Conceptual and numerical aspects of the mixed variational formulation of geometrically exact beam models

Peter Češarek*, **Dejan Zupan***, **Joachim Linn**#

* Faculty of Civil and Geodetic Engineering
University of Ljubljana
Jamova 2, 1000 Ljubljana, Slovenia
[peter.cesarek,dejan.zupan]@fgg.uni-lj.si

Department Mathematical Methods in Dynamics and Durabilty
Fraunhofer Institute for Industrial Mathematics ITWM
Fraunhofer Platz 1, 67633 Kaiserslautern, Germany
joachim.linn@itwm.fraunhofer.de

ABSTRACT

In our work, we highlight the conceptual and numerical aspects of mixed type formulations of geometrically exact beams. The governing set of equations obtained from mixed-type variational formulations consist of kinematic equations, constitutive equations, equilibrium equations and their boundary conditions, and a special set of equations denoted here as consistency conditions. The consistency conditions impose the requirement that the cross-sectional stress resultant forces and moments are equal to the constitutive forces and moments, respectively, along each point of the beam centerline. We present three beam formulations with assumed constant strains, which is equivalent to linear interpolation of displacements and helicoidal interpolation of rotations. Two of the formulations employ the consistency conditions as separate equations and are of the mixed-type, while in the third formulation the consistency conditions are eliminated from the governing system of equations. The configuration space of mixed-type formulations is simpler, which improves the performance (faster convergence, larger load or prescribed displacement steps) of the iterative solvers, and results in faster computational times. We demonstrate the performance of the formulations with a numerical example.

Keywords: geometrically exact beam, mixed formulation, configuration-based formulation.

1 INTRODUCTION

We concern ourselves with geometrically exact beam formulations and their numerical applications. A rigorous approach to derive the governing equations of the beam is to apply the principle of virtual work in the static case or its generalization for dynamics – d'Alembert's principle. An important part of deriving the governing equations is the treatment of the kinematic relations between displacements, rotations and strains. In the classical approach, the strains are taken as dependent variables and are obtained from the kinematic equations by means of interpolation of the configuration variables which are subsequently differentiated. This leads to formulations with displacement and rotational parameters as the primary variables – so called *configuration based formulations*. Equilibrium problems can likewise be formulated using the strains as independent variables, where the kinematic equations are added to the virtual work principle with the method of Lagrange multipliers. This leads to *mixed formulations* where the strains and Lagrange multipliers take the role of primary unknowns of the problem. Examples of such formulations for *three-dimensional geometrically exact beam* are the strain-based formulation of Zupan and Saje [1] and the intrinsic dynamic formulation presented by Hodges [2].

The history of mixed formulations for beams, shells and solids goes back to the 1960's, and since then they have been applied to many engineering problems. Their most attractive features are the accuracy of the numerically evaluated internal stresses, they successfully overcome the problems that arise in the elastic analysis of (nearly) incompressible solids, avoid the problem of locking in shells and beams and are natural for solving inelastic problems [3]. The main disadvantage of mixed formulations is the computational expense that comes with additional unknowns in the formulation. Because of that, the displacement-based formulations remained dominant, and reduced integration approaches were developed to overcome the locking problems. A lot of effort was put into establishing the connection between mixed and reduced/selective integration methods, e.g. [4], which constituted the reduced integration approach as a valid methodology, rather than a computational trick. For geometrically nonlinear elastic planar beams the equivalence between mixed and reduced integration methods has been discussed in [5], where mixed formulations are also claimed to be effective in comparison with displacement-based models.

In our work, we highlight the conceptual aspects of mixed formulations for finite-strain three-dimensional beam theory and compare the numerical efficiency of two formulations of finite elements based on the mixed formulations with the configuration-based finite elements. A particularly attractive choice of beam elements for flexible multibody applications are elements with assumed constant strains [6], or linear interpolation of displacements and helicoidal interpolation of rotations [7–9]. The strain-based formulation [6] employs configuration variables, strains and generalized forces as the primary unknowns and is of mixed-type (SB-mixed formulation). When the strains are eliminated from the set of primary variables using the analytical integration of the kinematic equations, it leads to a formulation where configuration variables and generalized forces are the only primary variables (CB-mixed formulation). A further reduction of such a formulation involves the extraction of generalized forces from the constitutive equations which are assumed to be satisfied. This leads to a classical configuration-based (CB) formulation where only displacement and rotational parameters are the primary unknowns.

2 BEAM FORMULATIONS

2.1 Beam kinematics and constitutive equations

Geometrically exact rod models [10–13] provide kinematical relations between configuration variables and resultant strain measures of the beam, regardless of the magnitude of displacements and rotations. The adopted strain measures are consistent with the virtual work principle and are defined by the first derivatives of configuration variables with respect to arc length parameter s of the beam centerline. The deformation of the centerline, determined by *position vector* $\vec{r}$, is given by the *translational strain measure* $\vec{\gamma}$

$$\vec{r}\,' = \vec{\gamma} - \vec{\gamma}^0. \tag{1}$$

The rate of change of orientation of the moving frame, attached to the beam centerline, with respect to the fixed frame, is given by the *rotational strain measure* $\vec{\kappa}$. For rotations represented with a *rotation matrix* Λ the corresponding kinematic relation reads

$$\Lambda' = \mathrm{S}\left(\vec{\kappa} - \vec{\kappa}^0\right)\Lambda, \tag{2a}$$

for rotations given in terms of a *rotational quaternion* $\hat{q}$ it reads

$$\hat{q}\,' = \frac{1}{2}\left(\vec{\kappa} - \vec{\kappa}^0\right) \circ \hat{q} \tag{2b}$$

and for rotations parametrized by the *rotational vector* $\vec{\vartheta}$ $\left(\Lambda = \Lambda\left(\vec{\vartheta}\right) \text{ or } \hat{q} = \hat{q}\left(\vec{\vartheta}\right)\right)$ the kinematic equation is given by

$$\vec{\vartheta}\,' = \mathrm{T}^{-1}\left(\vec{\vartheta}\right)\left(\vec{\kappa} - \vec{\kappa}^0\right), \tag{2c}$$

where S is the skew-symmetric operator, T is a tangent operator of the rotational map, see e.g. [6], and '$\circ$' denotes quaternion multiplication. $\vec{\gamma}^0$ and $\vec{\kappa}^0$ do not depend on the deformation of the beam and represent the initial strains which can be expressed from the known initial configuration of the beam. $\vec{\gamma}^0$ coincides with the normal of the cross section, while $\vec{\kappa}^0$ describes the initial curvature or twist of the beam centerline. The solution of the first order ODEs (1) and (2) is determined by initial values

$$\vec{r}(0) = \vec{r}_n \\ \Lambda(0) = \Lambda_n, \tag{3a}$$

and must satisfy the boundary conditions at $s = h$

$$\vec{r}(h) = \vec{r}_{n+1} \\ \Lambda(h) = \Lambda_{n+1}, \tag{3b}$$

where n and $n+1$ denote the boundary nodes and h is the length of the beam segment: $s \in [0, h]$.

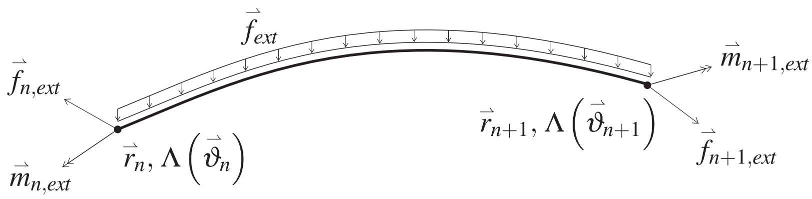
For the description of the material behaviour we assume a general (nonlinear elastic) form of the constitutive equations

$$\vec{f}^C = \mathrm{C}_F\left(\vec{\gamma}, \vec{\kappa}\right) \\ \vec{m}^C = \mathrm{C}_M\left(\vec{\gamma}, \vec{\kappa}\right), \tag{4}$$

where $\vec{f}^C$ and $\vec{m}^C$ are the resultant cross-sectional constitutive force and moment vectors, and the operators C_F and C_M are at least once differentiable with respect to $\vec{\gamma}$, $\vec{\kappa}$ and s.

2.2 Generalized virtual work principle

Next, we state the equilibrium problem for the beam presented in Figure 1. For simplicity we only consider the static case with a uniformly distributed external load $\vec{f}_{ext}$ and discrete loads $\vec{f}_{n,ext}$, $\vec{f}_{n+1,ext}$, $\vec{m}_{n,ext}$, $\vec{m}_{n+1,ext}$ applied at boundaries.



Constitutive equations:

$$\vec{f}^C = C_f\left(\vec{\gamma}, \vec{\kappa}\right) \\ \vec{m}^C = C_m\left(\vec{\gamma}, \vec{\kappa}\right)$$

Kinematic equations:

$$\vec{r}\,' = \vec{\gamma} - \vec{\gamma}^0 \\ \mathrm{T}\left(\vec{\vartheta}\right)\vec{\vartheta}\,' = \vec{\kappa} - \vec{\kappa}^0$$

Figure 1. Problem formulation

Before we proceed with the variational formulation we must first choose the parametrization of rotations. We could choose either the components of the rotation matrix or quaternion parameters, but such an approach requires that the equilibrium problem is formulated with additional constraints [14, 15]. However, we here choose the representation of rotations with a rotational vector, but we need to stress that the choice of the rotation parameters is not crucial for the comparisons of the formulations presented here.

The virtual work principle [11, 12] for the beam in Figure 1 states

$$\int_0^h \left(\vec{f}^C \cdot \delta\vec{\gamma} + \vec{m}^C \cdot \delta\vec{\kappa}\right) \mathrm{d}s = \int_0^h \left(\vec{f}_{ext} \cdot \delta\vec{r} + \vec{m}_{ext} \cdot \delta\vec{\vartheta}\right) \mathrm{d}s + \vec{f}_{n,ext} \cdot \delta\vec{r}(0) \\ + \vec{m}_{n,ext} \cdot \delta\vec{\vartheta}(0) + \vec{f}_{n+1,ext} \cdot \delta\vec{r}(h) + \vec{m}_{n+1,ext} \cdot \delta\vec{\vartheta}(h). \tag{5}$$

In the mixed approach we treat strains as independent quantities, therefore we enforce the kinematic equations (1) and (2c), which relate the configuration variables and strains, using the method of Lagrange multipliers. First, the constraining equations are multiplied by Lagrange multipliers $\vec{f}$ and $\vec{m}$ and integrated over the length of the beam

$$\begin{aligned} \int_0^h \vec{f} \cdot \left(\vec{\gamma} - \vec{\gamma}^0 - \vec{r}'\right) \mathrm{d}s = 0 \\ \int_0^h \vec{m} \cdot \left(\vec{\kappa} - \vec{\kappa}^0 - \mathrm{T}\left(\vec{\vartheta}\right) \vec{\vartheta}'\right) \mathrm{d}s = 0. \end{aligned} \tag{6}$$

Secondly, equations (6) are varied with respect to the kinematic variables $\delta\vec{r}$, $\delta\vec{\vartheta}$, $\delta\vec{\gamma}$, $\delta\vec{\kappa}$ and the Lagrange multipliers $\delta\vec{f}$ and $\delta\vec{m}$, and added to virtual work principle (5). This leads to the *modified principle of virtual work*

$$\begin{aligned} &\int_0^h \delta\vec{\gamma} \cdot \left(\vec{f}^C - \vec{f}\right) ds + \int_0^h \delta\vec{\kappa} \cdot \left(\vec{m}^C - \vec{m}\right) ds \\ &\quad - \int_0^h \delta\vec{r} \cdot \left(\vec{f}' + \vec{f}_{ext}\right) ds - \int_0^h \delta\vec{\vartheta} \cdot \left(\vec{m}' + \vec{r}' \times \vec{f}\right) ds \\ &\quad\quad - \int_0^h \delta\vec{f} \cdot \left(\vec{\gamma} - \vec{\gamma}^0 - \vec{r}'\right) ds - \int_0^h \delta\vec{m} \cdot \left(\vec{\kappa} - \vec{\kappa}^0 - \mathrm{T}\left(\vec{\vartheta}\right) \vec{\vartheta}'\right) ds \\ &\quad\quad\quad + \delta\vec{r}(0) \cdot \left(\vec{f}_{n,ext} + \vec{f}(0)\right) + \delta\vec{\vartheta}(0) \cdot \left(\vec{m}_{n,ext} + \vec{m}(0)\right) \\ &\quad\quad\quad\quad + \delta\vec{r}(h) \cdot \left(\vec{f}_{n+1,ext} - \vec{f}(h)\right) + \delta\vec{\vartheta}(h) \cdot \left(\vec{m}_{n+1,ext} - \vec{m}(h)\right) = 0. \end{aligned} \tag{7}$$

The coefficients at the independent variations in (7) must vanish, which results in the Euler-Lagrange equations of the three dimensional beam. They consist of:

(i) the kinematic equations

$$\vec{\gamma} - \vec{\gamma}^0 - \vec{r}' = \vec{0} \tag{8}$$

$$\vec{\kappa} - \vec{\kappa}^0 - \mathrm{T}\left(\vec{\vartheta}\right) \vec{\vartheta}' = \vec{0}, \tag{9}$$

(ii) the equilibrium equations

$$\begin{aligned} \vec{f}' + \vec{f}_{ext} = \vec{0} \\ \vec{m}' + \vec{r}' \times \vec{f} = \vec{0} \end{aligned} \tag{10}$$

and their boundary conditions

$$\begin{aligned} \vec{f}_{n,ext} + \vec{f}(0) = \vec{0} \qquad \vec{f}_{n+1,ext} - \vec{f}(h) = \vec{0} \\ \vec{m}_{n,ext} + \vec{m}(0) = \vec{0} \qquad \vec{m}_{n+1,ext} - \vec{m}(h) = \vec{0}, \end{aligned} \tag{11}$$

(iii) and a special set of equations here denoted as the *consistency conditions*

$$\begin{aligned} \vec{f}^C - \vec{f} = \vec{0} \\ \vec{m}^C - \vec{m} = \vec{0}. \end{aligned} \tag{12}$$

The equilibrium equations reveal the physical meaning of Lagrange multipliers in our approach, which are found to be the cross-sectional stress resultant force $\vec{f}$ and moment $\vec{m}$ of the beam, while the consistency conditions impose the requirement that $\vec{f}$ and $\vec{m}$ are equal to the constitutive force $\vec{f}^C$ and moment $\vec{m}^C$, respectively, along each point of the beam centerline.

2.3 A general solution approach

Usually it is assumed that the consistency conditions are exactly satisfied, such that they could be eliminated from the governing equations, which reduces the size of the problem. However, after the problem is discretized, the consistency is not necessarily preserved. From this perspective it is advantageous not to eliminate the consistency conditions from the problem, but to keep them as an independent member among the set of the governing equations. In mixed-type finite element formulations the generalized forces are members of the primary variables. This means that the equilibrium-based resultant forces $\vec{f}$ and moments $\vec{m}$, which take the role of Lagrange multipliers associated with the preservation of kinematic constraints, are demanded to be equal to the corresponding constitutive quantities obtained from the strains, using the constitutive equations. This approach completely avoids the shear locking problem without the necessity of any special numerical treatment of the governing equations [1].

Since the consistency equations are not eliminated in our formulation we need to solve the problem for a full set of unknowns: $\vec{r}$, $\vec{\vartheta}$, $\vec{f}$, $\vec{m}$, $\vec{\gamma}$ and $\vec{\kappa}$. The problem is solved by the finite-element method. The unknowns are approximated by discrete values and suitable shape functions. In general we could introduce the interpolation of all the unknowns of the formulation, or – similar to [16] – the interpolation of generalized strains and forces. Here, we follow the approach of Zupan and Saje [1] and select the strains to be the only interpolated variables, while we express the remaining unknowns with strains. By doing so, we minimize the number of unknowns of the mixed formulation. A general solution approach to solve the system of equations (8)–(12) is then as follows:

(i) introduce the interpolation of strain measures,

(ii) integrate the kinematic equations (8)–(9),

(iii) integrate the equilibrium equations (10),

(iv) the remaining equations (consistency conditions (12), static (11) and kinematic (3) boundary conditions) form the governing system of equations, which is solved for the unknown discrete values of interpolated strains, boundary values of the configuration variables, and the discrete-point values of stress-resultants.

The solution approach presented above summarises the conceptual aspect of the mixed (strain-based) formulation presented in [1, 6]. In this computational approach abstract vectors are replaced by their component representations with respect to the global or local – material basis. Strain measures in particular are naturally expressed in material frame. In the following, the vectors expressed in the fixed basis are marked by a lower case font and vectors expressed in the moving base are denoted by an upper case font. The relationship between the two component representations of a vector is established by the rotation matrix as: $u = \Lambda U$.

2.4 Formulations

Mixed strain based formulation

In general any interpolation function can be chosen for the interpolation of strains. Particularly interesting for the multibody applications [9] are the elements with assumed constant material strains

$$\Gamma(s) = \Gamma, \qquad K(s) = K.$$

When the material strains Γ and K are taken to be constant, the kinematic equations can be solved analytically. While the kinematic equation for rotations expressed with rotational vector (9) is not appropriate for exact integration, however, the solution of the equivalent equation (2a), written for the material rotational strain, is well known, e.g. [6]; it reads:

$$\Lambda(s) = \Lambda(0)\exp\left(s\left(K - K^0\right)\right) = \Lambda(0)\,\Lambda\left(s\left(K - K^0\right)\right). \tag{13}$$

Having a closed-form analytical expression for rotations, we can now integrate the kinematic equation (8) to obtain the analytical expression for the position vector

$$r(s) = r(0) + \int_0^s \Lambda \, \mathrm{d}s \left(\Gamma - \Gamma^0\right). \tag{14}$$

The kinematics of the beam is a boundary value problem, therefore we explicitly demand that the boundary conditions (3) are satisfied: $r(0) = r_n$, $r(h) = r_{n+1}$, $\Lambda(0) = \Lambda_n$, $\Lambda(h) = \Lambda_{n+1}$. After taking into account (13) and (14) this leads to

$$r_{n+1} - r_n - \int_0^h \Lambda(s) \, \mathrm{d}s \left(\Gamma - \Gamma^0\right) = 0 \tag{15}$$

$$\Lambda_{n+1} - \Lambda_n \Lambda\left(h\left(K - K^0\right)\right) = 0. \tag{16}$$

Equation (16) is a matrix equation in $SO(3)$ and consists of nine algebraic equations of which only 3 are independent. Because the exact integration rule for rotations (13) was used, the components of rotation matrix $\Lambda\left(h\left(K - K^0\right)\right)$ satisfy the unit length and orthogonality conditions, therefore the remaining six equations are automatically satisfied. The vector representation of equation (16) can be obtained using the extraction of rotational vectors from the rotation matrices. The extraction cannot be expressed by an explicit formula, thus we will use the symbolic notation

$$\left[\Lambda_{n+1} - \Lambda_n \Lambda\left(h K_{n+1/2}\right)\right]_{\mathbb{R}^3} = 0.$$

In the present mixed approach we express the stress-resultants with strains. To that end we integrate equilibrium equations (10):

$$\begin{aligned} f(s) &= f(0) - \int_0^s f_{ext} \, \mathrm{d}s \\ m(s) &= m(0) - \int_0^s r' \times f \, \mathrm{d}s = m(0) - \int_0^s \Lambda \cdot \left(\Gamma - \Gamma^0\right) \times f \, \mathrm{d}s. \end{aligned} \tag{17}$$

The remaining equations – consistency conditions (12), static boundary conditions (11) and kinematic conditions (15) and (16), now constitute the governing system of equations of the present mixed formulation. Because we used the lowest order of interpolation for strains, the integrals over the length of the beam can be evaluated using simple a midpoint rule. The integrated kinematic boundary condition (15) then simplifies to

$$r_{n+1} - r_n - h\Lambda_{n+1/2} \cdot \left(\Gamma - \Gamma^0\right) = 0. \tag{18}$$

where $\Lambda_{n+1/2} = \Lambda_n \Lambda\left(\frac{h}{2}\left(K - K^0\right)\right)$. After the discretization, the continuous consistency conditions (12) cannot be satisfied at an arbitrary point along the beam centerline. We employ the collocation method and satisfy them at a discrete point. In accord with the integration rule, we choose to satisfy the consistency conditions at the midpoint of the element, i.e. at $s = s_{n+1/2} = \frac{h}{2}$. We also choose the discrete values of stress-resultants at the midpoint of the beam $f_{n+1/2} = f\left(\frac{h}{2}\right)$ and $m_{n+1/2} = m\left(\frac{h}{2}\right)$ to be the primary variables and express the boundary values using (17):

$$\begin{aligned} f(0) &= \tfrac{h}{2} f_{ext} + f\left(\tfrac{h}{2}\right) \\ m(0) &= \tfrac{h}{2} \Lambda_{n+1/2}\left(\Gamma - \Gamma^0\right) \times \left(f\left(\tfrac{h}{2}\right) + \tfrac{h}{4} f_{ext}\right) + m\left(\tfrac{h}{2}\right) \\ f(h) &= -\tfrac{h}{2} f_{ext} + f\left(\tfrac{h}{2}\right) \\ m(h) &= -\tfrac{h}{2} \Lambda_{n+1/2}\left(\Gamma - \Gamma^0\right) \times \left(f\left(\tfrac{h}{2}\right) - \tfrac{h}{4} f_{ext}\right) + m\left(\tfrac{h}{2}\right). \end{aligned}$$

The final governing system of equations of the strain-based formulation of the lowest order now reads:

$$
\begin{aligned}
g_1 &= f^C_{n+1/2} - f_{n+1/2} = 0, \\
g_2 &= m^C_{n+1/2} - m_{n+1/2} = 0, \\
g_3 &= f_{n,ext} + \tfrac{h}{2} f_{ext} + f_{n+1/2} = 0, \\
g_4 &= m_{n,ext} + \tfrac{h}{2}\Lambda_{n+1/2}\left(\Gamma - \Gamma^0\right) \times \left(f_{n+1/2} + \tfrac{h}{4} f_{ext}\right) + m_{n+1/2} = 0, \\
g_5 &= f_{n+1,ext} + \tfrac{h}{2} f_{ext} - f_{n+1/2} = 0, \\
g_6 &= m_{n+1,ext} + \tfrac{h}{2}\Lambda_{n+1/2}\left(\Gamma - \Gamma^0\right) \times \left(f_{n+1/2} - \tfrac{h}{4} f_{ext}\right) - m_{n+1/2} = 0, \\
g_7 &= r_{n+1} - r_n - h\Lambda_{n+1/2}\left(\Gamma_{n+1/2} - \Gamma^0\right) = 0, \\
g_8 &= \left[\Lambda_{n+1} - \Lambda_n \Lambda\left(h K_{n+1/2}\right)\right]_{\mathbb{R}^3} = 0,
\end{aligned}
\tag{19}
$$

where $f^C_{n+1/2} = \Lambda_{n+1/2} \mathrm{C}_F(\Gamma, K)$ and $m^C_{n+1/2} = \Lambda_{n+1/2} \mathrm{C}_M(\Gamma, K)$ denote the spatial representations of the constitutive material stress-resultants. The unknowns of system (19) are the kinematic vectors r_n, ϑ_n, r_{n+1}, ϑ_{n+1}, the equilibrium stress resultants $f_{n+1/2}$, $m_{n+1/2}$ and the strain vectors $\Gamma_{n+1/2}$ and $K_{n+1/2}$.

Mixed configuration based formulation

Among the primary unknowns of the mixed formulation (19) are the strain vectors. We can eliminate them from the system using equations g_7 (18) and g_8 (16)

$$
\begin{aligned}
\Gamma_{n+1/2} &= \frac{1}{h}\Lambda^T_{n+1/2}\left(r_{n+1} - r_n\right) + \Gamma^0 \\
K_{n+1/2} &= \frac{1}{h}\theta + K^0, \quad \Lambda(\theta) = \Lambda^T_n \Lambda_{n+1}.
\end{aligned}
\tag{20}
$$

Not surprisingly, the discrete strain approximations (20) are equivalent to the ones obtained by the linear interpolation of displacements and helicoidal interpolation of rotations [17]. After the elimination of the strains from the mixed strain-based formulation (19), we obtain a new partially reduced formulation of the geometrically exact beam, which reads:

$$
\begin{aligned}
g_1 &= f^C_{n+1/2} - f_{n+1/2} = 0, \\
g_2 &= m^C_{n+1/2} - m_{n+1/2} = 0, \\
g_3 &= f_{n,ext} + \tfrac{h}{2} f_{ext} + f_{n+1/2} = 0, \\
g_4 &= m_{n,ext} + \tfrac{h}{2}\tfrac{r_{n+1} - r_n}{h} \times \left(f_{n+1/2} + \tfrac{h}{4} f_{ext}\right) + m_{n+1/2} = 0, \\
g_5 &= f_{n+1,ext} + \tfrac{h}{2} f_{ext} - f_{n+1/2} = 0, \\
g_6 &= m_{n+1,ext} + \tfrac{h}{2}\tfrac{r_{n+1} - r_n}{h} \times \left(f_{n+1/2} - \tfrac{h}{4} f_{ext}\right) - m_{n+1/2} = 0.
\end{aligned}
\tag{21}
$$

The system (21) represents a mixed-type formulation that needs to be solved for the kinematic vectors r_n, ϑ_n, r_{n+1}, ϑ_{n+1} and the equilibrium stress resultants $f_{n+1/2}$ and $m_{n+1/2}$.

Configuration based formulation

If we proceed and eliminate the equilibrium stress resultants $f_{n+1/2}$ and $m_{n+1/2}$ from equations g_1 and g_2 of the system (21) by employing the consistency conditions (12)

$$
\begin{aligned}
f_{n+1/2} &= f^C_{n+1/2} \\
m_{n+1/2} &= m^C_{n+1/2},
\end{aligned}
$$

we obtain the further reduced set of equilibrium equations

$$\begin{aligned}
g_3 &= f_{n,ext} + \tfrac{h}{2} f_{ext} + f^C_{n+1/2} = 0, \\
g_4 &= m_{n,ext} + \tfrac{h}{2} \tfrac{r_{n+1}-r_n}{h} \times \left(f^C_{n+1/2} + \tfrac{h}{4} f_{ext} \right) + m^C_{n+1/2} = 0, \\
g_5 &= f_{n+1,ext} + \tfrac{h}{2} f_{ext} - f^C_{n+1/2} = 0, \\
g_6 &= m_{n+1,ext} + \tfrac{h}{2} \tfrac{r_{n+1}-r_n}{h} \times \left(f^C_{n+1/2} - \tfrac{h}{4} f_{ext} \right) - m^C_{n+1/2} = 0,
\end{aligned} \tag{22}$$

in which only the kinematic vectors r_n, ϑ_n and r_{n+1}, ϑ_{n+1} are unknown. The formulation (22) can be identified as a classical configuration-based formulation, which is usually obtained using virtual work principle (5) where the strains are expressed with the configuration variables using the discrete kinematic equations (20).

2.5 Notes on numerical implementation

We solve the governing system of equations of the mixed strain-based formulation (19), the mixed configuration-based formulation (21) and the classical configuration-based formulation (22) with Newton's iterative method. To that end, the systems of equations $g(x) = 0$ are linearized as

$$\mathrm{K}\, \delta x = -g$$

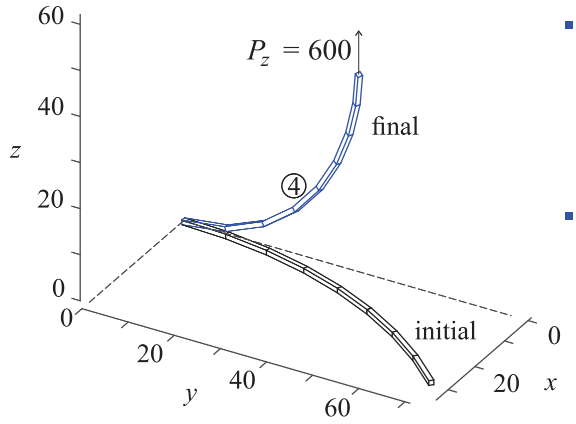
and then iteratively solved to obtain corrections of the primary variables, until the convergence criteria are satisfied. K denotes the Jacobian (tangent stiffness) matrix of the system. δx is the vector of unknowns, which consists of the nodal configuration variables δr_n, $\delta \vartheta_n$, δr_{n+1}, $\delta \vartheta_{n+1}$, and in case of mixed formulations, additional internal element variables $\delta f_{n+1/2}$, $\delta m_{n+1/2}$ – mixed configuration-based formulation (21), or $\delta f_{n+1/2}$, $\delta m_{n+1/2}$, $\delta \Gamma_{n+1/2}$, $\delta K_{n+1/2}$ – mixed strain-based formulation (19). In practice, the additional variables of the mixed methods are eliminated at the element level by static condensation [18], which can be executed at low computational costs. The efficiency of the static condensation, however, is not a part of this study. The importance of static condensation is twofold, first, it enhances the robustness of the linear solver, and second, by an elimination of the additional variables, the resulting system involves only (primal) configuration degrees of freedom, such that it could be easily incorporated into computational environments based on displacement-based methods.

3 NUMERICAL EXPERIMENT

Bending of 45°cantilever

In order to demonstrate the performance of the presented formulations (19) – SB-mixed, (21) – CB-mixed and (22) – CB, we analysed a 45°cantilever bend presented in [19]. The axis of the bending is in the form of the circular arc with the central angle 45°and radius $R = 100$. Bending is located in the horizontal plane (x, y) and subject to an out-of-plane point load $P = 600$ in the z direction at the free-end, see Figure 2, which triggers all modes of deformation of the structure: bending, shear, extension and torsion. The elastic material data and the geometric properties of the cross-section of the beam are: $E = 10^7$, $G = E/2$, $A_1 = 1$, $A_2 = A_3 = 5/6$, $J_1 = 1/6$, $J_2 = J_3 = 1/12$. The beam was discretized using 8 equal straight elements.

The first observation from the simulations is that all of the formulations return the same solution. Results for configuration parameters of the tip and internal forces at the midpoint of the fourth element are presented in Figure 2. Next, we compare the execution times for the calculation of element stiffness matrix and the residual vector, and the computational time required by the linear solver in each iteration. Relative values of the time measurements are presented in Figure 3. The linearization of the strain-based formulation is slightly more complex than the linearization of the configuration-based formulations, which results in a larger computational time for the element stiffness matrix (Figure 3(a)). On the other hand, tangent matrices of the mixed formulations are less dependent on the actual configuration, which results in a faster solution of the linearized system of equations (Figure 3(b)).



- position and orientation of the tip:

  $r_x = 15.80$ $\quad \vartheta_x = 2.0376$

  $r_y = 47.23$ $\quad \vartheta_y = -0.1390$

  $r_z = 53.37$ $\quad \vartheta_z = 1.5057$

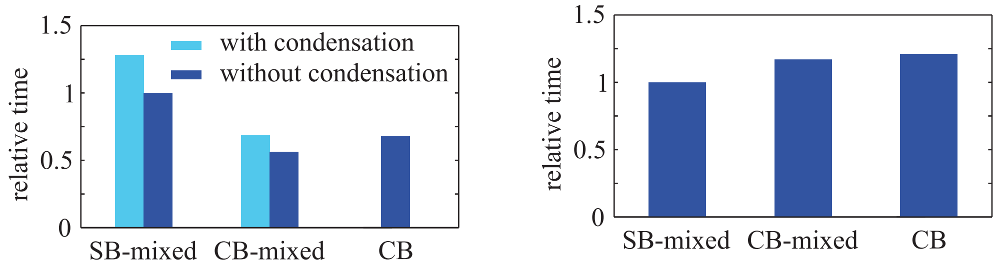
- internal forces at midpoint of ④th element:

  $F_X = 448$ $\quad M_X = -2549$

  $F_Y = 396$ $\quad M_Y = 1582$

  $F_Z = 41$ $\quad M_Z = 12588$

Figure 2. Bending of 45°cantilever: initial and deformed configuration, and converged results.



(a) Tangent matrix and residual vector

(b) Linear solver

Figure 3. Bending of 45°cantilever: relative computational times for calculation of element tangent matrix and residual, and relative computational times of linear solver.

The latter becomes more important if we observe Newton iterations and total computational times. Results for simulations with different numbers of equal load increments are presented in Table 1.

Table 1. Bending of 45°cantilever: convergence of Newton iteration scheme and relative computational times. The iteration stopping criterion was 10^{-7} for the Euclidean norm of the residual.

formulation	$n_{inc} = 1$			$n_{inc} = 4$			$n_{inc} = 10$			$n_{inc} = 40$		
	n_{iter}	$\sum n_{iter}$	t_{rel}	n_{iter}	$\sum n_{iter}$	t_{rel}	n_{iter}	$\sum n_{iter}$	t_{rel}	n_{iter}	$\sum n_{iter}$	t_{rel}
SB-mixed	7	7	1	5	20	1	5	50	1	4	160	1
CB-mixed	10	10	0.9	8	31	0.9	7	71	0.9	6	237	1
CB	/	/	/	13	52	1.8	8	85	1.2	8	310	1.1

n_{inc} – number of load increments, n_{iter} – typical number of iterations per increment, $\sum n_{iter}$ – total number of iterations, t_{rel} – relative computational time

We can observe that the mixed-strain based formulation shows the best convergence properties, however, due to the smaller computational effort, the mixed configuration-based formulation is faster. The classical configuration based formulation is most dependent on the configuration parameters and typically requires more iterations per load increment compared to the mixed formulations. This can in some cases result in a mediocre performance, as in the present case, where the configuration based formulation does not converge when the load is applied in one step, and is two times slower then the mixed formulations, if the load is applied in four increments. When the solution of the problem is required at a larger number of load stages, e.g. to obtain a smoother response for visualization purposes, all of the formulations have approximately the same computational efficiency.

4 CONCLUSIONS

We compared three geometrically exact beam formulations for large displacement analysis. The governing equations of the formulations were derived from a modified principle of virtual work. Two of the formulations are of mixed-type and employ stress-resultants and strains as independent members of the primary variables. These additional variables add some computational expense, yet they simplify the configuration space, which results in an excellent performance of the mixed formulations. The numerical results proved that all three formulations converge to the same solution and showed that mixed formulations are computationally not only comparably efficient, but can be more efficient as the classical displacement based formulations.

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