*_ε.

Now consider the convergences of the deformations $\{V^{(n)}_\varepsilon\}_\varepsilon$. Note that they satisfy

$$\|\nabla V^{(n)}_\varepsilon - \mathbf{I}_3\|_{L^\infty(\Omega_\varepsilon)} \leq C(n)\varepsilon,$$

which estimates the displacement gradient. This implies for ε small enough, that $\det(\nabla V^{(n)}_\varepsilon) > 0$ for all $n \in \mathbb{N}$ and all $x \in \Omega^*_\varepsilon$. This leads us to replace $\widehat{W} = Q$ (cf. Section 4.2) and together with the right-hand-side to

$$m_\varepsilon \leq \mathcal{J}_\varepsilon(V^{(n)}_\varepsilon). \tag{4.47}$$

Since the convergence of the deformation components are known, we obtain

$$\frac{1}{2\varepsilon^2}\mathfrak{T}_\varepsilon\big((\nabla V^{(n)}_\varepsilon)^T\nabla V^{(n)}_\varepsilon - \mathbf{I}_3\big) \longrightarrow \mathbf{E}(\mathcal{U}^{(n)}) + e_y\big(\widehat{u}^{(n)}\big) \quad \text{strongly in } L^2(\Omega\times\mathcal{Y}^*)$$

defined as in Lemma 4.4.4. This convergence gives rise to the convergence of the elastic energy

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^5}\mathcal{J}_\varepsilon(V^{(n)}_\varepsilon) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^5}\int_{\Omega\times\mathcal{Y}^*}\mathfrak{T}_\varepsilon\big(\widehat{W}\big(y,\nabla V^{(n)}_\varepsilon\big)\big)dx'dy\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^5}\int_{\Omega\times\mathcal{Y}^*}\mathfrak{T}_\varepsilon\big(Q\big(y,(\nabla V^{(n)}_\varepsilon)^T\nabla V^{(n)}_\varepsilon - \mathbf{I}_3\big)\big)dx'dy\\
&= \int_{\Omega\times\mathcal{Y}^*}Q\big(y,\mathbf{E}(\mathcal{U}^{(n)}) + e_y(\widehat{u}^{(n)})\big)dx'dy,
\end{aligned}$$

and the right-hand-side

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^5}\int_{\Omega\times\mathcal{Y}^*}\mathfrak{T}_\varepsilon\big(f_\varepsilon\cdot\big(V_\varepsilon^{(n)}-I_d\big)\big)dx'dy\longrightarrow|\mathcal{Y}^*|\int_\Omega f\cdot\mathcal{U}^{(n)}dx'dy.$$

Hence, with (4.47) we obtain

$$\limsup_{\varepsilon\to 0}\frac{m_\varepsilon}{\varepsilon^5}\leq\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^5}\mathcal{J}_\varepsilon(V_\varepsilon^{(n)})=\mathcal{J}\big(\mathcal{U}^{(n)},\widehat{u}^{(n)}\big).$$

Since this holds for every $n\in\mathbb{N}$, consider now the limit for n to infinity. The strong convergences (4.46) yield

$$\limsup_{\varepsilon\to 0}\frac{m_\varepsilon}{\varepsilon^5}\leq\lim_{n\to+\infty}\mathcal{J}\big(\mathcal{U}^{(n)},\widehat{u}^{(n)}\big)=\mathcal{J}\big(\mathcal{U}',\widehat{u}'\big),$$

which concludes the proof of (4.45).

Step 3. Combining *Step 1* and *2* we obtain for every $(\mathcal{U}',\widehat{u}')\in\mathbb{U}\times L^2(\Omega,H^1_{per}(\mathcal{Y}^*))$

$$\mathcal{J}(\mathcal{U},\widehat{u})\leq\liminf_{\varepsilon\to 0}\frac{m_\varepsilon}{\varepsilon^5}\leq\limsup_{\varepsilon\to 0}\frac{m_\varepsilon}{\varepsilon^5}\leq\mathcal{J}(\mathcal{U}',\widehat{u}'). \tag{4.48}$$

Thus, choosing $(\mathcal{U}',\widehat{u}')=(\mathcal{U},\widehat{u})$ gives

$$\mathcal{J}(\mathcal{U},\widehat{u})=\lim_{\varepsilon\to 0}\frac{m_\varepsilon}{\varepsilon^5}$$

and finally

$$\lim_{\varepsilon\to 0}\frac{m_\varepsilon}{\varepsilon^5}=\mathcal{J}(\mathcal{U},\widehat{u})=\min_{(\mathcal{U}',\widehat{u}')\in\mathbb{U}\times L^2(\Omega,H^1_{per}(\mathcal{Y}^*))}\mathcal{J}(\mathcal{U}',\widehat{u}').$$

□

4.4.3 The cell problems

Recall the energy (4.32):

$$\mathcal{J}(\mathcal{U},\widehat{u})=\frac{1}{2|\mathcal{Y}^*|}\int_\Omega\int_{\mathcal{Y}^*}a(y)\big(\mathbf{E}(\mathcal{U})+e_y(\widehat{u})\big)\ :\ \big(\mathbf{E}(\mathcal{U})+e_y(\widehat{u})\big)\ dydx'-\int_\Omega f\cdot\mathcal{U}\ dx'. \tag{4.49}$$

To obtain the cell problems consider the variational formulation for $\widehat{u}$ associated to the functional $\mathcal{J}$. For this we use the Euler-Lagrange equation (since it is a quadratic form in $e(\widehat{u})$ over a Hilbert-space) and we obtain:

$$\begin{aligned}&\text{Find }\widehat{u}\in L^2(\Omega,H^1(\mathcal{Y}^*))\text{ such that}\\&\int_{\Omega\times\mathcal{Y}^*}a(y)\big(\mathbf{E}(\mathcal{U})+e_y(\widehat{u})\big)\ :\ e_y(\widehat{w})\ dy=0,\qquad\text{for all }\widehat{w}\in L^2(\Omega,H^1_{per,0}(\mathcal{Y}^*)).\end{aligned}\tag{4.50}$$

Upon this, we use the periodicity w.r.t. $\mathcal{Y}^*$ to restrict the cell problems to

$$\begin{aligned}&\text{Find } \widehat{u} \in H^1_{per,0}(\mathcal{Y}^*) \text{ such that}\\ &\int_{\mathcal{Y}^*} a(y)\big(\mathbf{E}(\mathcal{U}) + e_y(\widehat{u})\big) \ : \ e_y(\widehat{w})\, dy = 0, \qquad \text{for all } \widehat{w} \in H^1_{per,0}(\mathcal{Y}^*).\end{aligned} \tag{4.51}$$

Hence, the fields $\widehat{u}$ depend linearly on $\mathbf{E}(\mathcal{U})$. Thus we are led to assume

$$\widehat{u} = \sum_{\alpha,\beta=1}^{2} \mathcal{Z}_{\alpha\beta}\, \widehat{\chi}^m_{\alpha\beta}(y) + \sum_{\alpha,\beta=1}^{2} \partial_{\alpha\beta}\mathcal{U}_3\, \widehat{\chi}^b_{\alpha\beta}(y). \tag{4.52}$$

This leads directly to the typical cell problems

$$\begin{aligned}&\text{Find } \left(\widehat{\chi}^m_{11}, \widehat{\chi}^m_{12}, \widehat{\chi}^m_{22}, \widehat{\chi}^b_{11}, \widehat{\chi}^b_{12}, \widehat{\chi}^b_{22}\right) \in H^1_{per,0}(\mathcal{Y}^*)^6 \text{ such that}\\ &\left.\begin{aligned}&\int_{\mathcal{Y}^*} a(y)(M^{\alpha\beta} + e_y(\widehat{\chi}^m_{\alpha\beta})) \ : \ e_y(\widehat{w})\, dy = 0,\\ &\int_{\mathcal{Y}^*} a(y)(-y_3 M^{\alpha\beta} + e_y(\widehat{\chi}^b_{\alpha\beta})) \ : \ e_y(\widehat{w})\, dy = 0,\end{aligned}\right\} \text{for all } \widehat{w} \in H^1_{per,0}(\mathcal{Y}^*)^3,\end{aligned} \tag{4.53}$$

where we denote

$$M^{11} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad M^{22} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad M^{12} = M^{21} = \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{4.54}$$

Then, set the homogenized coefficients

$$a^{hom}_{\alpha\beta\alpha'\beta'} = \frac{1}{|\mathcal{Y}^*|}\int_{\mathcal{Y}^*} a_{ijkl}(y)\left[M^{\alpha\beta}_{ij} + e_{y,ij}(\widehat{\chi}^m_{\alpha\beta})\right] M^{\alpha'\beta'}_{kl}\, dy, \tag{4.55}$$

$$b^{hom}_{\alpha\beta\alpha'\beta'} = \frac{1}{|\mathcal{Y}^*|}\int_{\mathcal{Y}^*} a_{ijkl}(y)\left[y_3 M^{\alpha\beta}_{ij} + e_{y,ij}(\widehat{\chi}^b_{\alpha\beta})\right] M^{\alpha'\beta'}_{kl}\, dy, \tag{4.56}$$

$$c^{hom}_{\alpha\beta\alpha'\beta'} = \frac{1}{|\mathcal{Y}^*|}\int_{\mathcal{Y}^*} a_{ijkl}(y)\left[y_3 M^{\alpha\beta}_{ij} + e_{y,ij}(\widehat{\chi}^b_{\alpha\beta})\right] y_3 M^{\alpha'\beta'}_{kl}\, dy. \tag{4.57}$$

Accordingly, the homogenized energy is defined by

$$\begin{aligned}\mathcal{J}^{hom}_{vK}(\mathcal{U}) = \frac{1}{2}\int_{\Omega} \big(a^{hom}_{\alpha\beta\alpha'\beta'} \mathcal{Z}_{\alpha\beta}\mathcal{Z}_{\alpha'\beta'} + b^{hom}_{\alpha\beta\alpha'\beta'} \mathcal{Z}_{\alpha\beta}\, \partial_{\alpha'\beta'}\mathcal{U}_3 + c^{hom}_{\alpha\beta\alpha'\beta'} \partial_{\alpha\beta}\mathcal{U}_3\, \partial_{\alpha'\beta'}\mathcal{U}_3\big)\, dx'\\ - \int_{\Omega} f \cdot \mathcal{U}\, dx'.\end{aligned} \tag{4.58}$$

with

$$\mathcal{Z}_{\alpha\beta} = e_{\alpha\beta}(\mathcal{U}) + \partial_\alpha \mathcal{U}_3 \partial_\beta \mathcal{U}_3.$$

By introducing in the limit energy the homogenized coefficients yield the homogenized energy in the theorem below.

Theorem 4.4.8. *Under the assumptions* (4.11)-(4.12) *the problem*

$$\min_{\mathcal{U}\in\mathbb{U}} \mathcal{J}_{vK}^{hom}(\mathcal{U}) \tag{4.59}$$

admits solutions. Moreover, one has

$$m = \lim_{\varepsilon\to 0}\frac{m_\varepsilon}{\varepsilon^5} = \min_{(\mathcal{U},\widehat{u})\in\mathbb{U}\times L^2(\Omega;H^1_{per}(\mathcal{Y}^*))^3} \mathcal{J}(\mathcal{U},\widehat{u}) = \min_{\mathcal{U}\in\mathbb{U}} \mathcal{J}_{vK}^{hom}(\mathcal{U}).$$

4.4.4 The linear problem

The analysis presented in this chapter is stated especially for the von-Kármán limit. Although, this is a nonlinear model, it is stated with displacements and not deformations, which usually arise in nonlinear elasticity. In fact the von-Kármán plate is the critical case for the choice of the geometric energy $\|dist(\nabla v, SO(3))\|_{L^2(\Omega^*_\varepsilon)} \sim C\varepsilon^{5/2}$ in between linear and nonlinear plates, as it can be seen in [4, 5, 7, 14, 20, 21, 36].

To obtain the linear problem one simply considers the symmetric strain tensor $e(u)$ instead of the Green-Lagrangian strain tensor $e(u) + \frac{1}{2}\nabla u(\nabla u)^T = \frac{1}{2}(\nabla v(\nabla v)^T - \mathbf{I}_3)$. All results in this chapter remain true, yet all $\mathcal{Z}_{\alpha\beta}(\mathcal{U})$ are replaced by $e_{\alpha\beta}(\mathcal{U})$ in the limit.

The resulting linear limit energy is

$$\mathcal{J}_{lin}(\mathcal{U},\widehat{u}) = \frac{1}{|\mathcal{Y}^*|}\int_\Omega\int_{\mathcal{Y}^*} \widehat{W}\big(y, \mathbf{E}^{lin}(\mathcal{U}) + e_y(\widehat{u})\big)\, dydx' - \int_\Omega f\cdot\mathcal{U}\, dx', \tag{4.60}$$

with

$$\mathbf{E}^{lin}(\mathcal{U}) = \begin{pmatrix} e_{11}(\mathcal{U}) - y_3\frac{\partial^2\mathcal{U}_3}{\partial x_1^2} & e_{12}(\mathcal{U}) - y_3\frac{\partial^2\mathcal{U}_3}{\partial x_1\partial_2} & 0\\ * & e_{22}(\mathcal{U}) - y_3\frac{\partial^2\mathcal{U}_3}{\partial x_2^2} & 0\\ 0 & 0 & 0\end{pmatrix} \tag{4.61}$$

found in (4.31).

Then, with the same steps as in Section 4.4.3 the cell problems are given by (4.53) and yield the homogenized linear plate equation also found in [16, Thm. 11.21].

Theorem 4.4.9. *Assume that the force satisfies* $f_\varepsilon = \varepsilon^{2+\nu}f_1\mathbf{e}_1 + \varepsilon^{2+\nu}f_2\mathbf{e}_2 + \varepsilon^{3+\nu}f_3\mathbf{e}_3$ *with* $f\in L^2(\Omega)$ *and* $\nu > 0$. *Then,* $\mathcal{J}^{lin}$ *is the unfolded limit energy. Furthermore, the cell problems are again given by* (4.53) *and yield the homogenized energy*

$$\mathcal{J}_{lin}^{hom}(\mathcal{U}) = \frac{1}{2}\int_\Omega \big(a^{hom}_{\alpha\beta\alpha'\beta'}e_{\alpha\beta}(\mathcal{U})e_{\alpha'\beta'}(\mathcal{U}) + b^{hom}_{\alpha\beta\alpha'\beta'}e_{\alpha\beta}(\mathcal{U})\,\partial_{\alpha'\beta'}\mathcal{U}_3 + c^{hom}_{\alpha\beta\alpha'\beta'}\partial_{\alpha\beta}\mathcal{U}_3\,\partial_{\alpha'\beta'}\mathcal{U}_3\big)dx' - \int_\Omega f\cdot\mathcal{U}\,dx', \tag{4.62}$$

and the minimizer of this functional satisfies the variational equality:

Find $\mathcal{U} \in \mathbb{U}$ *such that for all* $\mathcal{V} \in \mathbb{U}$*:*

$$\int_\Omega a^{hom}_{\alpha\beta\alpha'\beta'} e_{\alpha\beta}(\mathcal{U}) e_{\alpha'\beta'}(\mathcal{V}) + \frac{b^{hom}_{\alpha\beta\alpha'\beta'}}{2} \big(e_{\alpha\beta}(\mathcal{U})\, \partial_{\alpha'\beta'}\mathcal{V}_3 + e_{\alpha\beta}(\mathcal{V})\, \partial_{\alpha'\beta'}\mathcal{U}_3 \big) \\ + c^{hom}_{\alpha\beta\alpha'\beta'} \partial_{\alpha\beta}\mathcal{U}_3\, \partial_{\alpha'\beta'}\mathcal{V}_3 dx' = \int_\Omega f \cdot \mathcal{V}\, dx'. \quad (4.63)$$

Note that this is the same energy as for the problem presented in [16, Ch. 11] for the case $\theta = 1$. The existence and uniqueness of a solution for this linear problem is for instance investigated in [10, 12, 16, 38].

Remark 4.4.10. *The shown derivation of a homogenized von-Kármán plate is also valid for other micro-structures for which the extension in Section C holds true , e.g., shells whose mid-surfaces are developable surfaces.*

Remark 4.4.11. *It is also possible to derive the von-Kármán plate on the level of deformations with the decomposition of deformations, see [4, 5]. However, this needs a more involved analysis of the decomposed fields, since the decomposition of a deformation is more complex. This different approach yields some insights into nonlinear elasticity and the connection between nonlinear decomposition and linear decompositions (see also [5, 25]), yet the result is the same as presented here.*

4.4.5 The textile for isotopic, homogeneous fibers is orthotropic

Here we show that for isotropic homogeneous beams, the resulting homogenized textile is orthotropic. This is true for the von-Kármán and the completely linear regime, since this only depends on the cell problems.

Thus, let us here assume that the yarns are made from an isotropic and homogeneous material, whose Lamé's constants are λ, μ.

Lemma 4.4.12. *Under the above assumption on the material, one has*

$$b^{hom}_{\alpha\beta\alpha'\beta'} = 0 \qquad \forall(\alpha\beta\alpha'\beta') \in \{1,2\}^4 \quad (4.64)$$

and also

$$\begin{aligned} a^{hom}_{1111} &= a^{hom}_{2222} \qquad \text{and} \qquad a^{hom}_{\alpha\alpha 12} = 0, \qquad \alpha \in \{1,2\}, \\ c^{hom}_{1111} &= c^{hom}_{2222} \qquad \text{and} \qquad c^{hom}_{\alpha\alpha 12} = 0, \qquad \alpha \in \{1,2\}. \end{aligned} \quad (4.65)$$

Proof. Consider the following transformation:

$$\begin{aligned} &\phi \in H^1_{per}(\mathcal{Y}^*)^3 \;\longmapsto\; \widetilde{\phi} \in H^1_{per}(\mathcal{Y}^*)^3 \\ &\widetilde{\phi}(y) = -\phi_1(\widetilde{y})\mathbf{e}_1 + \phi_2(\widetilde{y})\mathbf{e}_2 + \phi_3(\widetilde{y})\mathbf{e}_3 \\ &\text{where} \quad \widetilde{y} = (2 - y_1)\mathbf{e}_1 + y_2\mathbf{e}_2 + y_3\mathbf{e}_3, \quad \text{for a.e. } y \in \mathcal{Y}^*. \end{aligned}$$

One has

$$\begin{cases} e_{y,ii}(\widetilde{\phi})(y) = e_{y,ii}(\phi)(\widetilde{y}), \quad i \in \{1,2,3\}, \\ e_{y,12}(\widetilde{\phi})(y) = -e_{y,12}(\phi)(\widetilde{y}), \\ e_{y,13}(\widetilde{\phi})(y) = -e_{y,13}(\phi)(\widetilde{y}), \\ e_{y,23}(\widetilde{\phi})(y) = e_{y,23}(\phi)(\widetilde{y}), \end{cases} \qquad \text{for a.e. } y \in \mathcal{Y}^*.$$

Using this transformation in problem $(4.53)_2$ gives

$$\begin{cases} \widehat{\chi}^b_{\alpha\alpha}(2 - y_1, y_2, y_3) = \widehat{\chi}^b_{\alpha\alpha}(y), \\ \widehat{\chi}^b_{12}(2 - y_1, y_2, y_3) = -\widehat{\chi}^b_{12}(y), \end{cases} \qquad \text{for a.e. } y \in \mathcal{Y}^*. \tag{4.66}$$

Since $\displaystyle\int_{\mathcal{Y}^*} y_3\,dy = 0$, one has

$$b^{hom}_{\alpha\beta\alpha'\beta'} = \frac{1}{|\mathcal{Y}^*|}\int_{\mathcal{Y}^*} \sigma_{\alpha'\beta'}(\widehat{\chi}^b_{\alpha\beta})\,dy.$$

Hence

$$b^{hom}_{\alpha\alpha 12} = 0, \quad \alpha \in \{1,2\}.$$

Now, from (4.66), we get

$$\begin{cases} \widehat{\chi}^b_{12,1}(2 - y_1, y_2, y_3) = \widehat{\chi}^b_{12,1}(y), \\ \widehat{\chi}^b_{12,2}(2 - y_1, y_2, y_3) = -\widehat{\chi}^b_{12,2}(y), \\ \widehat{\chi}^b_{12,3}(2 - y_1, y_2, y_3) = -\widehat{\chi}^b_{12,3}(y), \end{cases} \quad \text{for a.e. } y \in \mathcal{Y}^*,$$

$$\begin{cases} \widehat{\chi}^b_{\alpha\alpha,1}(2 - y_1, y_2, y_3) = -\widehat{\chi}^b_{\alpha\alpha,1}(y), \\ \widehat{\chi}^b_{\alpha\alpha,2}(2 - y_1, y_2, y_3) = \widehat{\chi}^b_{\alpha\alpha,2}(y), \\ \widehat{\chi}^b_{\alpha\alpha,3}(2 - y_1, y_2, y_3) = \widehat{\chi}^b_{\alpha\alpha,3}(y), \end{cases} \qquad \text{for a.e. } y \in \mathcal{Y}^*.$$

Equality (**??**) and the periodicity lead to equalities below of the traces

$$\begin{aligned} &\widehat{\chi}^b_{12,i}(0, y_2, y_3) = \widehat{\chi}^b_{12,i}(1, y_2, y_3) = \widehat{\chi}^b_{12,i}(2, y_2, y_3) = 0, \qquad i \in \{2,3\}, \\ &\widehat{\chi}^b_{\alpha\alpha,1}(0, y_2, y_3) = \widehat{\chi}^b_{\alpha\alpha,1}(1, y_2, y_3) = \widehat{\chi}^b_{\alpha\alpha,1}(2, y_2, y_3) = 0. \end{aligned}$$

Now using the symmetry with respect to the plane $y_2 = 1$, we obtain

$$\begin{cases} \widehat{\chi}^b_{\alpha\alpha}(y_1, 2 - y_2, y_3) = \widehat{\chi}^b_{\alpha\alpha}(y), \\ \widehat{\chi}^b_{12}(y_1, 2 - y_2, y_3) = -\widehat{\chi}^b_{12}(y), \end{cases} \qquad \text{for a.e. } y \in \mathcal{Y}^*. \tag{4.67}$$

Hence

$$\begin{aligned} &\widehat{\chi}^b_{12,i}(y_1, 0, y_3) = \widehat{\chi}^b_{12,i}(y_1, 1, y_3) = \widehat{\chi}^b_{12,i}(y_1, 2, y_3) = 0, \qquad i \in \{1,3\}, \\ &\widehat{\chi}^b_{\alpha\alpha,2}(y_1, 0, y_3) = \widehat{\chi}^b_{\alpha\alpha,2}(y_1, 1, y_3) = \widehat{\chi}^b_{\alpha\alpha,2}(y_1, 2, y_3) = 0. \end{aligned}$$

The results above allow to replace problem $(4.53)_2$ by the following ones:

$$\begin{cases}\text{Find } \widehat{\chi}^b_{12}\in \mathbf{G}(\mathcal{Y}^*) \text{ such that,}\\ \displaystyle\int_{\mathcal{Y}^*}\sigma_{y,ii}(\widehat{\chi}^b_{12})\, e_{y,ij}(\widehat{w})\,dy=\int_{\mathcal{Y}^*} y_3\,\sigma_{y,12}(\widehat{w})\,dy,\\ \text{for all } \widehat{w}\in \mathbf{G}(\mathcal{Y}^*)\end{cases}$$
$$\begin{cases}\text{Find } \widehat{\chi}^b_{\alpha\alpha}\in \mathbf{H}(\mathcal{Y}^*) \text{ such that,}\\ \displaystyle\int_{\mathcal{Y}^*}\sigma_{y,ii}(\widehat{\chi}^b_{\alpha\alpha})\, e_{y,ij}(\widehat{w})\,dy=\int_{\mathcal{Y}^*} y_3\,\sigma_{y,\alpha\alpha}(\widehat{w})\,dy,\\ \text{for all } \widehat{w}\in \mathbf{H}(\mathcal{Y}^*)\end{cases}\tag{4.68}$$

where $\mathcal{Y}^*$ is the part of the cell included in $(0,1)^2\times(-2\kappa,2\kappa)$ and ($i\in\{2,3\}$, $j\in\{1,3\}$)

$$\begin{aligned}\mathbf{G}(\mathcal{Y}^*)&=\big\{\phi\in H^1(\mathcal{Y}^*)^3 \mid \phi_i(0,y_2,y_3)=\phi_i(1,y_2,y_3)=0,\ \phi_j(y_1,0,y_3)=\phi_j(y_1,1,y_3)=0\big\},\\ \mathbf{H}(\mathcal{Y}^*)&=\big\{\phi\in H^1(\mathcal{Y}^*)^3 \mid \phi_1(0,y_2,y_3)=\phi_1(1,y_2,y_3)=0,\ \phi_2(y_1,0,y_3)=\phi_1(y_1,1,y_3)=0\big\}.\end{aligned}$$

Now, consider the transformation

$$\begin{aligned}&\phi\in\mathbf{H}(\mathcal{Y}^*)\ \longmapsto\ \overline{\phi}\in\mathbf{H}(\mathcal{Y}^*),\ \ (\text{resp. } \ \phi\in\mathbf{G}(\mathcal{Y}^*)\ \longmapsto\ \overline{\phi}\in\mathbf{G}(\mathcal{Y}^*))\\ &\overline{\phi}(y)=\phi_2(\overline{y})\mathbf{e}_1+\phi_1(\overline{y})\mathbf{e}_2-\phi_3(\overline{y})\mathbf{e}_3,\quad \text{where}\quad \overline{y}=y_2\mathbf{e}_1+y_1\mathbf{e}_2-y_3\mathbf{e}_3,\qquad \text{for a.e. } y\in\mathcal{Y}^*.\end{aligned}$$

One has

$$\left.\begin{aligned}&e_{y,11}(\overline{\phi})(y)=e_{y,22}(\phi)(\overline{y}), && e_{y,22}(\overline{\phi})(y)=e_{y,11}(\phi)(\overline{y}),\\ &e_{y,13}(\overline{\phi})(y)=-e_{y,23}(\phi)(\overline{y}), && e_{y,12}(\overline{\phi})(y)=e_{y,12}(\phi)(\overline{y}),\\ &e_{y,33}(\overline{\phi})(y)=e_{y,33}(\phi)(\overline{y}), && e_{y,23}(\overline{\phi})(y)=-e_{y,13}(\phi)(\overline{y}),\end{aligned}\right.\qquad \text{for a.e. } y\in\mathcal{Y}^*.$$

We use the above transformation in problems (4.68) that gives

$$\begin{aligned}\widehat{\chi}^b_{12}(y_2,y_1,-y_3)&=-\widehat{\chi}^b_{12}(y),\\ \widehat{\chi}^b_{11}(y_2,y_1,-y_3)&=-\widehat{\chi}^b_{22}(y),\end{aligned}\qquad \text{for a.e. } y\in\mathcal{Y}^*.\tag{4.69}$$

These equalities lead to

$$b^{hom}_{1212}=b^{hom}_{1122}=0,\qquad b^{hom}_{1111}=-b^{hom}_{2222}.$$

The last transformation

$$\begin{aligned}&\phi\in\mathbf{H}(\mathcal{Y}^*)\ \longmapsto\ \overline{\overline{\phi}}\in\mathbf{H}(\mathcal{Y}^*),\\ &\overline{\overline{\phi}}(y)=-\phi_2(\overline{\overline{y}})\mathbf{e}_1+\phi_1(\overline{\overline{y}})\mathbf{e}_2+\phi_3(\overline{\overline{y}})\mathbf{e}_3,\\ &\text{where } \overline{\overline{y}}=(1-y_2)\mathbf{e}_1+y_1\mathbf{e}_2+y_3\mathbf{e}_3,\ \text{for a.e. } y\in\mathcal{Y}^*.\end{aligned}$$

One has

$$\left.\begin{aligned}&e_{y,11}(\overline{\overline{\phi}})(y)=e_{y,22}(\phi)(\overline{\overline{y}}), && e_{y,22}(\overline{\overline{\phi}})(y)=e_{y,11}(\phi)(\overline{\overline{y}}),\\ &e_{y,13}(\overline{\overline{\phi}})(y)=-e_{y,23}(\phi)(\overline{\overline{y}}), && e_{y,12}(\overline{\overline{\phi}})(y)=-e_{y,12}(\phi)(\overline{\overline{y}}),\\ &e_{y,33}(\overline{\overline{\phi}})(y)=e_{y,33}(\phi)(\overline{\overline{y}}), && e_{y,23}(\overline{\overline{\phi}})(y)=-e_{y,13}(\phi)(\overline{\overline{y}}),\end{aligned}\right.\qquad \text{for a.e. } y\in\mathcal{Y}^*.$$

We use the above transformation in problem $(4.68)_2$ that gives

$$\widehat{\chi}^b_{11}(1-y_2,y_1,y_3)=\widehat{\chi}^b_{22}(y),\qquad \text{for a.e. } y\in\mathcal{Y}^*. \tag{4.70}$$

This equality gives

$$b^{hom}_{1111}=b^{hom}_{2222}$$

which ends the proof of (4.64). Similarly one obtains (4.65). □

As a consequence of the above Lemma in the expressions of the energy (4.58) and (4.62) there remain three coefficients a^{hom} (a^{hom}_{1111}, a^{hom}_{11122}, a^{hom}_{1212}) and c^{hom} (c^{hom}_{1111}, c^{hom}_{1122}, c^{hom}_{1212}).

4.5 Comparison to chapter 3

In fact, the cell problems derived here are the same as for the linear limit in chapter 3. Recall, that the linear case there corresponds to fixed junctions between the beams, i.e., the gap-function $g\equiv 0$. Hence, the domain for the cell problems is the same but only the point of view is different. In chapter 3 the domain is stated in the beams reference while in this section here the domain is stated in the coordinate system of the plate. The equivalence of both cell problems is concluded by the fact that the procedure presented in this section is obviously for both and thus yields the same cell problems. For this reason, the textile derived in chapter 3 with contact $g_\varepsilon\le C\varepsilon^{3+\delta}$, $\delta>0$ and isotropic homogeneous fibers is also orthotropic, as shown in Section 4.4.5.

Part II

Optimization of Buckling

Chapter 5

The Buckling of Textiles

The buckling phenomenon is in the mathematical context a bifurcation problem and characterizes a loss of stability of the structure. This means that the solution suddenly changes its state and usually can attain multiple states with equal probability. In elasticity the simplest example for this behavior is Euler's buckling beam (see [44]). The buckling occurs if the applied force exceeds the critical value

$$F_{crit}^{Euler} = \frac{\pi^2 EI}{(KL)^2} \tag{5.1}$$

for a beam of length L with second moment of inertia I, Young's modulus E and a factor K accounting for the boundary condition, see Figure 5.1. Note that there are two solutions possible, since the displacement can attain either of the two dashed forms.

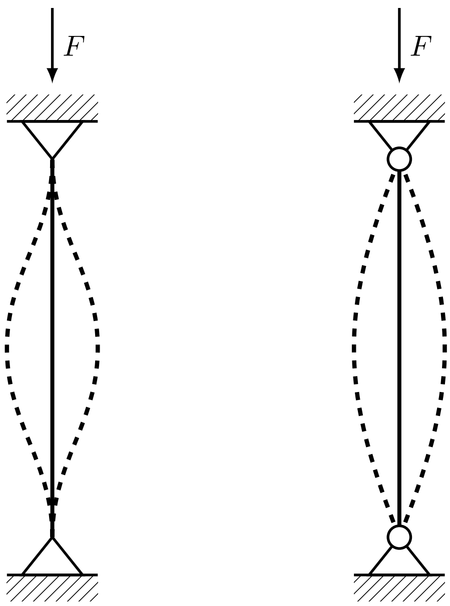


Figure 5.1: Euler's buckling beam. In the left picture translation and rotation are fixed ($K = 0.5$). In the right picture the translation is fixed while the rotation is free ($K = 1$).

Naturally, for plates the buckling analysis is more involved. In contrast to the beam example there are multiple states giving rise to buckling behavior. Hereafter, we restrict to an uniaxial force and orthotropic or isotropic materials. Even the isotropic case, admits interesting and promising results. Indeed, the buckling is not limited to only compressive forces but also occurs for tension. This is obviously impossible for the 1D-Euler beam. In fact, as derived for instance in [9, 42] an applied tension induces a compression in the lateral direction. This induced compression is comparable to a directly applied compression. For this reason, the numerical computations are reduced to a compressional setting.

5.1 Buckling of plates: a short introduction

The buckling of plates can be modeled in various ways. In this work we use the von-Kármán plate to model buckling arising from in-plane forces. The von-Kármán buckling is for instance investigated in [7, 34, 42]. The nonlinearity of the von-Kármán model allows to transfer in-plane forces to bending displacements, which is not possible for linear models. In [34] the authors show that linear and nonlinear stability are close to each other in the sense that the critical loads are equal for isotropic materials.

In the following, we chose the von-Kármán plate to model buckling behavior. First, an academic example from [3] is recalled. After that, we show results for the orthotropic textile and compare the results.

5.1.1 Example: Uniaxial compression of isotropic von-Kármán

For clarification of the buckling for a von-Kármán-plate we recall an example considered in [3]. For this academic example, assume that the plate is infinitely long in one direction and subjected to a force F in the other direction, see Figure 5.2. This leads to a one dimensional problem of an isotropic plate[1] of thickness h in the direction of the force with the Young's modulus E and poissons ratio ν. Denote by $\mathcal{U}$ the in-plane displacement and by $\mathcal{U}_3$ the bending displacement. As boundary condition assume simply $\mathcal{U}(-L) = \mathcal{U}_3(-L) = \mathcal{U}_3(L) = 0$ and a force F acting at $\mathcal{U}(L)$.

The von-Kármán energy for the plate reduced to the lateral direction reads as:

$$J_{vK}(\mathcal{U},\mathcal{U}_3) = \frac{1}{2}\int_{-L}^{L} \frac{Eh^3}{12(1-\nu^2)}(\partial_{xx}\mathcal{U}_3)^2 + \frac{Eh}{(1-\nu^2)}\mathcal{Z}^2\,dx - \frac{F}{2}\mathcal{U}(L), \tag{5.2}$$

with $\mathcal{Z} = \partial_x\mathcal{U} + \frac{1}{2}(\partial_x\mathcal{U}_3)^2$. Since only the bending displacement $\mathcal{U}_3$ is of interest, the energy is first minimized with respect to $\mathcal{Z}$. For this substitute the boundary value by

$$\mathcal{U}(L) = \int_{-L}^{L} \partial_x\mathcal{U}dx = \int_{-L}^{L}(\mathcal{Z} - \frac{1}{2}(\partial_x\mathcal{U}_3)^2))dx.$$

[1]For isotropic plate we have $c_{1111} = \frac{Eh^3}{12(1-\nu^2)}$, see [39].

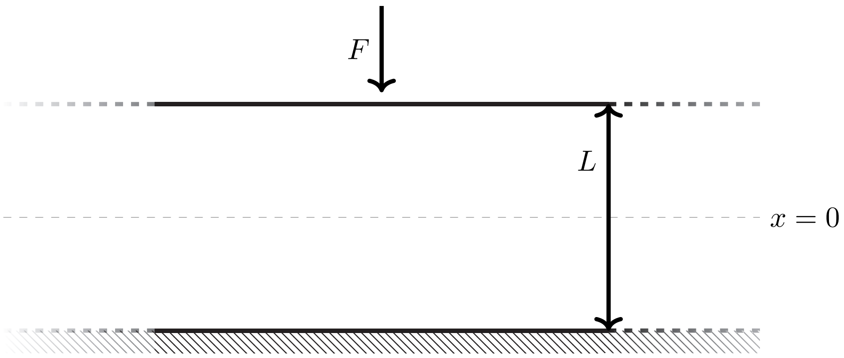


Figure 5.2: Plate domain of width $2L$ and infinite length, which is subjected to a compressive force F on the upper edge. The lower side is clamped.

Inserted into the energy this gives rise to a functional only depending on $\mathcal{U}_3$ and $\mathcal{Z}$

$$J(\mathcal{U}_3, \mathcal{Z}) = \frac{1}{2} \int_{-L}^{L} \frac{Eh^3}{12(1-\nu^2)} (\partial_{xx}\mathcal{U}_3)^2 + \frac{Eh}{(1-\nu^2)} \mathcal{Z}^2 dx - \frac{F}{2} \int_{-L}^{L} (\mathcal{Z} - \frac{1}{2}(\partial_x \mathcal{U}_3)^2)) dx, \quad (5.3)$$

which is easily minimized with respect to $\mathcal{Z}$ with the minimizer

$$\mathcal{Z}^* = \frac{F(1-\nu^2)}{Eh}. \quad (5.4)$$

Hence, we find the energy functional

$$J(u) = \frac{1}{2} \int_{-L}^{L} \frac{Eh^3}{12(1-\nu^2)} (\partial_{xx}\mathcal{U}_3)^2 + F(\partial_x \mathcal{U}_3)^2 dx.$$

This energy only depends on $\mathcal{U}_3$ and allows to distinguish between two cases: tension for $F > 0$ and compression for $F < 0$. For the tensional case the energy is always a convex functional and admits a unique solution, namely $\mathcal{U}_3 \equiv 0$. In the case of compression, it depends heavily on the magnitude of F. Indeed, the energy can be divided into two opposing parts: the bending part $\frac{Eh^3}{12(1-\nu^2)}(\partial_{xx}\mathcal{U}_3)^2$ and the rest. While for a small value of $|F|$ the bending dominates the energy, a large force $|F|$ leads to the domination of the second term. In this case the functional becomes non-convex, since the energy is no more bounded from below and violates the coercivity of the problem.

For the further investigation assume henceforth $F < 0$. The mathematical explanation uses the Poincaré-Wirtinger inequality

$$4\pi^2 \int_{-L}^{L} (\partial_x \mathcal{U}_3)^2 dx \leq (2L)^2 \int_{-L}^{L} (\partial_{xx}\mathcal{U}_3)^2 dx, \quad (5.5)$$

for the bending displacement $\mathcal{U}_3$. The inequality holds due to the boundary conditions $\mathcal{U}_3(-L) = \mathcal{U}_3(L) = 0$ implying $\int_{-L}^{L} \partial_x \mathcal{U}_3 \, dx = 0$. The Poincaré-Wirtinger inequality allows to estimate the energy from below

$$J(\mathcal{U}_3) \geq \frac{1}{2} \int_{-L}^{L} \left(\frac{Eh^3}{12(1-\nu^2)} - \frac{|F|L^2}{\pi^2} \right) (\partial_{xx}\mathcal{U}_3)^2 dx. \quad (5.6)$$

The coefficient in front of $\partial_{xx}\mathcal{U}_3$ determines the character of the energy. As long as $|F| \leq F_{crit}$ with

$$F_{crit} = \frac{Eh^3\pi^2}{12L^2(1-\nu^2)} \tag{5.7}$$

the energy is still convex and coercive. The only solution is in the tension case $\mathcal{U}_3 \equiv 0$. However, by exceeding this critical threshold the lower bound vanishes and the convexity and coerciveness is lost. Now assume the force $F = F_{crit}$ for which the functional is obviously still bounded from below, yet the solution is no more unique and the minimum of the energy is attained on a family of functions. These functions are characterized by the Poincaré-Wirtinger inequality (5.5) which has to be fulfilled with equality leading to

$$\mathcal{U}_3 = c\sin(\frac{\pi}{L}x), \tag{5.8}$$

for a constant amplitude c. In fact, the sign of c is typically not defined for bifurcation problems. The sign only defines the direction of the buckling, cf Figure 5.1.

The analysis of the regime $F > F_{crit}$ needs the full nonlinear theory, which is not done in this work. However, this would lead to higher modes of the solution depending again on the magnitude of the force.

5.2 Buckling of an orthotropic textile plate

In this section, we consider a textile under uniaxial compression. For this recall

$$\mathcal{J}_{vK}^{hom}(\mathcal{U}) = \int_\omega \big(a^{hom}_{\alpha\beta\alpha'\beta'}\mathcal{Z}_{\alpha\beta}\mathcal{Z}_{\alpha'\beta'} + c^{hom}_{\alpha\beta\alpha'\beta'}\partial_{\alpha\beta}\mathcal{U}_3\,\partial_{\alpha'\beta'}\mathcal{U}_3\big)\,dx' - \int_\omega f\cdot\mathcal{U}\,dx'.$$

the von-Kármán energy (4.55) derived in Chapter 4 for isotropic and homogeneous fibers. Note, that the homogenized textile plate is orthotropic, see section 4.4.5.

The new space of displacements is denoted by

$$\mathbb{U}_{Comp} = \left\{\mathcal{U} = \big(\mathcal{U}_1,\mathcal{U}_2,\mathcal{U}_3\big) \in H^1(\omega)^2 \times H^2(\omega)\ \mid\ \mathcal{U} = \partial_1\mathcal{U}_3 = 0 \quad \text{a.e. on } \Gamma_D\right\},$$

for $\Gamma_D = \{0,L\} \times (0,L)$.

Furthermore, the boundary conditions are inhomogeneous Dirichlet conditions, where the displacements satisfy

$$\mathcal{U}(0,x_2) = \begin{pmatrix} \frac{e^*L}{2} \\ 0 \\ 0 \end{pmatrix}, \qquad \mathcal{U}(L,x_2) = \begin{pmatrix} -\frac{e^*L}{2} \\ 0 \\ 0 \end{pmatrix}, \qquad \text{for a.e. } x_2 \in (0,L). \tag{5.9}$$

A displacement satisfying these conditions is

$$\widetilde{\mathcal{U}}(x') = e^* \Big(\frac{L}{2} - x_1\Big)\mathbf{e}_1 \quad \text{for a.e. } x' = (x_1, x_2) \in \omega, \tag{5.10}$$

with the symmetric strain tensor

$$e(\widetilde{\mathcal{U}}) = \begin{pmatrix} -e^* & 0 \\ 0 & 0 \end{pmatrix}.$$

Now, our aim is to minimize the functional for $\mathcal{U} \in \mathbb{U}_{Comp}$

$$\begin{aligned} \mathcal{J}^{hom}_{vK}(\mathcal{U} + \widetilde{\mathcal{U}}) = \int_\omega \Big(& a^{hom}_{1111}\big((\mathcal{Z}'_{11})^2 + (\mathcal{Z}'_{22})^2\big) + 4a^{hom}_{1212}(\mathcal{Z}'_{12})^2 + 2a^{hom}_{1122}\mathcal{Z}'_{11}\mathcal{Z}'_{22} \\ & + c^{hom}_{1111}\big((\partial_{11}\mathcal{U}_3)^2 + (\partial_{22}\mathcal{U}_3)^2\big) + 4c^{hom}_{1212}(\partial_{12}\mathcal{U}_3)^2 + 2c^{hom}_{1122}\partial_{11}\mathcal{U}_3\ \partial_{22}\mathcal{U}_3\Big)\, dx', \end{aligned} \tag{5.11}$$

with

$$\mathcal{Z}'_{\alpha\beta} = e_{\alpha\beta}(\mathcal{U} + \widetilde{\mathcal{U}}) + \partial_\alpha\mathcal{U}_3\partial_\beta\mathcal{U}_3 = \mathcal{Z}_{\alpha\beta} + \mathcal{Z}^*_{\alpha\beta},$$

where

$$\mathcal{Z}^*_{\alpha\beta} = e_{\alpha\beta}(\widetilde{\mathcal{U}}).$$

This means, we want to solve the minimization problem

$$\begin{cases} \text{Find } \mathcal{U}^* \in \mathbb{U}_{Comp} \text{ such that} \\ \min\limits_{\mathcal{U}\in\mathbb{U}_{Comp}} \mathcal{J}^{hom}_{vK}(\mathcal{U} + \widetilde{\mathcal{U}}) = \mathcal{J}^{hom}_{vK}(\mathcal{U}^* + \widetilde{\mathcal{U}}). \end{cases}$$

We know that the infimum of this functional on $\mathbb{U}_{Comp}$ is reached and the minimum exists, see chapter 4.

Remark 5.2.1. *The derivation in Chapter 4 is for applied forces. However, the energy minimization here is obtained analogously by considering*

$$m_\varepsilon = \inf_{v\in\mathbf{V}_\varepsilon} J_\varepsilon(v + \widetilde{v}),$$

as initial minimization problem, where $\widetilde{v} = I_d + \widetilde{\mathcal{U}}$*. The homogenization procedure is exactly the same.*

Note that a displacement corresponding to buckling admits out-of-plane displacements.

To identify the conditions for buckling, suppose for the time being that there is no buckling. Then the solutions of the above minimization are in fact the solution of the following minimization problem of a linear plate:

$$\begin{cases} \text{Find } \mathcal{U}^{*lin} \in \mathbb{U}_{Comp} \text{ such that} \\ \min\limits_{\mathcal{U}\in\mathbb{U}_{Comp}} \mathcal{J}^{hom}_{lin}(\mathcal{U} + \widetilde{\mathcal{U}}) = \mathcal{J}^{hom}_{lin}(\mathcal{U}^{*lin} + \widetilde{\mathcal{U}}) \end{cases} \tag{5.12}$$

where

$$\mathcal{J}^{hom}_{lin}(\mathcal{U}+\widetilde{\mathcal{U}}) = \int_\omega \Big(a^{hom}_{1111}\big((e_{11}(\mathcal{U}+\widetilde{\mathcal{U}}))^2 + (e_{22}(\mathcal{U}+\widetilde{\mathcal{U}}))^2\big) \\ + 4a^{hom}_{1212}(e_{12}(\mathcal{U}+\widetilde{\mathcal{U}}))^2 + 2a^{hom}_{1122}e_{11}(\mathcal{U}+\widetilde{\mathcal{U}})e_{22}(\mathcal{U}+\widetilde{\mathcal{U}})\Big)\, dx' \quad (5.13)$$

This is the energy for a linear orthotropic plate, where no bending occurs. The minimization problem (5.12) admits a unique solution $\mathcal{U}^{*lin}$, a pure in-plane displacement, which satisfies

$$0 < \mathcal{J}^{hom}_{lin}(\mathcal{U}^{*lin}+\widetilde{\mathcal{U}}) = C^*(e^*)^2 \leq \mathcal{J}^{hom}_{lin}(\widetilde{\mathcal{U}}) = |\omega|(e^*)^2 a^{hom}_{1111}{}^2. \quad (5.14)$$

Then, it is possible to characterize the buckling by the existence of a displacement of type $\mathcal{V}_3\mathbf{e}_3$ such that

$$\mathcal{J}^{hom}_{vK}(\mathcal{V}_3\mathbf{e}_3+\widetilde{\mathcal{U}}) < \mathcal{J}^{hom}_{lin}(\mathcal{U}^{*lin}+\widetilde{\mathcal{U}}) = \mathcal{J}^{hom}_{vK}(\mathcal{U}^{*lin}+\widetilde{\mathcal{U}}).$$

Taking into account the boundary conditions, the displacement of interest is independent of x_2 and has the form

$$\mathcal{V}_3(x') = V_3(x_1) \quad \text{for a.e. } x' = (x_1, x_2) \in \omega$$

with $V_3 \in H^2_0(0, L)$. It is easily checked that $\mathcal{V}_3\mathbf{e}_3 \in \mathbb{U}_{Comp}$. Computing the strain tensor

$$\begin{aligned} e_{11}(\widetilde{\mathcal{U}}) + (\partial_1\mathcal{V}_3)^2 &= -e^* + (V_3'(x_1))^2, \\ (\partial_{11}\mathcal{V}_3)^2 &= (V''{}_3(x_1))^2, \\ e_{12}(\widetilde{\mathcal{U}}) + \partial_1\mathcal{V}_3\partial_2\mathcal{U}_3 = e_{22}(\widetilde{\mathcal{V}}) + \partial_2\mathcal{V}_3\partial_2\mathcal{U}_3 &= 0, \\ (\partial_{22}\mathcal{V}_3)^2 = (\partial_{12}\mathcal{V}_3)^2 = \partial_{11}\mathcal{V}_3\ \partial_{22}\mathcal{V}_3 &= 0. \end{aligned}$$

yields the energy

$$\mathcal{J}^{hom}_{vK}(\mathcal{V}_3\mathbf{e}_3+\widetilde{\mathcal{U}}) = \int_0^L L\, a^{hom}_{1111}\big((e^*)^2 - 2e^*(V_3'(x_1))^2 + (V_3'(x_1))^4\big)dx_1 \\ + \int_0^L Lc^{hom}_{1111}(V''{}_3(x_1))^2 dx_1. \quad (5.15)$$

A necessary condition to obtain a buckling is $\mathcal{J}^{hom}_{vK}(\mathcal{V}_3\mathbf{e}_3+\widetilde{\mathcal{U}}) < (e^*)^2 a^{hom}_{1111}L^2$. Hence

$$\int_0^L a^{hom}_{1111}(V_3'(x_1))^4 dx_1 + \int_0^L c^{hom}_{1111}(V_3''(x_1))^2 dx_1 < 2e^* \int_0^L a^{hom}_{1111}(V_3'(x_1))^2 dx_1.$$

Now choose the function

$$V_3(x_1) = \sin^2\Big(\frac{\pi x_1}{L}\Big) \qquad \text{for all } x_1 \in [0, L].$$

[2] To get the exact value of the constant C^* we have to solve the corresponding linear problem.

Then a straight forward calculation leads to

$$a_{1111}^{hom}\frac{3\pi^4}{8L^3} + c_{1111}^{hom}\frac{2\pi^4}{L^3} < 2e^* a_{1111}^{hom}\frac{\pi^2}{2L},$$

and yields

$$e^* > \frac{\pi^2}{L^2}\frac{3a_{1111}^{hom} + 16c_{1111}^{hom}}{8a_{1111}^{hom}},$$

as condition on the applied strain.

For the sufficient condition, we give a lower bound on C^* in (5.14). The coercivity of the problem yields

$$c\|e(\mathcal{U}^{*lin} - \widetilde{\mathcal{U}})\|^2 \leq \mathcal{J}_{lin}^{hom}(\mathcal{U}^{*lin} - \widetilde{\mathcal{U}}). \tag{5.16}$$

Furthermore, it is clear that $\|e(\mathcal{U}^{*lin} - \widetilde{\mathcal{U}})\| > 0$ since the fields satisfy different boundary conditions. Indeed, we know that by the Korn-inequality and the trace-estimation give the following inequality:

$$\|e(\mathcal{U}^{*lin} - \widetilde{\mathcal{U}})\|_{L^2(\omega)} \geq c\|\mathcal{U}^{*lin} - \widetilde{\mathcal{U}}\|_{H^1(\omega)} \geq c\|\mathcal{U}^{*lin} - \widetilde{\mathcal{U}}\|_{L^2(\Gamma)} \geq cL^2 e^* > 0 \tag{5.17}$$

where c does not depend on ω.

Remark 5.2.2. *A weaker condition on* e^**, recovering conditions derived in [3, 34, 42], is given by another energy bound from below:*

$$\mathcal{J}_{vK}^{hom}(\mathcal{V}_3\mathbf{e}_3 + \widetilde{\mathcal{U}}) \geq \int_0^L L\left[c_{1111}^{hom} - e^*\frac{L^2}{2\pi^2}\right](V_3''(x_1))^2 dx_1. \tag{5.18}$$

Here we used the Poincaré inequality

$$\int_0^L (V_3'(x_1))^2\, dx_1 \leq \frac{L^2}{(2\pi)^2}\int_0^L (V_3''(x_1))^2\, dx_1. \tag{5.19}$$

The necessary condition is that the energy is coercive. This is satisfied if

$$e^* \geq \frac{\pi^2 c_{1111}^{hom}}{2L^2 a_{1111}^{hom}}. \tag{5.20}$$

For an isotropic plate of thickness h *the coefficients* $a_{1111}^{hom,iso} = \frac{Eh}{1-\nu^2}$ *and* $c_{1111}^{hom,iso} = \frac{Eh^3}{12(1-\nu^2)}$ *are derived in [39]. Then this condition degenerates to*

$$e^* \geq \frac{\pi^2 h^2}{24L^2}. \tag{5.21}$$

This corresponds to the critical force $F_{crit} = a_{1111}^{hom}e^* = \frac{Eh}{1-\nu^2}e^* = \frac{2Eh^3\pi^2}{24(1-\nu^2)L^2}$ *derived in [3, 34].*

5.2.1 Buckling under tension

In contrary to the section before we now consider the textile subject to uniaxial tension. For this assume that all displacements satisfy the following inhomogeneous Dirichlet conditions:

$$\mathcal{U}(0,x_2) = \begin{pmatrix} -\dfrac{e^*L}{2} \\ \nu e^*\Big(\dfrac{L}{2} - x_2\Big) \\ 0 \end{pmatrix}, \qquad \mathcal{U}(L,x_2) = \begin{pmatrix} \dfrac{e^*L}{2} \\ \nu e^*\Big(\dfrac{L}{2} - x_2\Big) \\ 0 \end{pmatrix}, \qquad \text{for a.e. } x_2 \in (0,L), \tag{5.22}$$

with $e^* > 0$ to obtain tension in $\mathbf{e}_1$-direction. Set

$$\widetilde{\mathcal{U}}(x') = -e^*\Big(\frac{L}{2} - x_1\Big)\mathbf{e}_1 + \nu e^*\Big(\frac{L}{2} - x_2\Big)\mathbf{e}_2 \quad \text{for a.e. } x' = (x_1,x_2) \in \omega. \tag{5.23}$$

This displacement satisfies the above Dirichlet conditions and one has

$$e(\widetilde{\mathcal{U}}) = \begin{pmatrix} e^* & 0 \\ 0 & -\nu e^* \end{pmatrix}.$$

Denote the new space of displacement

$$\mathbb{U}_{Tension} = \Big\{\mathcal{U} = \big(\mathcal{U}_1,\mathcal{U}_2,\mathcal{U}_3\big) \in H^1(\omega)^2 \times H^2(\omega) \ \mid \ \mathcal{U} = \partial_1\mathcal{U}_3 = 0 \text{ a.e. on } \Gamma_D \\ \text{and } \mathcal{U}_3 = \partial_2\mathcal{U}_3 = 0 \text{ a.e. on } \partial\omega \setminus \Gamma_D\Big\}.$$

Remark 5.2.3. *To obtain this as limit in the homogenization consider the initial problem with the boundary conditions $v|_{\Gamma_D} = Id$ and $v_3|_{\partial\omega\setminus\Gamma_D} = Id$ on the deformation, for which the homogenization works exactly as before.*

Furthermore, consider as above an energy of the form

$$\mathcal{J}^{hom}_{Vk}(\mathcal{U}) = \frac{1}{2}\int_\omega a_{\alpha\beta\alpha'\beta'}\mathcal{Z}'_{\alpha\beta}\mathcal{Z}'_{\alpha\beta} + c_{\alpha\beta\alpha'\beta'}\partial_{\alpha\beta}\mathcal{U}_3\partial_{\alpha'\beta'}\mathcal{U}_3 \, dydx' - \int_\omega 2a_{\alpha\beta\alpha'\beta'}e_{\alpha\beta}(\widetilde{\mathcal{U}})e_{\alpha'\beta'}(\mathcal{U})dx'dy. \tag{5.24}$$

where the force term is replaced by $a_{\alpha\beta\alpha'\beta'}e_{\alpha\beta}(\widetilde{\mathcal{U}})e_{\alpha'\beta'}(\mathcal{U})$ solely acting in-plane.

Recall that $e_{\alpha\beta}(\mathcal{U}) = \mathcal{Z}_{\alpha\beta} - \dfrac{1}{2}\partial_\alpha\mathcal{U}_3\partial_\beta\mathcal{U}_3$ and neglecting the constant or vanishing terms:

$$\mathcal{J}^{hom}_{vK}(\mathcal{U}) = \\ \frac{1}{2}\int_\omega \Big(a^{hom}_{\alpha\beta\alpha'\beta'}\mathcal{Z}_{\alpha\beta}\mathcal{Z}_{\alpha'\beta'} + c^{hom}_{\alpha\beta\alpha'\beta'}\partial_{\alpha\beta}\mathcal{U}_3\,\partial_{\alpha'\beta'}\mathcal{U}_3 - 2a^{hom}_{\beta\beta\beta'\beta'}e_{\beta\beta}(\widetilde{\mathcal{U}})(\mathcal{Z}_{\beta'\beta'} - \frac{1}{2}\partial_{\beta'}\mathcal{U}_3\partial_{\beta'}\mathcal{U}_3)\Big)\, dx'. \tag{5.25}$$

Instead of solving the problem completely, consider the functional depending on $\mathcal{J}^{hom}_{vK}(\mathcal{Z},\mathcal{U}_3)$. This allows to minimize the functional with respect to $\mathcal{Z}$, as in section 5.1.1, yielding the minimizer

$$\mathcal{Z}^* = e(\widetilde{\mathcal{U}}). \tag{5.26}$$

The remaining functional depends only on $\mathcal{U}_3$. Indeed, inserting $\mathcal{Z}^*$ yields

$$\mathcal{J}_{vK}^{hom,3}(\mathcal{U}_3) = \frac{1}{2}\int_\omega \left(c^{hom}_{\alpha\beta\alpha'\beta'}\partial_{\alpha\beta}\mathcal{U}_3\,\partial_{\alpha'\beta'}\mathcal{U}_3 - a^{hom}_{\beta\beta\beta'\beta'}e_{\beta\beta}(\widetilde{\mathcal{U}})\partial_{\beta'}\mathcal{U}_3\partial_{\beta'}\mathcal{U}_3\right)\,dx'. \tag{5.27}$$

For further simplification we assume that $\mathcal{U}_3(x') = \sin^2(\frac{\pi x_1}{L})\mathcal{V}_3(x_2)$, which satisfies the boundary condition of $\mathbb{U}_{Tension}$. Hence, the boundary conditions on $\mathcal{V}_3$ are $\mathcal{V}_3(0) = \mathcal{V}_3(L) = d_2\mathcal{V}_3(0) = d_2\mathcal{V}_3(L) = 0$. These boundary conditions can of course be replaced by other suitable conditions for the given problem.

Consequently, by inserting we obtain the functional only for x_2-direction

$$\mathcal{J}^{(3)}_{vK,x_2}(\mathcal{V}_3) = \frac{1}{2}\int_0^L A(\mathcal{V}_3)^2 + B(\partial_2\mathcal{V}_3)^2 + C(\partial_{22}\mathcal{V}_3)^2 dx_2 \tag{5.28}$$

with coefficients

$$\begin{aligned}
A &= -(a^{hom}_{1111}e_{11}(\widetilde{\mathcal{U}}) + a^{hom}_{2211}e_{22}(\widetilde{\mathcal{U}}))\frac{\pi^2}{2L} + c^{hom}_{1111}\frac{2\pi^4}{L^3},\\
&= -(a^{hom}_{1111}e^* - a^{hom}_{2211}\nu e^*)\frac{\pi^2}{2L} + c^{hom}_{1111}\frac{2\pi^4}{L^3},\\
B &= -(a^{hom}_{1122}e_{11}(\widetilde{\mathcal{U}}) + a^{hom}_{2222}e_{22}(\widetilde{\mathcal{U}}))\frac{3L}{8} + (2c^{hom}_{1122} + 4c^{hom}_{1212})\frac{\pi^2}{2L},\\
&= -(a^{hom}_{1122}e^* - a^{hom}_{2222}\nu e^*)\frac{3L}{8} + (2c^{hom}_{1122} + 4c^{hom}_{1212})\frac{\pi^2}{2L},\\
C &= c^{hom}_{2222}\frac{3L}{8}.
\end{aligned}$$

Obviously, we have $C > 0$. The coefficients A and B may become negative if e^* exceeds a critical threshold. If all coefficients are positive the only solution to the functional is $\mathcal{V}_3 \equiv 0$. The critical e^* depends on the material coefficients $a_{\alpha\alpha\beta\beta}$ and $c_{\alpha\alpha\beta\beta}$ for $(\alpha,\beta) \in \{1,2\}^2$. Suppose now $A \leq 0$ and $B \leq \frac{L^2}{\pi^2}|A|$. This becomes a critical case for e^* large enough as we show below. In fact, the assumption on A and B are chosen such that the following analysis works. Other cases may yield buckling too, which is why the actual critical strain may be different. Then, with the Poincaré-inequalities

$$\pi^2\int_0^L(\mathcal{V}_3)^2dx_2 \leq L^2\int_0^L(\partial_2\mathcal{V}_3)^2dx_2, \qquad \pi^2\int_0^L(\partial_2\mathcal{V}_3)^2dx_2 \leq L^2\int_0^L(\partial_{22}\mathcal{V}_3)^2dx_2, \tag{5.29}$$

it is possible to calculate the lower bound to the functional (5.28):

$$\mathcal{J}^{(3)}_{vK,x_2}(\mathcal{V}_3) \geq \frac{1}{2}\int_0^L\left(\frac{L^4}{\pi^4}A + \frac{L^2}{\pi^2}B + C\right)(\partial_{22}\mathcal{V}_3)^2dx'$$

We see that the energy remains positive, when

$$\frac{L^2}{\pi^2}\left(-\frac{1}{2}(a^{hom}_{1111}+a^{hom}_{2211}\nu)e^*+\frac{3}{8}(-a^{hom}_{1122}+a^{hom}_{2222}\nu)e^*\right)+(2c^{hom}_{1111}+\frac{3}{8}c^{hom}_{2222}+c^{hom}_{1122}+2c^{hom}_{1212})\geq 0.$$

This occurs if the strain does not exceed the critical value

$$e^*_{crit}=\frac{\pi^2}{L^2}\frac{16c^{hom}_{1111}+3c^{hom}_{2222}+8c^{hom}_{1122}+16c^{hom}_{1212}}{4a^{hom}_{1111}-3\nu a^{hom}_{2222}+a^{hom}_{1122}(3-4\nu^{hom})}. \tag{5.30}$$

This e^*_{crit} satisfies both assumption $A \leq 0$ and $B \leq \frac{L^2}{\pi^2}|A|$ used above.

As conclusion of the buckling analysis we want to mention that the critical force or strain is heavily dependent on the applied boundary conditions.

5.3 Optimization of the Buckling

The optimization of buckling is obviously an optimization with a partial differential equation as constraint, since the buckling has to satisfy a plate-equation. Although the numerical treatment below is without further analysis, we refer to [31, 35] for literature on optimization problems with PDE-constraints.

For the numerical simulation of the macroscopic plate we restrict to the one-dimensional compression problem recalled in section 5.1.1 and [3]. Thus, consider an isotropic plate of thickness h and denote the bending stiffness $C = c_{1111} = \frac{Eh^3}{12(1-\nu^2)}$ with the Young's modulus E and Poisson's ratio ν. Consider for the force $F > 0$ the energy

$$J(u)=\frac{1}{2}\int_{-L}^{L} C(\partial_{xx}u)^2 - F(\partial_x u)^2 dx.$$

This energy is used in [3] and also gives a condition for the textile on the critical force, cf. Remark 5.2.2.

For the problem to be well posed we assume a force $F < F_{crit} = \frac{\pi^2 C}{L^2} = \frac{\pi^2 h^3 E}{12(1-\nu^2)L^2}$.

As objective for the optimization we consider the delay and the optimization of the buckling shape. The latter is addressed by a typical tracking term with a given goal shape of the first mode. Hereafter, the different objectives are explained and constraints coming from industry are discussed.

5.3.1 Maximization of the critical force

The delay of buckling is equivalent to the maximization of the critical force F_{crit}. To compute the critical force, consider the generalized Rayleigh-quotient [40, 41] given by

$$R(u)=\frac{\frac{1}{2}\int C(x)(d_{xx}u)^2 - F(d_x u)^2\,dx}{\int (d_x u)^2\,dx}=\frac{J(u)}{\int (d_x u)^2\,dx}, \tag{5.31}$$

for $u \in H_0^2(-L,L) = \{u \in H^2(-L,L) \mid u(\pm L) = d_x u(\pm L) = 0\}$. Note that the denominator defines a norm on this space.

Remark 5.3.1. *The usual Rayleigh quotient reads as*

$$R(x) = \frac{\langle x, Ax\rangle}{\langle x, x\rangle} \tag{5.32}$$

with an operator A and the scalar product $\langle\cdot,\cdot\rangle$ of the underlying space. The generalized version takes another normalization into account such that

$$R_g(x) = \frac{\langle x, Gx\rangle}{\langle x, Hx\rangle}. \tag{5.33}$$

In our case the operators are

$$G(u) = d_{xx}(\frac{C}{2} d_{xx}u) + \frac{F}{2} d_{xx}u, \qquad H(u) = -d_{xx}u \tag{5.34}$$

whereupon we use the partial integration theorem to transfer the derivatives. Since $u \in H_0^2(-L,L)$ the denominator is positive definite.

Rewriting the Rayleigh quotient and taking the minimum yields

$$\min_u R(u) = \frac{\int C(x)(d_{xx}u)^2\,dx}{2\int (d_x u)^2\,dx} - \frac{F}{2}. \tag{5.35}$$

To show that this is an observable for the critical force, assume for the time being that C is constant. Then, together with the result of section 5.1.1 saying that the solutions satisfy (5.19) with equality we obtain for the Rayleigh quotient

$$\min_u R(u) = \frac{1}{2}\left[\frac{C\pi^2}{L^2} - F\right]. \tag{5.36}$$

Setting $\min_u R(u) = 0$ yields directly the critical force $F_{crit} = \frac{C\pi^2}{L^2} = \frac{\pi^2 E h^3}{12(1-\nu^2)L^2}$, which was already found before in Remark 5.2.2 and [3, 34, 42].

Hence, the generalized Rayleigh quotient is a suitable way to express the critical force. With this it is not necessary to compute the critical force explicitly to maximize it. Instead by keeping the force F constant, a maximization of $\min_u R(u)$ itself yields the same result and is easier to handle and implement.

5.3.2 Optimization of the buckling shape: Tracking term

For the optimization of the shape, the typical tracking term is used. In this case we consider the L^2-metric and a goal-function u_g. For simplicity, chose $L = 1$. Then, the tracking term is given by

$$\|u - u_g\|_{L^2(\Omega)}. \tag{5.37}$$

The goal-function u_g has to be chosen a priori and is usually constrained and defined by the application. In fact, u_g does not have to satisfy the underlying plate equation. If this is not the case equality $u = u_g$ is excluded and the tracking term does not vanish. Such not attainable goal-functions still have a reason for they allow to define a desired shape and obtain a solution close to the preferred design via the optimization.

We consider a setting, where the goal-function u_g is said to be flat at the boundaries and we define

$$\overline{u}_g(x) = \begin{cases} 0 & x \in (-1, -\ell), \\ \cos(\frac{\pi x}{\ell}) + 1 & x \in (-\ell, \ell), \\ 0 & x \in (\ell, 1). \end{cases} \tag{5.38}$$

The actual goal function $u_g(x) = \frac{\overline{u}_g(x)}{\|\overline{u}_g\|_{L^2(-1,1)}}$ is normalized.

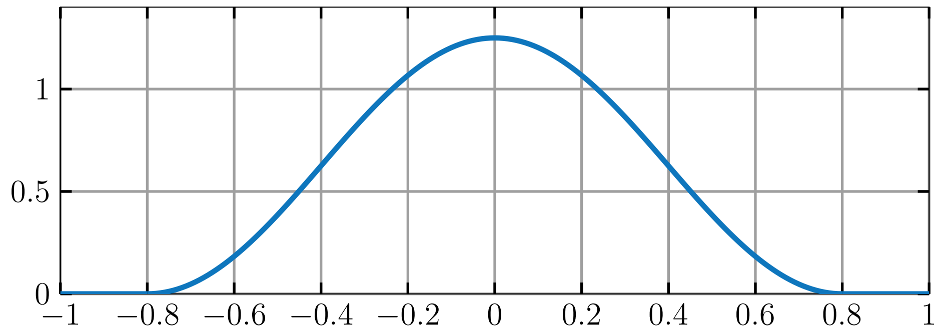


Figure 5.3: The normalized goal-function $\overline{u}_g$ with $\ell = 0.8$.

The goal function $\overline{u}_g$ is continuously differentiable and satisfies the boundary conditions by construction. It is certainly no solution to the plate equation.

5.3.3 Further constraints on the optimization

Box constraints

Typical further constraints to the optimization are box-constraints on the design space. This means that the parameters are only viable in a certain range. For the buckling optimization, the bending stiffness C is only allowed to take values in the interval $I = (a, b)$, with $0 < a \leq b < +\infty$. These constraints are necessary since otherwise unphysical states like $C = +\infty$ or $C = 0$ could arise during the optimization. These two states correspond either to a total rigid material or no material at all.

Further constraints from the textile industry

Of course there are a lot of additional constraints possible. The constraints coming from industrial projects often include requirements on structure and symmetries of the textile.

For instance, consider the following constraints, which are also implemented in the numerical example:

- The preservation of the mean bending stiffness, i.e.,

$$\mathcal{M}_C = \int_{-1}^{1} C(x)\, dx \tag{5.39}$$

  has to remain constant or is only allowed up to a deviation of a given percentage. Hereafter, we assume a deviation of 10% of the initially given value M_0. Treated with care, this constraint can also replace either the lower or upper bound of the box constraints, but usually not both. This condition is a nonlocal box constraint.

- The bending stiffness has to be symmetric with respect to the mid of the specimen, i.e. $C(x) = C(-x)$.

- The bending stiffness is piecewise constant. This corresponds to the fact that continuous textiles are not producible and the industry can not vary every fiber. However, they allow for a stripe like structure, where stripes of specified width have the same fibers used.

5.3.4 The optimal control problem

The full optimal control problem consists of an objective functions $J(u)$ with auxiliary conditions and constraints. In our case we consider the objective functional

$$J_\gamma(u) = \gamma \|u - u_g\|_{L^2(\Omega)} - (1-\gamma)\lambda_C(u) \tag{5.40}$$

with an weighting-factor γ, the goal-function u_g and the minimal Rayleigh-quotient $\lambda_C(u)$. Note that the the second term is negative, such that the minimization of J_γ maximizes $\lambda_C(u)$.

The weighting-factor $\gamma \in [0,1]$ is necessary to change the weighting of the two contributions to the objective functional. The optimal value of J_γ and the minimizers itself are of course highly dependent on this factor.

Collecting the constraints the optimization problem is given by

$$\begin{aligned} &\min_C J_\gamma(u), \\ &s.t.\ a \le C(x) \le b, \\ &\qquad \lambda_C = \min_u R(u), \\ &\qquad 0.9 \le \mathcal{M}_C/\mathcal{M}_0 \le 1.1, \\ &\qquad C \text{ symmetric.} \end{aligned} \tag{5.41}$$

Furthermore, the bending stiffness is assumed partitioned and piecewise constant in each segment.

5.4 Numerics

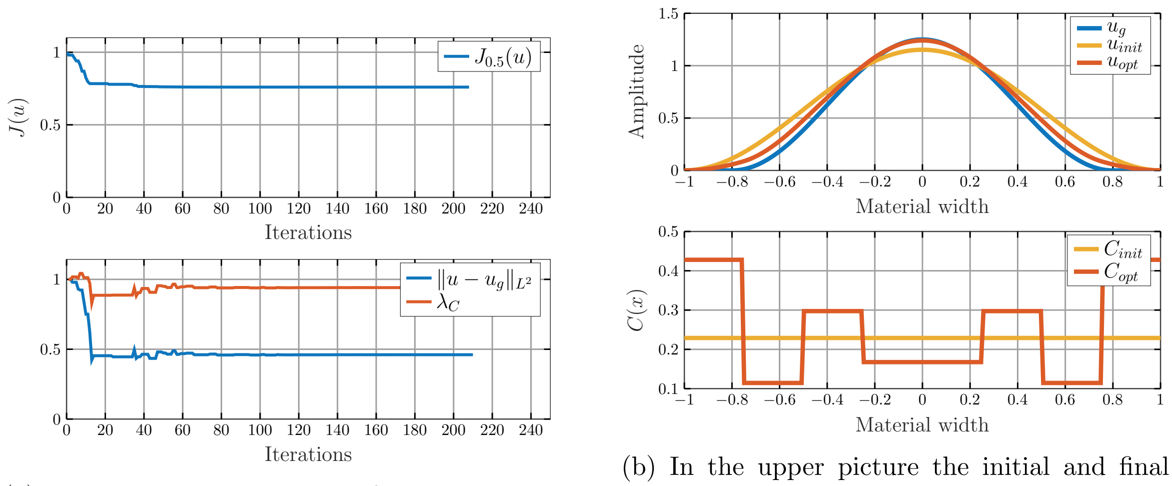
The optimization problem above was implemented in MATLAB together with the MATLAB-internal functions `eig` to calculate the minimal eigenvalue and `fminsearch` for the minimization. Although the function `fminsearch` is inefficient for searching minima compared to gradient methods, yet it suffices for the considered task and does not require differentiability. The implementation is based on finite differences.

The implementation of the full micro-structure optimization is divided into two steps. First the macroscopic optimization problem gives rise to macroscopic values of the bending stiffness C. In the second step these optimal macroscopic are fitted by a second optimization which finds the corresponding micro-structure. For more complex structure the second step is done with the help of software from the Fraunhofer ITWM in order to obtain cell problems depending on a prior chosen parametrization of the textile structure, see [30].

The parameters used in the simulations below are taken from actual industrial projects. We consider the length $L = 1m$ the thickness $h = 1mm$, the Poisson's ratio $\nu = 0.3$, Young's modulus $E = 2.5 \cdot 10^9 Pa$ and $F = \frac{F_{crit}}{2}$ close to the critical force $F_{crit} = \frac{\pi^2 E h^3}{12(1-\nu^2)L^2} \approx 0.564 N/m$. The box constraint for the bending stiffness is only one-sided, i.e. $C(x) \geq 0.11$.

5.4.1 The balanced case

The first simulation is for the balanced case, where both parts of the objective functional are of the same importance. This corresponds to $\gamma = 0.5$.



(a) The normalized objective functional with $\gamma = 0.5$ and the evolution of the two parts.

(b) In the upper picture the initial and final buckling mode as well as the reference goal-shape is depicted. In the lower picture the initial and final bending stiffness C is shown.

Figure 5.4: Optimization of buckling for $\gamma = 0.5$.

In Figure 5.4 the simulation results of the optimization are depicted. The objective functional $J_{\gamma=0.5}(u)$ is minimized and converges to a minimum, at least a local one as we see in the next simulations below. The two terms, of which the objective functional consists, are effectively not equally contributing. This means in hindsight that the minimization of the tracking term is more efficient in terms of cost than the maximization of λ_C. The tracking term is improved by more than 50%, while λ_C did not improve more that 6%.

The final buckling mode u_{opt} is shown in Figure 5.4b and it is visible that the optimal shape approached the goal function u_g. The corresponding initial and optimal bending stiffnesses are depicted in the lower graph. The high bending stiffness at the ends directly represent the choice of the goal-function u_g to be completely flat at the ends.

5.4.2 The Pareto front: Varying γ

The parameter γ in the objective functional parametrizes the Pareto-front and gives control over the contribution of the two single objectives. The Figure 5.5 shows the percentage of improvement of the single objectives for varying $\gamma \in [0, 1]$. Note that the figure only shows the individual improvements and not the overall improvement of $J(u)$, which is depicted in Figure 5.6. However, the parameter γ is a possibility to change the weight of the two objectives and heavily affects the output. It is crucial to identify the requirement for the optimization and adjust γ accordingly.

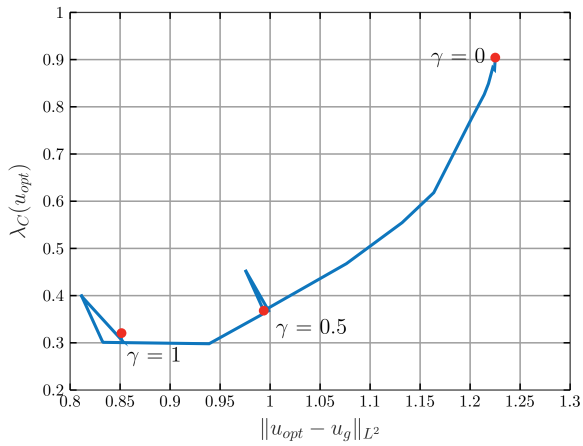


Figure 5.5: The influence of γ on the two competing terms in the objective functional. For $\gamma = 0$ only the maximization of λ_C contributes, while for $\gamma = 1$ the objective functional only consists of the tracking term $\|u_b - u_g\|_{L^2}$. This characterizes the Pareto front for this problem.

The minimal value of the full objective functional J_γ is depicted in Figure 5.6. The figure shows that the pure minimization of the tracking term for $\gamma = 1$ yields better overall results than the maximization of λ_C.

The artifacts on the Pareto-curve in Figure 5.5 come most likely by the MATLAB-built-in minimization scheme `fminsearch`.

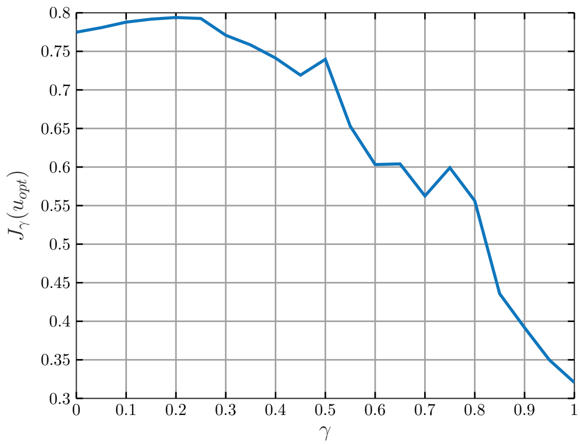
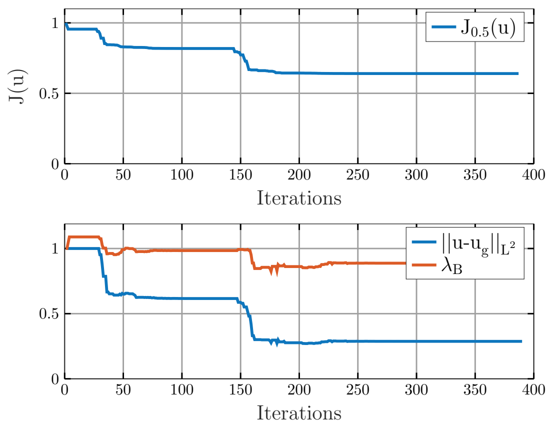


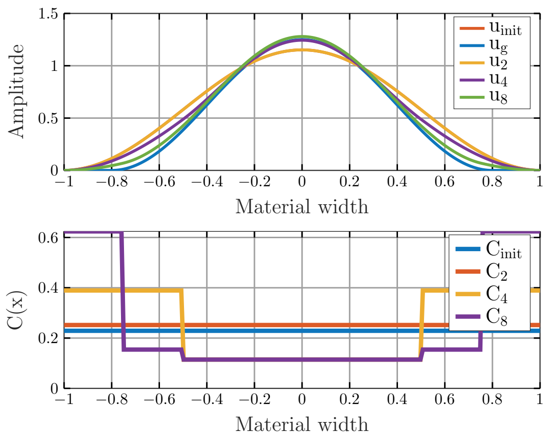
Figure 5.6: The optimal value of the objective functional $J_\gamma(u_{opt})$ with respect to the parameter $\gamma \in [0, 1]$.

5.4.3 Hierarchical approach

A promising approach for these kind of optimization problems is to solve hierarchically ordered problems and use intermediate results as initial condition for the subsequent optimization step. For the given problem we consider increasing numbers of segments with different bending stiffness. Specifically, we start with two segments and every hierarchical step the segments are doubled.



(a) Development of the objective functional $J_{0.5}$ in (upper figure) and its components (lower figure) for the hierarchical computation.

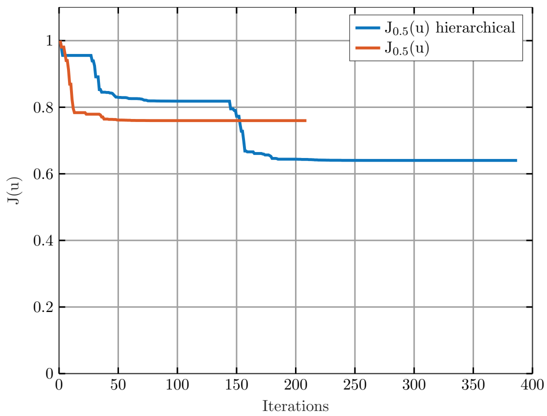


(b) The upper figure shows the development of the optimal mode-shapes for each hierarchical step. The lower figure shows the optimal bending stiffnesses for the different hierarchical steps.

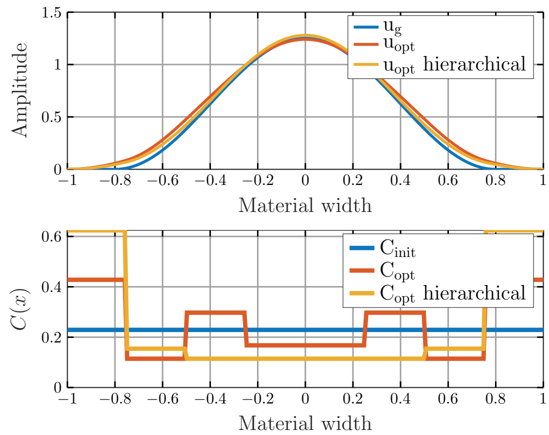
Figure 5.7: Results of the hierarchical optimization

The Figure 5.7 summarizes the intermediate and final results arising from the hierarchical approach. Specifically, in Figure 5.7a the development of the objective functional is depicted. Noticeable is the effect that usually the tracking term is preferred in the optimization, as after every segment incrementation it is optimized accompanied with a deterioration of λ_C.

The comparison of the hierarchical result with the direct result obtained beforehand in section 5.4.1 yields interesting insights into the problem. First of all the hierarchical approach is slightly better, as we see in Figure 5.8a. Since both schemes, the direct and the hierarchical, converge up the a relative error of 10^{-8} it is clear that the problem admits different local minima. Since the weighting factor $\gamma = 0.5$ remains for both simulations, it certainly depends on the starting value for the bending stiffness. The difference of the minimizers is shown in the lower part of Figure 5.8, where both minimizers are clearly distinct. The mode-shapes though, are both very close to u_g and differ only slightly from each other.



(a) The development of objective functionals $J_{0.5}$ of hierarchical and direct approach.



(b) The upper figure compares the optimal mode-shapes with u_g as reference. In the lower figure the optimal bending stiffnesses are compared.

Figure 5.8: Comparison of hierarchical and direct computation.

5.4.4 The micro-structure-optimization

The final step of the optimization is to find the corresponding micro-structure for the optimal bending stiffnesses found by the procedure above. Of course the bending stiffness has to satisfy some bounds, which is accounted for in the box constraints on C.

As example for the micro-structure optimization for given values of the bending stiffness C we consider an academic example. This example consist of a very simple open grid structure. This structure is well investigated and admits an analytic solution for the bending stiffness, see [47].

The computations in [47, Section 7.2.3] show that the bending stiffness in the respective directions are

$$C = \frac{Eh^3 b_\alpha}{12 t_\alpha} \tag{5.42}$$

where E denotes Young's modulus of the beams in the structure. Hence, for this simple structure it is easily possible to derive the correct micro-structure to the optimal macroscopic bending stiffness. Note, that the directions are independent, which allows to optimize one direction without hindering the other one.

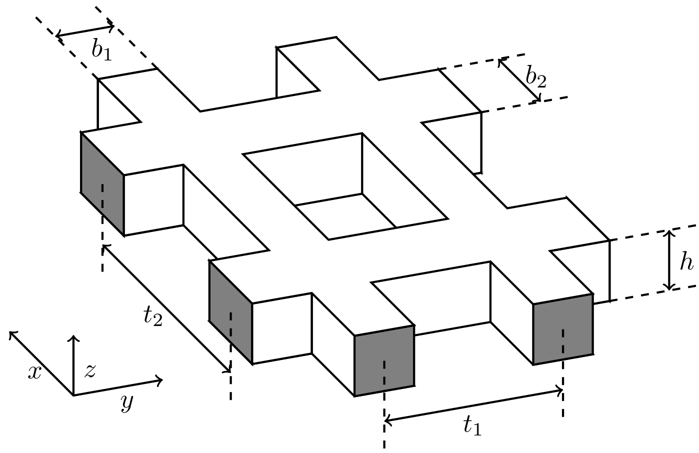


Figure 5.9: An open grid-structure of height h with the periodicities t_1, t_2 and the widths b_1, b_2 in the respective direction.

The optimal micro structure for each stripe is different and depends on the design variables, which can be chosen from the parameters in (5.42).

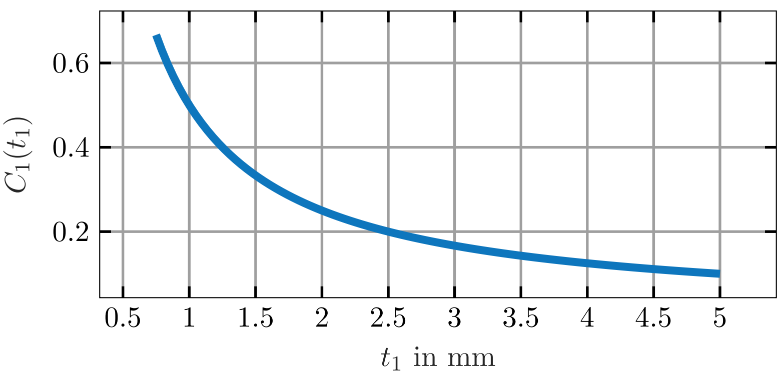


Figure 5.10: The homogenized bending stiffness C_1 with respect to the distance t_1 between beams. The other parameters are the Young's modulus $E = 6 \cdot 10^9 Pa$ and $b_1 = h = 1mm$.

As example consider the macroscopic result presented in section 5.4.1. Recall, the resulting optimal bending stiffness $C_{opt}(x)$ takes values in the interval $(0.1, 0.6)$, see Figure 5.4b. The Figure 5.9 shows that these values are covered by varying the distance between the beams $t_1 \in [0.7mm, 5mm]$. Numerically, the fitting is done via a gradient method and yields different distances between the fibers for each stripe of the textile. For the textile industry these stripe-wise changes are easily achieved by small modifications in the production.

Other possible design parameters are the height h, the Young's modulus E of the beams and their width b_1. The height is usually not considered as design parameter, as it changes the textile texture and surface to much.

For more complex structures simulation tools are necessary to solve the cell-problems and obtain the effective properties. This can be done symbolically for the variables in design space, which allows to use standard projected gradient methods to find the micro-structure to a given bending stiffness C. Due to nonlinearities in the dependencies of effective properties on the design space it is possible that local minima disturb the gradient methods. This is in general the case for more complex structures. For this more involved optimization we refer to [30, 43].

Conclusion

Chapter 6

Conclusion

In this thesis we derived the homogenization and dimension reduction for textiles within two different energy regimes: the linear elasticity and the von-Kármán regime. The Korn-type estimates for both regimes are established with the help of an textile-adapted decomposition of displacements. The derivation of the limit is done with the a unfolding-rescaling operator, accounting for the beam-structure.

In particular, the homogenization of the textile in the regime of linear elasticity is augmented by a Signorini's condition to include the influence of contact between the yarns, see Ch. 3. The contact condition leads to different limit problems, where we concentrated in this thesis on the cases $\|g_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{3+\delta}$ with $\delta = 0$ and $\delta > 0$, corresponding to a Leray-Lions-type or standard linear elasticity problem, respectively. From these different limit and the estimates before it is obvious that the contact between fibers plays a crucial role in terms of regularity of the problem. Other, weaker assumptions on the contact condition tremendously change the limit-problem and need further investigations.

A second homogenization in the von-Kármán regime, starting from nonlinear elasticity with rigid contact between the fibers, yields a homogenized von-Kármán-plate in the limit. Again the unfolding-rescaling operator is used to derive the limit-displacements. Although, starting in nonlinear elasticity the limit derived by Γ-convergence arguments gives rise to linear cell problems equivalent to the cell problems for the standard homogenized linear plate. Eventually, it is proven that in the case of isotropic homogeneous fibers, the resulting homogenized textile is orthotropic for the full linear elastic regime and the von-Kármán regime. For all identified homogenized limit problems the existence of solutions is shown, while the uniqueness of solution is only be achieved for the linear limit plate.

The last part is dedicated to the buckling textiles, which illustrates the application of the obtained results. We derive the critical force for buckling for the orthotropic textile under both, compression and tension. In a final step the buckling of textiles is optimized with respect to delay of buckling and the mode shape. Different numerical examples are presented.

Altogether, this thesis provides the asymptotic analysis allowing to investigate textiles as macroscopic two dimensional plates for two different energy regimes.

Appendix

A Important Results used in the thesis

Theorem A.1 (Poincaré inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then every function $u \in H^1(\Omega)$ satisfies*

$$\|u - \mathcal{M}_\Omega(u)\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)} \tag{6.1}$$

with the constant C only depending on Ω. Moreover, if $u \in H^1_\Gamma(\Omega)$, i.e. the trace $u_{|\Gamma} = 0$ vanishes, it satisfies

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)} \tag{6.2}$$

with the constant C only depending on Ω and Γ.

Proof. See [15, Ch. 3]. □

Theorem A.2 (First Korn inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then every function $u \in H^1_0(\Omega)$ satisfies*

$$\|\nabla u\|^2_{L^2(\Omega)} \leq 2\|e(u)\|^2_{L^2(\Omega)} \tag{6.3}$$

Proof. See [38, Ch. 2]. □

Theorem A.3 (Second Korn inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then every function $u \in H^1_0(\Omega)$ satisfies*

$$\|\nabla u\|^2_{L^2(\Omega)} \leq C\left(\|u\|^2_{L^2(\Omega)} + \|e(u)\|^2_{L^2(\Omega)}\right), \tag{6.4}$$

with the constant C only depending on Ω

Proof. See [38, Ch. 2]. □

B Appendix homogenization

Lemma B.1. *Let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence of functions in $L^2(\Omega)$ satisfying*

$$\|\phi_\varepsilon\|_{L^2(\Omega)} + \varepsilon\Big\|\frac{\partial \phi_\varepsilon}{\partial z_1}\Big\|_{L^2(\Omega)} \le C$$

where C does not depend on ε. Then, up to a subsequence, there exists $\widehat{\phi} \in L^2(\Omega \times \mathcal{Y})$ such that $\widehat{\phi}$ is 2-periodic with respect to X_1 and $\dfrac{\partial \widehat{\phi}}{\partial X_1} \in L^2(\Omega \times \mathcal{Y})$. Moreover

$$\begin{aligned}
&\mathcal{T}_\varepsilon(\phi_\varepsilon) \rightharpoonup \widehat{\phi} \quad \text{weakly in } L^2(\Omega \times \mathcal{Y}),\\
&\varepsilon\mathcal{T}_\varepsilon\Big(\frac{\partial \phi_\varepsilon}{\partial z_1}\Big) \rightharpoonup \frac{\partial \widehat{\phi}}{\partial X_1} \quad \text{weakly in } L^2(\Omega \times \mathcal{Y}),\\
&\mathcal{T}_\varepsilon(\phi_\varepsilon)_{|X_1=a} \rightharpoonup \widehat{\phi}_{|X_1=a} \quad \text{weakly in } L^2\big(\Omega \times (\overline{\mathcal{Y}} \cap \{X_1 = a\})\big).
\end{aligned}$$

Lemma B.2. *One has*

$$\begin{aligned}
&\big\|\mathbb{U} - \mathcal{V}_0\big\|^2_{L^2(0,r)} + \big\|\mathbb{U} - \mathcal{V}_{2N_\varepsilon}\big\|^2_{L^2(L-r,L)} + \sum_{p=1}^{2N_\varepsilon-1} \big\|\mathbb{U} - \mathcal{V}_p\big\|^2_{L^2(p\varepsilon-r,p\varepsilon+r)} \le C\|e(u)\|^2_{L^2(\mathcal{P}_\varepsilon)},\\
&\big\|\mathcal{R} - \mathcal{R}(0)\big\|^2_{L^2(0,r)} + \big\|\mathcal{R} - \mathcal{R}(L)\big\|^2_{L^2(L-r,L)}\\
&\qquad\qquad + \sum_{p=1}^{2N_\varepsilon-1} \big\|\mathcal{R} - \mathcal{R}(p\varepsilon)\big\|^2_{L^2(p\varepsilon-r,p\varepsilon+r)} \le \frac{C}{r^2}\|e(u)\|^2_{L^2(\mathcal{P}_\varepsilon)},
\end{aligned} \tag{6.5}$$

where the rigid motion in the knots is denoted by

$$\mathcal{V}_p(z_1) = \mathbb{U}(p\varepsilon) + (z_1 - p\varepsilon)\mathcal{R}(p\varepsilon) \wedge \mathbf{e}_1 \tag{6.6}$$

for a.e. $z_1 \in (p\varepsilon - r, p\varepsilon + r) \cap (0, L)$, $p \in \{0, \dots, 2N_\varepsilon\}$.

Proof. From (3.16) and the Poincaré inequality $(6.5)_2$ is obtained. For $(6.5)_1$ observe the identity

$$d_1\mathbb{U} - d_1\mathcal{V}_p - \big(\mathcal{R} - \mathcal{R}(p\varepsilon)\big) \wedge \mathbf{e}_1 = d_1\mathbb{U} - \mathcal{R} \wedge \mathbf{e}_1$$

Then, from (3.18) it is clear that

$$\begin{aligned}
&\Big\|d_1\mathbb{U} - d_1\mathcal{V}_0 - \big(\mathcal{R} - \mathcal{R}(0)\big) \wedge \mathbf{e}_1\Big\|^2_{L^2(0,r)} + \Big\|d_1\mathbb{U} - d_1\mathcal{V}_{2N_\varepsilon} - \big(\mathcal{R} - \mathcal{R}(L)\big) \wedge \mathbf{e}_1\Big\|^2_{L^2(L-r,L)}\\
&\qquad + \sum_{p=1}^{2N_\varepsilon-1} \Big\|d_1\mathbb{U} - d_1\mathcal{V}_p - \big(\mathcal{R} - \mathcal{R}(p\varepsilon)\big) \wedge \mathbf{e}_1\Big\|^2_{L^2(p\varepsilon-r,p\varepsilon+r)} \le \frac{C}{r^2}\|e(u)\|^2_{L^2(\mathcal{P}_\varepsilon)}
\end{aligned}$$

Finally, using $(6.5)_2$ yields

$$\big\|d_1\mathbb{U}-d_1\mathcal{V}_0\big\|^2_{L^2(0,r)}+\big\|d_1\mathbb{U}-d_1\mathcal{V}_{2N_\varepsilon}\big\|^2_{L^2(L-r,L)}\\ +\sum_{p=1}^{2N_\varepsilon-1}\big\|d_1\mathbb{U}-d_1\mathcal{V}_p\big\|^2_{L^2(p\varepsilon-r,p\varepsilon+r)}\le\frac{C}{r^2}\|e(u)\|^2_{L^2(\mathcal{P}_\varepsilon)},$$

from which the Poincaré inequality leads to $(6.5)_1$. $\square$

This Lemma shows that the displacements in the contact can be replaced by rigid motions.

C Extension results for deformations and displacements

In this section, Ω and Ω' are two bounded domains in $\mathbb{R}^n$ containing the origin with Lipschitz boundaries and such that $\Omega\subset\Omega'$. For every $\varepsilon>0$, we denote $\Omega_\varepsilon=\varepsilon\,\Omega$ and $\Omega'_\varepsilon=\varepsilon\,\Omega'$. In Lemma C.1 we prove an extension result for deformations in $H^1(\Omega_\varepsilon)^n$.

Lemma C.1. *For every deformation v in $H^1(\Omega_\varepsilon)^n$ there exists a deformation $\widetilde{v}$ in $H^1(\Omega'_\varepsilon)^n$ satisfying*

$$\widetilde{v}_{|\Omega_\varepsilon}=v,\qquad \textit{and}\qquad \big\|dist\big(\widetilde{v},SO(n)\big)\big\|_{L^2(\Omega'_\varepsilon)}\le C\big\|dist(v,SO(n))\big\|_{L^2(\Omega_\varepsilon)}. \tag{6.7}$$

The constant C does not depend on ε.

Proof. First, some classical recalls and then the proof.

(i) Since Ω is a bounded domain with Lipschitz boundary, there exist $N\in\mathbb{N}^*$, R_1 and R_2 two strictly positive constants and a finite set $\{\mathcal{O}_1,\dots,\mathcal{O}_N\}$ of open subsets of Ω, each of diameter less than R_1 and star-shaped with respect to a ball of radius R_2 ($B(A_i,R_2)$, $A_i\in\mathcal{O}_i$) such that

$$\Omega=\bigcup_{k=1}^N\mathcal{O}_k.$$

As a consequence, there exists r such that for every $\mathcal{O}_i$, $i\in\{1,\dots,N\}$ there exists a chain from $\mathcal{O}_1$ to $\mathcal{O}_i$

$$\mathcal{O}_{l_1}=\mathcal{O}_1,\ \ \mathcal{O}_{l_2},\ \ \dots,\ \ \mathcal{O}_{l_p}=\mathcal{O}_i,\qquad p\in\{1,\dots,N\}$$

such that, if $p>1$ one has $\mathcal{O}_{l_j}\cap\mathcal{O}_{l_{j+1}}$, $j\in\{1,\dots,p-1\}$, contains a ball of radius r.

(ii) Let $\mathcal{O}$ be an open set in $\mathbb{R}^n$ included in the ball $B(A;R_1)$ and star-shaped with respect to the ball $B(A,R_2)$, $R_1>0$, $R_2>0$. Theorem II.1.1 in [6] claims that for every deformation $v\in H^1(\mathcal{O})^n$, there exist a matrix $\mathbf{R}\in SO(n)$ and $\mathbf{a}\in\mathbb{R}^n$ such that

$$\|v-\mathbf{a}-\mathbf{R}\,x\|_{L^2(\mathcal{O})}\le CR_1\|dist(\nabla v,SO(3))\|_{L^2(\mathcal{O})},\\ \|\nabla v-\mathbf{R}\|_{L^2(\mathcal{O})}\le C\|dist(\nabla v,SO(n))\|_{L^2(\mathcal{O})}.$$

The constant C depends only on $\dfrac{R_1}{R_2}$ and n.
Transform $\mathcal{O}$ by a dilation of ratio $\varepsilon > 0$ and center A, the above result gives: for every deformation $v \in H^1(\mathcal{O}_\varepsilon)^n$ where $\mathcal{O}_\varepsilon \doteq \varepsilon\mathcal{O}$, there exist a matrix $\mathbf{R} \in SO(n)$ and $\mathbf{a} \in \mathbb{R}^n$ such that

$$\begin{aligned}
&\|v - \mathbf{a} - \mathbf{R}\,x\|_{L^2(\mathcal{O}_\varepsilon)} \le C\varepsilon\|dist(\nabla v, SO(n))\|_{L^2(\mathcal{O}_\varepsilon)},\\
&\|\nabla v - \mathbf{R}\|_{L^2(\mathcal{O}_\varepsilon)} \le C\|dist(\nabla v, SO(n))\|_{L^2(\mathcal{O}_\varepsilon)}.
\end{aligned}$$

The constant C does not depend on ε.

(iii) Ω and Ω' being two bounded domains in $\mathbb{R}^n$ with Lipschitz boundaries and such that $\Omega \subset \Omega'$. There exists a continuous linear extension operator $\mathcal{P}'$ from $H^1(\Omega)^n$ into $H^1(\Omega')^n$ satisfying

$$\begin{aligned}
&\mathcal{P}'(v)_{|\Omega} = v,\\
&\|\mathcal{P}'(v)\|_{L^2(\Omega')} \le C\|v\|_{L^2(\Omega)}, \qquad \forall v \in H^1(\Omega)^n,\\
&\|\mathcal{P}'(v)\|_{H^1(\Omega')} \le C\|v\|_{H^1(\Omega)}.
\end{aligned}$$

If we transform Ω and Ω' by the same dilation of ratio ε (and center $O \in \Omega$), this extension operator induces an extension operator $\mathcal{P}'_\varepsilon$ from $H^1(\Omega_\varepsilon)^3$ into $H^1(\Omega'_\varepsilon)^3$ satisfying

$$\forall v \in H^1(\Omega_\varepsilon)^3, \quad \begin{cases}
\mathcal{P}'_\varepsilon(v)_{|\Omega_\varepsilon} = v,\\
\|\mathcal{P}'_\varepsilon(v)\|_{L^2(\Omega'_\varepsilon)} \le C\|v\|_{L^2(\Omega_\varepsilon)},\\
\|\mathcal{P}'_\varepsilon(v)\|_{L^2(\Omega'_\varepsilon)} + \varepsilon\|\nabla\mathcal{P}'_\varepsilon(v)\|_{L^2(\Omega'_\varepsilon)} \le C\big(\|v\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|\nabla v\|_{L^2(\Omega_\varepsilon)}\big).
\end{cases}$$

The constants do not depend on ε.

Now, consider a deformation $v \in H^1(\Omega_\varepsilon)^n$. We apply (ii) with the open sets $\mathcal{O}_{i,\varepsilon} = A_i + \varepsilon(\mathcal{O}_i - A_i)$, there exist matrices $\mathbf{R}_i \in SO(n)$ and vectors $\mathbf{a}_i \in \mathbb{R}^n$ such that

$$\begin{aligned}
&\|v - \mathbf{a}_i - \mathbf{R}_i\,x\|_{L^2(\varepsilon\mathcal{O}_{i,\varepsilon})} \le C\varepsilon\|dist(\nabla v, SO(n))\|_{L^2(\varepsilon\mathcal{O}_{i,\varepsilon})},\\
&\|\nabla v - \mathbf{R}_i\|_{L^2(\varepsilon\mathcal{O}_{i,\varepsilon})} \le C\|dist(\nabla v, SO(n))\|_{L^2(\varepsilon\mathcal{O}_{i,\varepsilon})}.
\end{aligned}$$

The constant C does not depend on ε.

Then, using the second part of (i), we compare $\mathbf{R}_i$ to $\mathbf{R}_1$ as well as $\mathbf{a}_i$ to $\mathbf{a}_1$, $i \in \{1, \dots, N\}$. As a consequence, one obtains that

$$\begin{aligned}
&\|v - \mathbf{a}_1 - \mathbf{R}_1\,x\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon\|dist(\nabla v, SO(3))\|_{L^2(\Omega_\varepsilon)},\\
&\|\nabla v - \mathbf{R}_1\|_{L^2(\Omega_e)} \le C\|dist(\nabla v, SO(n))\|_{L^2(\Omega_\varepsilon)}.
\end{aligned}$$

The constants do not depend on ε.

Now, we define the extension of v. We set

$$\widetilde{v} = \mathcal{P}'_\varepsilon(v - \mathbf{a}_1 - \mathbf{R}_1\,x) + \mathbf{a}_1 + \mathbf{R}_1\,x \qquad \text{a.e. in } \Omega'_\varepsilon.$$

We easily check (6.7). $\square$

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Curriculum Vitae

Deutsch:

1996-2000	Besuch der Grundschule Pallien (Trier)
2000-2009	Besuch des Hindenburg Gymnasium Trier (HGT)
2010-2013	Bachelor of Science: Mathematik mit Nebenfach Physik an der Technischen Universität Kaiserslautern
2013-2016	Master of Science: Mathematik mit Nebenfach Physik an der Technischen Universität Kaiserslautern
Seit 2016	Doktorand am Fraunhofer-Institut für Techno- und Wirtschaftsmathematik ITWM in Kaiserslautern

English:

1996-2000	Elementary school of Pallien (Trier)
2000-2009	Secondary school: Hindenburg Gymnasium Trier (HGT)
2010-2013	Bachelor of Science: Mathematics with Physics minor at Technischen Universität Kaiserslautern
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Since 2016	PhD-student at Fraunhofer Institute for Industrial Mathematics ITWM in Kaiserslautern

The topic of this thesis originated in the task to optimize a belt-shaped textile structure with respect to buckling. The approach is divided into two main steps: asymptotic analysis and optimization.

First, we show the simultaneous homogenization and dimension reduction for the textile elasticity problem using the unfolding method. In particular, the effective model is derived for different energy regimes, depending on periodicity, applied force and fiber-to-fiber contact. The different energy regimes use different approaches, where for linear elasticity variational inequalities are used, while the nonlinear von-Karman regimes requires arguments of the Gamma-Convergence to derive the limit problem.

In the second part the resulting homogenized problem gives rise to an effective textile plate problem and the corresponding cell problems. The cell problems connect macroscopic textile properties and the microscopic ones, namely the fibers, their contact and weaving pattern. This allows to use a macroscopic buckling model for further investigation and optimization. Eventually, the cell problems yield the corresponding microscopic properties for a buckling optimized textile.

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