Scale-related evolution of BRDFs

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Technical Report IES-2016-09

Abstract: In computer graphics, objects are often modeled as polygonal meshes with flat facets, the reflection of light from facets is described by the so-called bi-directional reflectance distribution function (BRDF). Along with the intrinsic optical properties of the material, it captures the effects due to surface microstructure, i.e. shape features smaller than the facet size (and therefore not representable by the mesh model). If the typical facet size changes, some structures may switch from the “responsibility domain” of BRDFs to that of the geometric model, or vice versa. Therefore, for sufficiently complex surfaces, BRDFs are inherently scale-dependent. If one measures a BRDF at one scale, and applies at another, it must be adjusted (“evolved”) according to the specific surface structure. In this report, we derive explicit BRDF evolution equations and discuss their implications.

1 Introduction

The geometrical complexity of a model that describes an arbitrary surface (for example, a desk) in general depends on the resolution of the chosen observation method. If we sample surface points at a distance of a few centimeters from each other, the resulting model will be rather smooth (e.g., the desk will appear almost flat). If the distance between the samples is of the order of micrometers, the model becomes extremely complex, with overlapping peaks and valleys. At even smaller scales, the object is no longer a surface but a collection of bound and free particles performing some complicated motions.

If we study the reflection of light from the model towards some detector (e.g., a camera pixel), we may usually limit ourselves with the geometrical optics (i.e. with scales much greater than the wavelength of light). To every scene and every camera setup, we may also associate some effective observation scale μ. It is typically determined by the resolution of the surface model as desribed above or by the size of the features resolved by the camera (whichever is greater). An efficient practical approach to capture the response of such a scene is to represent the "coarse" geometry (features of size larger than μ) by the mesh (re-deriving it from the original model, if necessary), and incapsulate the microscopic appearance into bi-directional reflectance distribution functions (BRDFs) [Bas01], associated with the flat facets of the mesh. BRDFs, therefore, provide an efficient way to characterize materials: they may be derived or measured once and used to render images of multiple objects under various conditions.

If the observation scale μ were fixed for all scenes and observation conditions, one could create a universal library of BRDFs for various materials (as is typically done in the development of video games). However, in reality the effective observation scale μ depends on the scene and the camera setup. At the same time, the surface may have features at some "characteristic" scale μ_c that significantly contribute to its optical appearance (consider, e.g., "orange-skin finish", "roughness", or "polishing artefacts"). Depending on whether μ is greater or smaller than μ_c, the BRDFs must significantly differ. One solution to this problem is to measure BRDFs at multiple scales and interpolate between them [TLQ$^+$02]. One needs then a massive database of measurements. An alternative is to derive the rules that govern the μ-related "evolution" of BRDFs, and generate them based on the measurements defined at some scale μ_0.

In this report, we formalize the notion of the scale-dependent evolution of BRDFs and suggest a simple integral equation connecting them with the surface statistics. For the simple case of isotropic (invariant with respect to rotations about the surface normal vector) BRDFs on nearly-planar surfaces, we manage to integrate the BRDF evolution equation in closed form. For the more general case of anisotropic surfaces, we demonstrate that upon a Fourier transform over the group of rotations, the equation factorizes and can be efficiently integrated numerically up to any finite frequency. Finally, we discuss the new exotic contributions in BRDFs ("plus-distributions") that may arise e.g. due to the scale evolution over some specially micro-structured surfaces.

2 Notations

Let us consider a small surface element ϵ of area dS with some 3D orientation Γ, illuminated with a parallel beam of light coming from the direction $\hat{i}$. The incident intensity of the illumination (radiation power per cross-section area, watts per square meter) is L_i (Fig. 2.1). Our goal is to measure the directional distribution of the reflected and scattered light. To that end, we place a detector very far

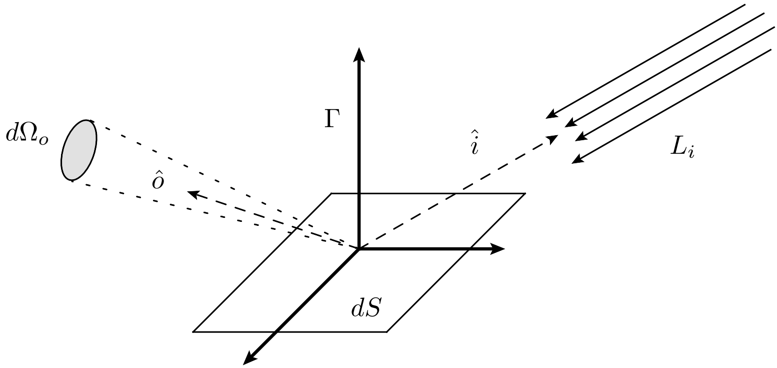

Figure 2.1: Reflection of light from a small surface element.

away from ϵ so that the direction towards the detector is $\hat{o}$ and its angular size as seen from ϵ is $d\Omega_o$. The BRDF $\beta(\hat{\imath}, \hat{o}, \Gamma)$ describes then the intensity I_o of light (power per unit of solid angle, watts per steradian) received by the detector:

$$I_o = L_i\ \beta(\hat{\imath}, \hat{o}, \Gamma)\ \left(\hat{\imath}^T \cdot \hat{n}(\Gamma)\right)\ d\Omega\, dS. \tag{2.1}$$

The geometrical factor $(\hat{\imath}^T \cdot \hat{n}(\Gamma))$ equals the cosine of angle between the incoming light direction $\hat{\imath}$ and the surface normal vector $\hat{n}(\Gamma)$ and accounts for the trivial change in the surface irradiation density as the surface incline changes. For convenience, in what follows we absorb his factor into the definition of BRDFs and define a function ρ as follows: $\rho(\hat{\imath}, \hat{o}, \Gamma) \equiv \beta(\hat{\imath}, \hat{o}, \Gamma)\ \left(\hat{\imath}^T \cdot \hat{n}(\Gamma)\right)$.

In a more detailed picture, ρ may further depend on the light wavelength λ, the incoming and the reflected light polarizations, coherence length, etc. We will focus here only on the dependence of ρ on the scale μ that separates "shape" from the "material properties", i.e. our primary object of interest will be the function $\rho(\hat{\imath}, \hat{o}, \Gamma, \mu)$.

The precise nature of the orientation Γ will be elaborated later. Here we only note that for isotropic surfaces and BRDFs, the actual orientation parameter is not Γ but the normal surface vector $\hat{n}(\Gamma)$. In particular, if the surface is an ideal flat mirror, its BRDF is

$$\rho_{\mathrm{mirror}}(\hat{\imath}, \hat{o}, \hat{n}) = A\delta\left(\hat{n} - \hat{n}_s(\hat{\imath}, \hat{o})\right),\ \text{ where }\ \hat{n}_s(\hat{\imath}, \hat{o}) = \frac{\hat{\imath} + \hat{o}}{\|\hat{\imath} + \hat{o}\|},$$

and A is some normalization constant.

3 General BRDF consistency equation

Let us now choose some macroscopic surface patch E, illuminated with a parallel beam of light of uniform intensity L_i coming from the direction $\hat{i}$, and observe it with an infinitely distant detector along the direction $\hat{o}$. There are two alternative ways to describe the observed light intensity in the detector.

In the first case, we follow the picture above and consider the entire patch E as a single element with its global orientation Γ_∞ and the respective BRDF $\rho(\hat{i}, \hat{o}, \Gamma, \infty)$. The Eq. (2.1) then applies without modifications.

In the second case, we split E into many pieces of typical size μ, each reflecting the light according to Eq. (2.1) with the respective microscopic BRDF $\rho(\hat{i}, \hat{o}, \Gamma, \mu)$. The resulting detector response I_o will then be a sum of contributions from all the elements, each having its specific orientation.

If we knew the complete micro-geometry of the surface, we could directly compute this sum. More often, however, we deal with unknown geometries but know the specific nature of the surface, such as wood, leather, polished metal, etc. The geometry in this case may be characterized statistically in terms of some function $p(\Gamma_\infty, \Gamma, \mu)$ that denotes the probability to find an element with orientation Γ in the surface E given its global orientation Γ_∞. In particular, p must be consistent with the definition of the global orientation: $p(\Gamma_\infty, \Gamma, \mu) \overset{\mu\to\infty}{\rightarrow} \delta(\Gamma - \Gamma_\infty)$.

Regardless of our representation of the surface, the amount of light collected by the detector should not change, which leads us to the following condition[1]:

$$\rho(\hat{i}, \hat{o}, \Gamma_\infty, \infty) = \int \rho(\hat{i}, \hat{o}, \Gamma, \mu)\, p(\Gamma_\infty, \Gamma, \mu)\, d\Gamma. \tag{3.1}$$

The left hand side of Eq. (3.1) is independent of the scale μ. We therefore arrive at the basic consistency equation for BRDFs:

$$\partial_\mu \int \rho(\hat{i}, \hat{o}, \Gamma, \mu)\, p(\Gamma_\infty, \Gamma, \mu)\, d\Gamma = 0, \tag{3.2}$$

where we define $\partial_\mu \equiv \partial/\partial\mu$ and the integration runs over all distinct orientations in the 3D space.

[1] This description can be compared to the microfacet BRDF model without the masking and shading effects, where some generic microscopic BRDF $\rho(\hat{i}, \hat{o}, \Gamma, \mu)$ is used instead of the Fresnel reflectance. As such, it applies only to non-grazing angles for $\hat{i}$ and $\hat{o}$, and nearly-flat surfaces: if an element has an incline that differs strongly from that of the global surface, the relevant probabilities will be dependent on the directions $\hat{i}$ and $\hat{o}$ via geometric factors.

4 Evolution for isotropic surfaces

As noted above, isotropic BRDFs and the statistics of isotropic surfaces depend only on the direction of the surface normal vector $\hat{n}$ and not on the rotation about $\hat{n}$. The re-formulated consistency condition in this case is

$$\partial_\mu \int \rho(\hat{i}, \hat{o}, \hat{n}, \mu)\, p(\hat{n}_\infty, \hat{n}, \mu)\, d\hat{n} = 0. \tag{4.1}$$

In order to decouple the evolution of the BRDF from that of the surface statistics, let us in addition assume that the surface only slightly deviates from a plane at all scales. This approximation can be formalized as follows. Let us choose the global vertical direction $\hat{z}$ and re-define ρ and p in terms of vectors $\vec{m}$ replacing the respective normal vectors $\hat{n}$:

$$\vec{m} = (\hat{n}^T \hat{z})^{-1} \hat{n} - \hat{z}.$$

In other words, $m_z \equiv 0$ and $(\vec{m} + \hat{z})$ is collinear with $\hat{n}$. Instead of $\hat{n}_\infty$, we now use $\vec{m}_\infty$, and instead of $\hat{n}$, we use $\vec{m}$, so that Eq. (4.1) becomes

$$\partial_\mu \int \rho(\hat{i}, \hat{o}, \vec{m}, \mu)\, p(\vec{m}_\infty, \vec{m}, \mu)\, dm_x dm_y = 0,$$

where the integration runs over the entire two-dimensional plane.

The condition that a surface is almost flat means that all relevant normal vectors are only slightly deviating from the z-direction, i.e. $\|\vec{m}_\infty\| \ll 1$ and $p(\vec{m}_\infty, \vec{m}, \mu) = 0$ for all $\|\vec{m}\| > m_{max}$ with some $m_{max} \ll 1$. In this case, the two-dimensional distribution of the normal vectors as a function of $\vec{m}$ will have a peak centered at the point $\vec{m}_\infty$ and in the first approximation will preserve its shape as $\vec{m}_\infty$ varies:

$$p(\vec{m}_\infty, \vec{m}, \mu) = q(\vec{m}_\infty - \vec{m}, \mu).$$

The consistency equation now takes the form of a convolution:

$$\partial_\mu \int \rho(\hat{i}, \hat{o}, \vec{m}, \mu)\, q(\vec{m}_\infty - \vec{m}, \mu)\, d\vec{m} = \partial_\mu \left((\rho \star q)(\vec{m}_\infty) \right) = 0. \tag{4.2}$$

If we denote the 2D Fourier images of the functions $\rho(\hat{i}, \hat{o}, \vec{m}, \mu)$ and $q(\vec{m}, \mu)$ with respect to $\vec{m}$ as $R(\hat{i}, \hat{o}, \vec{W}, \mu)$, and $Q(\vec{W}, \mu)$, respectively, then Eq. (4.2) is equivalent to

$$\partial_\mu \left(R(\hat{i}, \hat{o}, \vec{W}, \mu) Q(\vec{W}, \mu) \right) = 0, \quad \text{or} \tag{4.3}$$

$$\partial_\mu \log R(\hat{i}, \hat{o}, \vec{W}, \mu) = -\partial_\mu \log Q(\vec{W}, \mu).$$

(A trivial technical requirement here is that neither R nor Q may vanish inside the relevant domain of $\vec{W}$).

The solution of the Eq. (4.3) is trivial:

$$R(\hat{\imath}, \hat{o}, \vec{W}, \mu) = R(\hat{\imath}, \hat{o}, \vec{W}, \mu_0) \frac{Q(\vec{W}, \mu_0)}{Q(\vec{W}, \mu)}, \tag{4.4}$$

where μ_0 is some scale at which the BRDF is known. Using Eq. (4.4) and the actual statistics of the surface, one may easily produce BRDFs at any scale μ.

5 Evolution for anisotropic BRDFs and surfaces

In order to solve Eq. (3.2) in a more general case, we need to choose some explicit parameterization of orientations Γ in a three-dimensional space (in other words, choose some representation of the Lie group $SO(3)$ of 3D rotations). The most well-known representation of $SO(3)$ is the set of orthogonal 3×3 matrices with determinant $+1$. According to Euler, any 3D rotation may be represented as a function of three angles α, β, and γ such that $0 \leq \alpha, \gamma < 2\pi$, $0 \leq \beta < \pi$:

$$\Gamma(\alpha, \beta, \gamma) = U(\alpha) \cdot A(\beta) \cdot U(\gamma) \tag{5.1}$$

with

$$U(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } A(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}. \tag{5.2}$$

Now let us consider the distribution $p(\Gamma_\infty, \Gamma, \mu)$. It is clear that a simultaneous rotation of the global orientation Γ_∞ and the element orientation Γ by some rotation matrix $\Delta \in SO(3)$ is equivalent to a global rotation of space, and the fraction of elements oriented at $\Delta \cdot \Gamma$ relative to $\Delta \cdot \Gamma_\infty$ remains invariant. We therefore have:

$$p(\Delta \cdot \Gamma_\infty, \Delta \cdot \Gamma, \mu) = p(\Gamma_\infty, \Gamma, \mu), \text{ or, equivalently,}$$
$$p(\Gamma_\infty, \Gamma, \mu) = q(\Gamma^{-1} \cdot \Gamma_\infty, \mu) \text{ for some function } q.$$

The respective evolution equation

$$\partial_\mu \int \rho(\hat{\imath}, \hat{o}, \Gamma, \mu)\, q(\Gamma^{-1} \cdot \Gamma_\infty, \mu)\, d\Gamma = \partial_\mu \left((\rho \star q)(\Gamma_\infty) \right) = 0 \tag{5.3}$$

now contains the canonical convolution over the group $SO(3)$. In order to transform this convolution into a product, we need to briefly recall the properties of the Fourier transform over the group of rotations (for more details, see [KR08]).

Any function $f(\alpha,\beta,\gamma)$ of the three angles α, β, and γ defined in Eqs. (5.1) and (5.2) can be represented as an infinite sum:

$$
\begin{aligned}
f(\alpha,\beta,\gamma) &= \sum_{J=0}^{\infty}\sum_{M=-J}^{J}\sum_{M'=-J}^{J} f^J_{MM'}\, D^J_{MM'}(\alpha,\beta,\gamma), \quad \text{where} \\
f^J_{MM'} &= \langle f, D^J_{MM'}\rangle = \frac{2J+1}{8\pi^2}\times \\
&\quad\times \int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta\, d\beta \int_0^{2\pi} d\gamma\, f(\alpha,\beta,\gamma)\ \left(D^J_{MM'}(\alpha,\beta,\gamma)\right)^* .
\end{aligned}
$$

The basis functions here are the so-called Wigner D-functions:

$$
D^J_{MM'}(\alpha,\beta,\gamma) = e^{-iM\alpha}\, e^{-iM'\gamma}\, d^J_{MM'}(\cos\beta),
$$

and

$$
\begin{aligned}
d^J_{MM'}(t) &= (-1)^{J-M'}\, 2^{-J} \sqrt{\frac{(J+M)!}{(J+M')!(J-M')!(J-M)!}} \\
&\quad\times (1-t)^{-(M-M')/2}(1+t)^{-(M+M')/2} \\
&\quad\times \frac{d^{J-M}}{dt^{J-M}}\left[(1-t)^{J-M'}(1+t)^{J+M'}\right].
\end{aligned}
$$

The Fourier transform of a convolution may be found as follows. If $f(\Gamma)\mapsto f^J_{MM'}$, and $g(\Gamma)\mapsto g^J_{MM'}$, then $(f\star g)(\Gamma)\mapsto h^J_{MM'}$, where

$$
h^J_{MM"} = \sum_{k=-J}^{J} f^J_{Mk}\, g^J_{kM'}.
$$

This equation can be interpreted as follows. The Fourier coefficients of any function $f(\Gamma)$ can be arranged in the form of an (infinite) block-diagonal matrix, where the J-th block has the dimensions $(2J+1)\times(2J+1)$ and contains the coefficients $f^J_{MM'}$. The coefficients of the convolution of two functions are simply obtained via the matrix multiplication of the original coefficient matrices.

Finally, if the coefficient matrices corresponding to the Fourier transform of functions $\rho(\hat{\imath},\hat{o},\Gamma,\mu)$ and $q(\Gamma,\mu)$ are $R(\hat{\imath},\hat{o},\mu)$ and $Q(\mu)$, respectively, then the relation Eq. (5.3) assumes the following form:

$$
\Big(\partial_\mu R(\hat{\imath},\hat{o},\mu)\Big)\cdot Q(\mu) + R(\hat{\imath},\hat{o},\mu)\cdot\partial_\mu Q(\mu) = 0. \tag{5.4}
$$

For the J-th block of coefficients, Eq. (5.4) represents a closed system of $(2J+1)\times(2J+1)$ linear differential equations on the same number of functions of μ. The solution of such homogeneous linear differential equations are straightforward and involve matrix exponentials. In other words, this system can be solved numerically in each block, and we can solve it up to any fixed maximum cutoff frequency J.

6 Nearly-specular reflection

Let us consider the isotropic case and assume that each microfacet at the scale μ_0 is a perfect mirror, and that the surface at the scale μ is completely flat:

$$\rho(\hat{i},\hat{o},\vec{m},\mu_0) \sim \delta(\vec{m}-\vec{m}_s(\hat{i},\hat{o})), \text{ and } p(\vec{m}_\infty,\vec{m},\mu) \sim \delta(\vec{m}-\vec{m}_\infty).$$

Then Eq. (4.4) leads to the the following statement:

$$R(\hat{i},\hat{o},\vec{W},\mu) = Q(\vec{W},\mu_0)F \tag{6.1}$$

with some phase factor F. Eq. (6.1) represents a well-known relation between the metal surface roughness and its BRDF [Har86], but it also means, that starting from a trivial BRDF, one may prepare arbitrary BRDFs using special micro-structured surfaces.

In particular, let us imagine a surface whose normal vector distribution $p(\vec{m}_\infty,\vec{m},\mu_0) = h(\|\vec{m}_\infty-\vec{m}\|)$ is given by the following generalized function:

$$h(x) = \left\lfloor \frac{\log^n x}{x} \right\rfloor_+ ,$$

whose action on some probe function $\psi(x)$ is given by

$$\langle h,\psi\rangle = \int_0^\infty \frac{\log^n x}{x}\,(\psi(x)-\psi(0))\,dx.$$

Such "plus-distributions" are well-known in particle physics, where they describe scattering at small angles. Via Eq. (6.1), they may enter the BRDFs and give rise to exotic "nearly-specular" kind of reflection, which is neither specular, nor diffuse. The corresponding contributions for each exponent n may be measured and characterized by the dedicated experiments, and used to improve the stability of BRDF measurements[2].

[2] It is possible that the surfaces exhibiting such "near-specular" reflection are relatively common, but its contributions have always been confused with the specular or the diffuse components, leading to non-reproducible results and poor agreement with theoretical expectations.

So far, this "nearly-specular" reflection remains a theoretical prediction in a need of further research.

7 Summary

In this report, we have demonstrated how the BRDFs may be adjusted to the observation scale of a specific scene and detector setup, and derived a general evolution equation. We have further demonstrated a closed-form solution of the evolution equation in the isotropic case, and suggested a simple way to integrate the anisotropic evolution up to any finite angular frequency. Finally, we demonstrated a mechanism that may generate arbitrary contributions to BRDFs via special surface micro-structures, and hypothesized the existence of the novel "nearly-specular" type of reflection.

In the future, we plan to verify the presented equations with the real and simulated experiments, and extend the evolution to a more general class of surfaces (e.g., to those exhibiting significant microscopic masking and shading). We also plan to implement the evolution-based tools for computer graphics and BRDF measurements.

Bibliography

[Bas01] Michael Bass. *Handbook of optics, Volumes 1-2*. McGraw-Hill, 2 edition, 2001.

[Har86] J. E. Harvey. Light-scattering characteristics of optical surfaces. Proc. SPIE 645, pages 107 – 115, 1986.

[KR08] Peter J. Kostelec and Daniel N. Rockmore. FFTs on the rotation group. *Journal of Fourier Analysis and Applications*, 14:145 – 179, 2008.

[TLQ+02] Ping Tan, Stephen Lin, Long Quan, Baining Guo, and Heung-Yeung Shum. Filtering and rendering of resolution-dependent reflectance models. *IEEE Transactions on Visualization and Computer Graphics*, 1:1 – 14, 2002.