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The ordered gradual covering location problem on a network

Vorwort

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Prof. Dr. Dieter Prätzel-Wolters
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The Ordered Gradual Covering Location Problem on a Network

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Abstract

In this paper we develop a network location model that combines the characteristics of ordered median and gradual cover models resulting in the Ordered Gradual Covering Location Problem (OGCLP). The Gradual Cover Location Problem (GCLP) was specifically designed to extend the basic cover objective to capture sensitivity with respect to absolute travel distance. Ordered Median Location problems are a generalization of most of the classical locations problems like p-median or p-center problems. They can be modeled by using so-called ordered median functions. These functions multiply a weight to the cost of fulfilling the demand of a customer which depends on the position of that cost relative to the costs of fulfilling the demand of the other customers. We derive Finite Dominating Sets (FDS) for the one facility case of the OGCLP. Moreover, we present efficient algorithms for determining the FDS and also discuss the conditional case where a certain number of facilities are already assumed to exist and one new facility is to be added. For the multi-facility case we are able to identify a finite set of potential facility locations *a priori*, which essentially converts the network location model into its discrete counterpart. For the multi-facility discrete OGCLP we discuss several Integer Programming formulations and give computational results.

Keywords: Gradual Covering, Ordered Median Function, Network Location

1 Introduction

In this paper we develop a network location model that combines the characteristics of ordered median and gradual cover models. As discussed below, this provides a "unifying structure" for the standard location models, and allows us to consider combined objectives that are sensitive to both relative and absolute customer-to-facility travel distances.

The three "classical" objective functions in location models are the median, specifying that the total travel distance from customers to facilities be minimized, the center, minimizing the maximum travel distance (i.e., the travel distance for the customer who has to travel the furthest to get to a facility), and the cover, maximizing the number of customers covered by the facilities, where customers are considered to be covered if they are within a certain coverage radius of a facility. Good recent overview of location models on networks can be found in [4].

Since each of these objectives covers some important aspects of the underlying location problem, there has also been a considerable interest in combinations of these objectives (e.g., the "cent-dian" objective, which is a convex combination of the median and center objectives - see [13] and references therein). An important step in this direction has been the recent development of the Ordered Median Location Problem (OMP) that provides a unifying framework for the location models with median and center objectives, as well as the objectives that combine aspects of the two. (The OMP on networks was first introduced in [14] and the discrete version in [12]; see [13] for a comprehensive treatment.)

This unifying framework is accomplished as follows. Assuming there are n customers located at the nodes of the network, for given facility locations the median objective can be thought of as a two-step process: (1) compute the distance from each customer to the closest facility, and (2) add up the components of this vector to obtain the total travel distance. The facility locations are then chosen so as to minimize this total travel distance. The OMP interjects two additional steps into this process: (1A) "sorting", where customer travel distances are sorted from smallest to largest, and (1B) "weighting" where the i-th smallest customer travel distance is multiplied by the weight λ_i; the weighted costs are now added up as in step (2) above. This allows us to represent the standard median objective (by choosing weights $\lambda_i = 1$ for all i), the center objective (by choosing weights $\lambda_i = 0$ for the first $n-1$ components and $\lambda_n = 1$ for the last component, we obtain the largest travel distance), and the cent-dian objective (to obtain a convex combination of the median and center objectives with weight $\alpha \in (0,1)$, set $\lambda_i = \alpha$ for all the first $n-1$ components and $\lambda_n = 1$ for the last component). Thus, the median and the center models, as well as their combinations, are special cases of the OMP. In addition, by using different weight vectors, many other objectives can be represented (see [13]).

Note however, that one shortcoming of the OMP model is that it can only represent objectives based on *relative* travel distances - i.e., the travel distance of one customer relative to the other customers. In many settings, the absolute travel distances may be more relevant (e.g., a customer located over 10 kilometers from a supermarket is unlikely to patronize it, even if she happens to be the closest customer to this store). In fact, retailers typically define their trading areas in terms of the number of potential customers within a certain

distance from the store. Unfortunately, it is not possible to capture sensitivity with respect to absolute travel distances (e.g., assign higher weight to customers within 2 km from a facility) within the standard OMP framework.

The Gradual Cover Location Problem (GCLP), described in [1], [3], was specifically designed to extend the basic cover objective to capture sensitivity with respect to absolute travel distance. This model replaces the fixed coverage radius of the cover objective with a "coverage decay function" which assigns a coverage weight (a value between 0 and 1) to each customer based on the customer's distance from the closest facility. The objective is to maximize the weighted sum of covered customers. For example, the coverage decay function may specify two coverage levels $l < u$ with the stipulation that customers that are further than u from the closest facility are not covered at all (have coverage weight of 0), customers with travel distance between l and u are partially covered (weight of $1/2$) and customers closer than l from the closest facility are fully covered (weight of 1). Other forms of the coverage decay function may include linear decay, exponential decay, step function (representing multiple coverage radii instead of a single one in the cover objective), etc. In fact, by using a linear cover decay function with lower radius $l = 0$ and the upper radius u equal to the maximum distance between any two nodes, we obtain the median objective (details are provided below). Thus, the gradual cover framework allows us to represent the median objective, the cover objective and the intermediate objectives with various degrees of sensitivity with respect to the absolute travel distance. It can be seen as the counterpart of the OMP where the sorting and weighting steps (1A) and (1B) above are replaced with the "coverage weight" step (1C): for each customer determine the coverage weight by applying the coverage decay function to the travel distance to the closest facility, followed by step (2'): maximize the sum of the coverage weights (instead of minimizing the total weighted distance as in step (2) above).

However, the gradual cover framework is not capable of representing objectives that depend on relative distances since it is missing the sorting step (1A). Thus, it cannot capture the center objective, the cent-dian objective or other objectives related to relative distances that are easily representable within the OMP framework.

The goal of the current paper is to define and analyze a new model, the Ordered Gradual Covering Location Problem (OGCLP), that combines the features of the OMP and GCLP models. This new model is defined by performing step (1) above, followed by steps (1C), (1A), (1B) and (2') - i.e., the sorting and weighting steps are inserted into the gradual cover framework. The resulting model provides a unifying structure with respect to all three classical location objectives described earlier and is capable of capturing sensitivity with respect to both, the absolute and relative travel distances. This has practical implications since certain aspects of the underlying real-life problems - e.g., equity - are best represented in terms of relative travel distances, while others - e.g., definition of primary and secondary trading areas - are most naturally captured in terms of the absolute distances. In addition, OGCLP is of theoretical importance since any results established for this model are directly applicable to the three standard location objectives. We restrict our attention to network and discrete versions of OGCLP, leaving the study of planar version to future research.

The plan for the paper is as follows. Necessary notations are introduced and gradual decay

functions discussed in Section 1.1. The OGCLP is formally defined in Section 1.2. The single-facility version of the OGCLP is considered in detail in Section 2. In particular, we derive Finite Dominating Sets (FDS's) for models with non-negative weights, special classes of non-negative weights that yield simpler FDS's (Section 2.1), and general weights (Section 2.2). Moreover, we present efficient algorithms for determining the FDS. We also discuss the "conditional case" where a certain number of facilities are already assumed to exist and one new facility is to be added (Section 2.3). In Section 3, the FDS results are extended to multi-facility models. These results allow us to discretize the network model by determining a finite set of potential facility locations *a priori*, which essentially converts the network location model into its discrete counterpart. The multi-facility discrete location problem is addressed in Section 4 where we discuss several Integer Programming formulations for the OGCLP. The computational experiments analyzing the performance of these formulations are presented in Section 4.1. Some concluding remarks are presented in Section 5.

1.1 Preliminaries

Let $N = (G, \ell)$ be a network with underlying undirected graph $G = (V, E)$ and edge length ℓ. $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$ denote the set of nodes and edges, respectively, of the graph. An edge $e \in E$ is denoted $e = [v_i, v_j]$ with $i < j$. A point $x \in G$ on an edge $e = [v_i, v_j]$ of the network is denoted $x = (e, t)$, $0 \leq t \leq 1$, where t is the relative distance of x from node v_i. Let w_i be the demand associated with node $v_i \in V$ (which can be interpreted as the number of customers at node v_i), and $d_i(x) = d(v_i, x)$ be the shortest distance between node v_i and $x \in G$.

Assume that we wish to locate $p \geq 1$ facilities and that the facilities can be located at nodes or along the edges of the network. Suppose, for the moment, that the locations have already been chosen and let $S = \{x_1, \ldots, x_p\} \subset G$, $|S| = p$ be the *location set*. Define $d_i(S) = \min_{x \in S} d_i(x)$.

Let (l_i, u_i) be a pair of radii associated with node $v_i \in V$. Node v_i is fully covered (not covered) if $d_i(S) \leq l_i$ ($d_i(S) > u_i$). For $l_i < d_i(S) \leq u_i$, node v_i is partially covered.

Let $f_i(t)$ be a non-increasing function for $t \in [l_i, u_i]$ with $f_i(l_i) = 1$ and $f_i(u_i) = 0$. The function f_i is called the *coverage decay function*.
The demand of node v_i that is covered by S is defined as

$$c_i(d_i(S)) = \begin{cases} w_i & \text{if } d_i(S) \leq l_i \\ w_i \, f_i(d_i(S)) & \text{if } l_i < d_i(S) \leq u_i \\ 0 & \text{if } u_i < d_i(S) \end{cases}$$

We call $c_i(d_i(S))$ the *coverage function*. For short we write in the following $c_i(S)$ instead of $c_i(d_i(S))$.

Remark: Note that $c_i(S) = \max_{x \in S} c_i(x)$

The Gradual Cover Location Problems (GCLP) is to find

$$\max\left\{\sum_{i\in V} c_i(S) \mid S \subset G, |S| = p\right\}.$$

Some examples of GCLP models with different coverage decay functions are provided below:

1. Linear decay function

$$f_i(t) = 1 - \frac{1}{\alpha}t, \ i \in V$$

where $\alpha = \max_{i,j\in V} d_i(j)$ is a constant. If $u_i = \alpha$ and $l_i = 0$ for all $i \in V$, we have

$$\sum_{i\in V} c_i(S) = \sum_{i\in V} w_i f_i(d_i(S)) = \sum_{i\in V} w_i - \frac{1}{\alpha}\sum_{i\in V} d_i(S)w_i.$$

Since the first term is a constant and the second term is a (constant multiple of the) standard $p-$median objective, the GCLP with this coverage function is equivalent to the $p-$median problem.

2. Stepwise decay function

$$f_i(t) = \alpha_i^k \quad \text{if } t \in (r_i^{k-1}, r_i^k], \quad k = 1, \ldots, K_i$$

where $1 > \alpha_i^2 > \ldots > \alpha_i^{K_i} = 0$ and $l_i = r_i^0 < r_i^1 < \ldots < r_i^{K_i} = u_i$. For this type of coverage decay function, the values α_i^k are known as coverage levels and r_i^k as coverage radii. Clearly, in case of a single coverage level and a single coverage radius, the problem becomes equivalent to the standard maximum cover problem.

3. Piecewise linear decay function

$$f_i(t) = \beta_i^k - \alpha_i^k \cdot t \quad \text{if } t \in (r_i^{k-1}, r_i^k], \quad k = 1, \ldots, K_i$$

where $\beta_i^1, \ldots, \beta_i^{K_i}, \alpha_i^1, \ldots, \alpha_i^{K_i} > 0$ and $l_i = r_i^0 < r_i^1 < \ldots < r_i^{K_i} = u_i$.

1.2   OGCLP: Formulation and Relationship to the Ordered Median Problem

Let $\sigma \in \mathcal{P}(1 \ldots n)$ be a permutation of the index set $\{1 \ldots n\}$ that sorts the values of $c_i(S)$ in non-decreasing order:

$$c_{\sigma(1)}(S) \leq c_{\sigma(2)}(S) \leq \ldots \leq c_{\sigma(n)}(S) \tag{1}$$

Let $\lambda \in \mathbb{R}^n$ be a real-valued vector. λ is called the *modeling vector* and its entries the *modeling weights*.

The *Ordered Gradual Covering Location Problem* (OGCLP) is defined as follows:

$$\max\{\sum_{i=1}^{n} \lambda_i\, c_{\sigma(i)}(S) | S \subset G; |S| = p\}.$$

For $S \subset G$, $|S| = p$, we call

$$g(S) = \sum_{i=1}^{n} \lambda_i\, c_{\sigma(i)}(S)$$

the *Ordered Gradual Covering Function* (OGCF).

Note that if $c_i(S) = -d_i(S)w_i$ for all $i \in N$, the above formulation is exactly equivalent to the OMP on the network G (see [14]). However, this would require that the coverage decay function be $f_i(t) = -t$, which does not satisfy the requirements of a valid coverage decay function (recall that $f_i(t)$ has to be non-negative, equal to 1 at u_i and 0 at l_i). However, as the following result shows, with a suitable alteration of the network and the coverage decay function, the OMP can indeed be represented as a special case of OGCLP.

Let network $G' = (V, E)$ be identical to G (i.e., have the same nodes, edges and the distance function), but equipped with a unitary node demand vector $\mathbf{w}' = (1, \ldots, 1)$ instead of the original vector $\mathbf{w}$. For $i \in V$, let $j(i) = \arg\max_{j \in V} d_i(j)$ and let $i^* = \arg\max_{k \in V} w_k d_k(j(k))$. Define the coverage decay function

$$f_i(t) = 1 - \frac{w_i}{\alpha'} t, \ i \in V \tag{2}$$

where, $\alpha' = w_{i^*} d_{i^*}(j(i^*))$. Let $u_i = \alpha'/w_i$ and $l_i = 0$ for all $i \in V$. Intuitively, the original node weight vector has been incorporated into the coverage decay function. We have the following result:

Theorem 1.1 *For any modeling vector of $\lambda \in \mathbb{R}^n$ and any integer number of facilities $p \geq 1$, the OMP on network G is equivalent to the OGCLP on network G'.*

Proof
First note that $f_i(t)$ defined above is a proper coverage decay function since $f_i(u_i) = 0$, $f_i(l_i) = f_i(0) = 1$ and for any $t \in [l_i, u_i]$, $f_i(t) \geq 0$ must hold since $t < u_i$ implies that $tw_i < \alpha'$.

Let $S \subset G, |S| = p$ be a location set. For any $v_i \in V$, $d_i(S) \leq d_i(j(i)) \leq (1/w_i) * max_{k \in V} w_k d_k(j(k)) = u_i$. Thus $d_i(S) \in [l_i, u_i]$ and

$$c_i(S) = 1 - \frac{w_i d_i(S)}{\alpha'}, \ i \in V.$$

Therefore for $v_i, v_j \in V$,

$$c_i(S) \leq c_j(S) \Leftrightarrow w_i d_i(S) \geq w_j d_j(S). \tag{3}$$

6

Let σ' be a permutation defined by (1) with respect to the network G' and σ be the OMP-defining permutation on the network G (i.e., $d_{\sigma(1)}(S)w_{\sigma(1)} \leq \ldots \leq d_{\sigma(n)}(S)w_{\sigma(n)}$). By (3), $\sigma'(k) = \sigma(n-k+1)$ must hold for all $k \in \{1, \ldots, n\}$. Define $\lambda'_k := \lambda_{n-k+1}$. Thus

$$
\begin{aligned}
\sum_{k=1}^{n} \lambda'_k \, c_{\sigma'(k)}(S) &= \sum_{k=1}^{n} \lambda_{n-k+1} \, c_{\sigma(n-k+1)}(S) \\
&= \sum_{k=1}^{n} \lambda_k \, c_{\sigma(k)}(S) = \sum_{k=1}^{n} \lambda_k - \frac{1}{\alpha'} \sum_{k=1}^{n} \lambda_k w_{\sigma(k)} d_{\sigma(k)}(S).
\end{aligned}
$$

Since the first term above is constant and the second term is a scalar multiple of the OMP objective function, maximizing the OGCLP objective is equivalent to minimizing the OMP objective.

$\square$

The preceding result shows that any problem that can be represented as an OMP (i.e., $p-$median, $p-$center, cent-dian, etc.) can be solved via the OGCLP, proving that OGCLP indeed provides the unifying framework for all classical objectives in location models. OGCLP with different modeling weight vectors may also be interesting in its own right, as discussed in the following examples.

Examples of OGCLP's with different modeling vectors λ

Median: $\lambda = (1, \ldots, 1)$

In this case we have

$$
\sum_{i=1}^{n} \lambda_i \, c_{\sigma(i)}(S) = \sum_{i=1}^{n} c_i(S)
$$

that is, the problem reduces to the standard GCLP. As discussed earlier, both the median and the maximum cover location problems are special cases of the GCLP.

k-Centra: $\lambda^k = (0, \overset{n-k}{\ldots}, 0, 1, 0, \ldots, 0)$

We have

$$
\sum_{i=1}^{n} \lambda_i \, c_{\sigma(i)}(S) = \lambda_{n-k+1} c_{\sigma(n-k+1)}(S).
$$

This can be viewed as an extension of the center objective to gradual cover setting. The center objective calls for maximizing the coverage of the worst-covered customer node, which can be achieved by setting $\lambda^n = (1, 0, \ldots, 0)$. However, in a "true" cover setting, we would normally not expect to be able to extend the coverage to all customer nodes - thus the coverage level of some nodes would be 0; rendering the modeling vector λ^n not very useful.

One of the standard motivations for maximum cover problems is that the coverage of the worst-covered nodes will be sub-contracted to another service provider. The

$k-centra$ objective calls for maximizing the coverage of the worst-covered node among the k nodes receiving the best coverage. Thus, if the intention is to sub-contract the $n-k \geq 0$ worst-covered nodes to another provider, then the objective above maximizes the coverage for the worst-covered node which will be served from the facility set S.

k-Cover: $\lambda = (0, \overset{n-k}{\ldots}, 0, 1, .\overset{k}{.}., 1)$

In this case we have

$$\sum_{i=1}^{n} \lambda_i \, c_{\sigma(i)}(S) \;=\; \max_{\substack{V' \subset V \\ |V'|=k}} \sum_{i \in V'} c_i(S) \,.$$

i.e., we concentrate on providing the best possible coverage to the k best-covered nodes (again, under the assumption that the $n-k$ worst-covered nodes will be sub-contracted to another service provider).

k-Centdian-cover: $\lambda = (0, \overset{n-k}{\ldots}, 0, 1, \alpha, \overset{k-1}{\ldots}, \alpha), \alpha \in (0,1)$

In this case we have

$$\sum_{i=1}^{n} \lambda_i \, c_{\sigma(i)}(S) \;=\; \alpha \max_{\substack{V' \subset V \\ |V'|=k}} \sum_{i \in V'} c_i(S) + (1-\alpha) c_{\sigma(k)}(S) \,.$$

i.e., a convex combination of the $k-cover$ and the $k-centra$ objectives, allowing us to put extra weight on the k-th worst-covered customer nodes in the gradual $k-cover$ setting.

Trimmed-Cover: $\lambda = (0, .\overset{k_1}{.}., 0, 1, \ldots, 1, 0, .\overset{k_2}{.}., 0)$

In this case we have

$$\sum_{i=1}^{n} \lambda_i \, c_{\sigma(i)}(S) \;=\; \sum_{i=k_1+1}^{n-k_2} c_{\sigma(i)}(S)$$

which is the so-called (k_1, k_2)-trimmed mean, leaving aside the k_1 best and k_2 least covered nodes. This is an alternative to the k-centra objective which may be useful when it has been decided that the service of k_1 nodes will be subcontracted to another provider, and k_2 best-covered node are excluded from the calculation on the grounds that they will automatically receive adequate coverage.

Anti-Cover: $\lambda = (-1, \ldots, -1)$

In this case we have

$$\sum_{i=1}^{n} \lambda_i \, c_{\sigma(i)}(S) \;=\; -\sum_{i=1}^{n} c_i(S)$$

that is, we minimize the coverage of the clients. The possible applications include the location of obnoxious facilities, e.g, waste disposals. All clients within a given distance l of a facility experience the full impact of the obnoxious effect of the facility, the

clients further than u away experience negligible effect, with the impact of the facility gradually decreasing between these two radii.

Equity Objective: $\lambda = (1, .\overset{k}{.}., 1, 0, \ldots, 0, -1, .\overset{k}{.}., -1)$

In this case we have

$$\sum_{i=1}^{n} \lambda_i \, c_{\sigma(i)}(S) \;=\; -\big(\max_{\substack{V' \subset V \\ |V'|=k}} \sum_{i \in V'} c_i(S) \;-\; \min_{\substack{V' \subset V \\ |V'|=k}} \sum_{i \in V'} c_i(S)\big)$$

that is, we are trying to minimize the difference in total coverage received by the k best-covered and k worst-covered clients. This objective may be useful in location of non-emergency public service facilities, such as schools, where equity considerations are important.

2 Single Facility

For the ease of understanding we will first discuss the single facility case, i.e., $S = \{x\}$. Then

$$c_i(x) \;=\; \begin{cases} w_i & \text{if } d_i(x) \leq l_i \\ w_i \, f_i(d_i(x)) & \text{if } l_i < d_i(x) \leq u_i \\ 0 & \text{if } u_i < d_i(x) \end{cases}$$

As already observed, the Ordered Gradual Decay Function

$$g(x) = \sum_{i=1}^{n} \lambda_i \, c_{\sigma(i)}(x)$$

is defined point-wise. Whenever two coverage functions c_i and c_j intersect, the permutation of coverage functions changes and therefore the representation of the objective function.

Following ([3]) we define the following sets. For all $v_i \in V$, let

$$L_i = \{x \in G \mid d_i(x) = l_i\} \quad \text{and} \quad U_i = \{x \in G \mid d_i(x) = u_i\} \,.$$

and

$$L = \bigcup_{v_i \in V} L_i \quad \text{and} \quad U = \bigcup_{v_i \in V} U_i \,.$$

Note that L and U have $O(nm)$ elements each, as a distance function $d_i(x)$ can attain the values l_i and u_i at most twice on each edge.

Next, we will show that if f_i is convex and continuous for all $v_i \in V$, the intersection points of coverage functions, together with the node set V and the set L comprise a finite dominating set (FDS) for the single-facility problem.

Definition 2.1 Equilibrium and Bottleneck points

Let $v_i, v_j \in V$, $v_i \neq v_j$, and $w_i \cdot w_j \neq 0$. Define

$$EQ'_{ij} := \{ x \in G \; : \; c_i(x) \; = \; c_j(x) \; \} \, .$$

For $v_i = v_j$ or $w_i \cdot w_j = 0$, we set $EQ'_{ij} := \emptyset$. Let EQ_{ij} be the boundary of EQ'_{ij} and let $EQ := \bigcup_{i,j, i \neq j} EQ_{ij}$. The points in EQ are called equilibrium points.

A point $x = (e, t)$ on an edge $e = [v_i, v_j] \in E$ is called a bottleneck point if there exists some node v_k with $w_k \neq 0$, such that

$$d(x, v_k) = d(x, v_i) + d(v_i, v_k) = d(x, v_j) + d(v_j, v_k) \, .$$

Denote by BN the set of bottleneck points on N.

Note that the set of bottleneck points BN has $O(nm)$ elements. The number of equilibrium points on an edge $e \in E$ depends on the characteristics of the functions f_i, as we will show in the following example.

Example 2.1 *1. Assume the coverage decay functions $f_i(t)$ are linear for all $i \in V$.*

Let $v_i \in V$ and $e \in E$. If $d_i(x)$ does not have a bottleneck point on e, then $d_i(x)$ is linear on e. Therefore, $c_i(x)$ has at most two breakpoints on e (for $d_i(x) = l_i$ and $d_i(x) = u_i$). If $d_i(x)$ has a bottleneck point $y \in e$, then $d_i(x)$ is linear left and right of y and hence has at most four breakpoints on e. See left hand side picture in Figure 1. (Note that $c_i(x)$ has breakpoints either at $d_i(x) = u_i$ or at y but not both.)
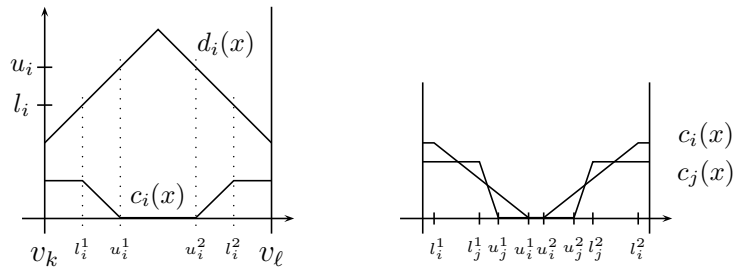
Figure 1: Linear decay function.

As the two coverage functions $c_i(x)$ and $c_j(x)$ are linear between consecutive breakpoints of c_i and c_j on e, they intersect in at most six points. (We can ignore situations where both are constant.) Hence, there are $O(n^2)$ equilibria on an edge and thus EQ has $O(mn^2)$ elements.

Consider the undirected network and the lower and upper bounds l_i and u_i depicted in Figure 2.

10
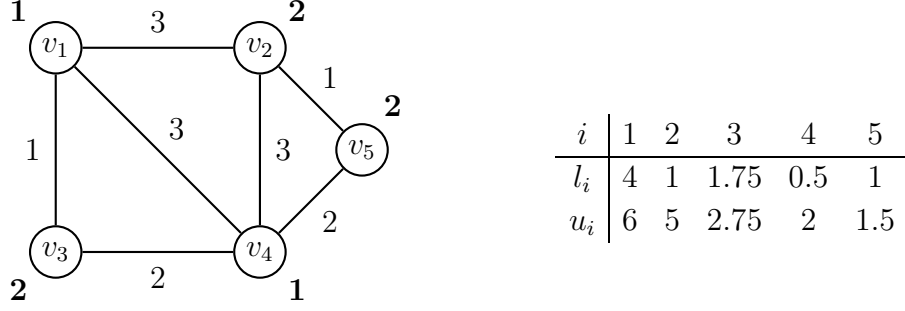
Figure 2: Network and table of upper and lower bounds (see Example 2.1).

Figure 3 shows the distance functions $d_i(x)$ (left hand side) and the coverage functions (right hand side) on the edge $e = [v_4, v_5]$. The point $x = ([v_4, v_5], 0.33)$ is an equilibrium of the coverage functions $c_3(x)$ and $c_5(x)$, i.e., $x = EQ_{35}$.

2. *Assume the coverage decay functions $f_i(t)$ are piecewise linear with K_i breakpoints for all $i \in V$.*

   If $d_i(x)$ is linear (has a breakpoint) on e, $c_i(x)$ has at most $K_i + 1$ ($2(K_i + 1)$) breakpoints on e. Therefore, $c_i(x)$ and $c_j(x)$ intersect in at most $O(K_i + K_j)$ points. Hence, $|EQ| = O(Kmn^2)$, where $K = \max_{i \in V} K_i$.

If all f_i are continuous, the permutation of the coverage functions can only change at equilibrium points. This observation leads to the following theorem.

Theorem 2.1 *Let N be an undirected network, $w, \lambda \geq 0$, and f_i be convex and continuous for all $i \in V$. Then, $V \cup EQ \cup L$ is a finite dominating set.*

Before proving this result we will need the following lemma. Observe that if f_i is convex,
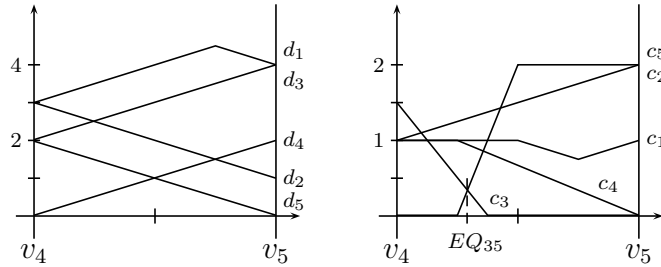
Figure 3: Distance and coverage functions on edge $[v_4, v_5]$ of the network in Figure 2.

11

then also

$$\bar{f}_i(t) \;=\; \begin{cases} f_i(t) & \text{if } l_i < t \le u_i \\ 0 & \text{if } t > u_i \end{cases}$$

is convex for $t > l_i$, since $f(u_i) = 0$.

Lemma 2.1 *Let f_i be convex and continuous for all $v_i \in V$, $S \subset G$, and $z_1, z_2 \in e$, $e \in E$, be two consecutive elements of $V \cup L$ on edge e, i.e., $[z_1, z_2] \cap (V \cup L) = \{z_1, z_2\}$. Then, $c_i(S \cup \{x\})$ is convex for $x \in [z_1, z_2]$.*

Proof
We have that $d_i(S \cup \{x\})$ is concave for $x \in [z_1, z_2]$, since $d_i(x)$ is concave on an edge, $d_i(S)$ is constant with respect to x, and $d_i(S \cup \{x\}) = \min\{d_i(S), d_i(x)\}$. Moreover, we have that

$$c_i(S \cup \{x\}) \;=\; \max\{c_i(S), c_i(x)\}$$

with $c_i(S)$ being constant with respect to x.

By assumption on z_1 and z_2, either $d_i(x) \le l_i$ or $d_i(x) > l_i$ for all $x \in [z_1, z_2]$. In the former case, $c_i(x) = w_i$ is constant. In the latter case, $c_i(x) = w_i \bar{f}_i(d_i(x))$ is a composition of a concave and a convex non-increasing function and therefore convex. Hence, $c_i(S \cup \{x\})$ is convex as a maximum over convex functions.

$\square$

We are now ready to prove Theorem 2.1.

Proof
Augment G by inserting the elements of $EQ \cup L$ as new nodes. Denote the augmented graph $G' = (V', E')$.

Let $e \in E'$ be an edge of the augmented graph. As we added the equilibrium points to the finite dominating set and all coverage decay functions are continuous, the order of the functions $\{c_i(x)\}_{i=1}^n$ will not change on the edge. Thus, the objective function reduces to a weighted sum of the coverage functions

$$g(x) = \sum_{i=1}^{n} \lambda_i \, c_{\sigma_e(i)}(x)$$

where σ_e denotes the corresponding permutation of the coverage functions on e. As the coverage functions are convex (Lemma 2.1) and $\lambda \ge 0$, $g(x)$ is also convex as a weighted sum of convex functions. Therefore, the objective function attains its maximum at one of end nodes of e and the result follows.

$\square$

As $|L| = O(nm)$, the size of the finite dominating set is of order $O(mn + |EQ|)$, where $|EQ|$ depends on the actual representations of the decay functions, see Example 2.1.

If the decay functions f_i are not convex, the above result no longer holds. However, we can extend it to stepwise as well as continuous piecewise linear decay functions as follows. Let r_i^k, $k = 0, \ldots, K_i$, with $l_i = r_i^0 < r_i^1 < \ldots < r_i^{K_i} = u_i$ denote the breakpoints of the decay functions f_i, see also Section 1.1. Define $R_i = \{r_i^k \mid k = 0, \ldots, K_i\}$. Each breakpoint of f_i induces one or more breakpoints of the coverage functions c_i. Denote by

$$BP_i \;=\; \{x \in G \mid d_i(x) = r \in R_i\}$$

the set of breakpoints of the coverage functions c_i induced by the breakpoints of the decay functions. Define $BP = \bigcup_{i \in V} BP_i$ as the set of all these breakpoints. Note that $L, U \subset BP$.

Theorem 2.2 *Let N be an undirected network and $w, \lambda \geq 0$.*

1. *For continuous piecewise linear decay functions, $V \cup BP \cup EQ$ is a finite dominating set.*

2. *For stepwise decay functions, $V \cup BP$ is a finite dominating set.*

Proof
We use the same ideas as in the proof of Theorem 2.1.

1. Augment G by inserting the elements of $EQ \cup BP$ as new nodes. As a result, the decay functions f_i reduce to linear functions on each edge of the augmented graph. Hence, the coverage functions c_i are convex on each edge as a composition of a concave and a linear non-increasing function.

   Moreover, as the order of the functions $\{c_i(x)\}_{i=1}^n$ will not change on an edge, the objective function reduces to a weighted sum of convex functions. Therefore, $g(x)$ attains its maximum at one of the end nodes of the edges of the augmented graph.

2. For stepwise functions, augment G by inserting the elements of BP as new nodes. Then, the coverage functions c_i are constant on the interior of each edge of the augmented graph. As the order of the functions $\{c_i(x)\}_{i=1}^n$ will not change on the interior, the objective function will also be constant. As $w, \lambda \geq 0$, $g(x)$ will always attain its maximum at least at one of the end nodes of the edges of the augmented graph. See Figure 4 for an illustration.

Hence, the result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Concerning the size of the finite dominating sets, observe that $|BP_i| = O(mK_i)$ as each breakpoint $r \in R_i$ may induce a breakpoint of a coverage function on each edge. Hence, $|BP| = O(mnK)$, where $K = \max_{i \in V} K_i$. Moreover, we have $|EQ| = O(mn^2K)$. Thus, the size of the finite dominating sets is $O(mn^2K)$ and $O(mnK)$, respectively.

Unfortunately, it does not appear possible to characterize FDS for the case of general coverage decay functions and unrestricted modeling vectors λ. However, we will see in the following sections that FDS can be characterized for many important special cases.

Efficient Algorithms

After identifying finite dominating sets, we now discuss how to efficiently compute them and, subsequently, solve the corresponding problems. For the sake of brevity, we restrict ourselves to stepwise and piecewise linear decay functions.

We start by considering the case of continuous piecewise linear decay functions. To solve the problem, we have to determine the sets BP and EQ. BP can be computed in $O(mnK)$ time, as a distance function d_i consists of at most two linear pieces on an edge, and we can solve the equations $d_i(x) = r_i^k$, $k = 0, \ldots, K_i$, in $O(K_i)$ time. Now we turn to EQ. For $i, j \in V$ and $k \in E$ we can compute the intersection points of the coverage functions c_i and c_j on edge e_k in $O(K_i + K_j)$ time, since c_i (c_j) has at most $O(K_i)$ ($O(K_j)$) breakpoints on e_k. To determine the set EQ we have to intersect pairwise all coverage functions on all edges, which requires in total $O(mn^2K)$ time.

To find the optimal solution, we straight forwardly evaluate the objective function for all elements of the finite dominating set. For a point $x \in G$, we can compute the objective function value $g(x)$ in $O(n \log n)$ time. Therefore, the overall complexity for solving the problem is

$$O(mnK + mn^2K + (n + mnK + mn^2K)n \log n) = O(mn^3 \log n \, K).$$

For stepwise decay functions this reduces to $O(mn^2 \log n \, K)$.

However, for continuous piecewise linear decay functions we can adapt the efficient algorithm for the single-facility Ordered Median Problem presented in Kalcsics et al. [9] to obtain a lower complexity algorithm. Observe, that the only breakpoints of the coverage functions $c_i(x)$ occur at elements of BP_i or at bottleneck points of the distance functions $d_i(x)$.

For an edge $e \in E$, we first compute the set of bottleneck points, equilibria, and elements of BP on e. Afterward, we sort them in nondecreasing order from one of the end nodes. Denote the elements of the sorted sequence by $\{x_1, \ldots, x_Q\}$. Then, for any x in the interior of the subedge connecting two consecutive elements x_q and x_{q+1}, the sorting of the coverage
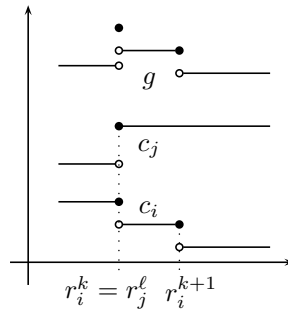
Figure 4: Illustration for stepwise functions.

14

functions c_i does not change; moreover, the c_i are linear on the subedge $s_q = [x_q, x_{q+1}]$ (as we included the bottleneck and breakpoints) and hence also the objective function. Therefore, if we know the objective value at x_q and the slope of the objective function on s_q, we can obtain $g(x_{q+1})$ in constant time. Moreover, we can update the slope of g at each x_q also in constant time (see Kalcsics [8] for more details). Only for the first element, x_1, we have to explicitly evaluate the objective function value $g(x_1)$ and the slope of $g(x)$ on s_1 in $O(n \log n)$ time. Therefore, the overall complexity of the algorithm is $O(mn^2 K \log(nK))$, as we have $O(n^2 K)$ bottleneck points, equilibria, and elements of BP on an edge.

2.1   Special Cases

For certain modeling vectors λ we can obtain smaller finite dominating sets, as the following result shows.

Corollary 2.1 *Let N be an undirected network, w, $\lambda \geq 0$, and f_i be convex for all $i \in V$. Moreover, let the modeling weights be non-decreasing, i.e., $\lambda_1 \leq \ldots \leq \lambda_n$. Then, $V \cup L$ is a finite dominating set.*

Proof
Augment G by inserting the elements of L as new nodes. Denote the augmented graph $G' = (V', E')$. From the definition of L follows that all coverage functions $c_i(x)$, $i \in V$, are convex on each edge of the augmented graph.
For $\lambda_1 \leq \ldots \leq \lambda_n$, we have

$$g(x) = \sum_{i=1}^{n} \lambda_i \, c_{\sigma(i)}(x) = \max_{\pi \in P(1\ldots n)} \sum_{i=1}^{n} \lambda_i \, c_{\pi(i)}(x)$$

as the permutation $\sigma \in P(1 \ldots n)$ that sorts the coverage functions c_i in nondecreasing order for a given $x \in e'$, $e' \in E'$, is identical to the permutation π^* for which the maximum on the right hand side is obtained (see, e.g., Hardy et al. [7]). As the coverage functions $c_{\pi(i)}(x)$ are convex on each edge of G', so is the right hand side expression as a maximum of a weighted sum of convex functions. Therefore, $g(x)$ obtains its maximum at an end node of e' and the result follows.

$\square$

For this special case, the size of the finite dominating set reduces to $O(mn)$.

2.2 Real-valued Modeling Weights: the Case of (Semi-) Obnoxious Facilities

First, we consider problems where the modeling weights are strictly non-positive, i.e., $\lambda \leq 0$ (alternatively, we could assume the node weights to be non-positive). Intuitively, this corresponds to the case where customer benefit is maximized when their coverage is as low as possible - which occurs in case of facilities like garbage dumps or nuclear waste sites, i.e., facilities one would rather not be covered by. Such facilities are typically referred to as "obnoxious facilities" in the location literature (see, e.g., Erkut and Neumann [6], Eiselt and Laporte [5]). Since $\lambda \leq 0$, we have,

$$\max_{x \in G} \sum_{i=1}^{n} \lambda_i \, c_{\sigma(i)}(x) \;=\; - \min_{x \in G} \sum_{i=1}^{n} |\lambda_i| \, c_{\sigma(i)}(x).$$

Observe that if f_i is concave, then the function

$$\bar{f}_i(t) \;=\; \begin{cases} w_i & \text{if } d_i(S) \leq l_i \\ f_i(t) & \text{if } l_i < d_i(S) \leq u_i \end{cases}$$

is concave for $t \leq u_i$, since $f(l_i) = w_i$.

Lemma 2.2 *Let f_i be concave for all $i \in V$ and z_1, z_2 be two consecutive elements of the set $V \cup BN \cup U$ on an edge, i.e., $[z_1, z_2] \cap (V \cup BN \cup U) = \{z_1, z_2\}$. Then, $c_i(x)$ is concave for $x \in [z_1, z_2]$.*

Proof
As we included the bottleneck points, $d_i(x)$ is linear for $x \in [z_1, z_2]$. By assumption on z_1 and z_2, either $d_i(x) \leq u_i$ or $d_i(x) > u_i$ for all $x \in [z_1, z_2]$. In the latter case, $c_i(x)$ is constant. In the former case, $c_i(x)$ is a composition of a linear and a concave non-increasing function and therefore concave.

□

Now we can state a result analogous to Theorem 2.1.

Theorem 2.3 *Let N be an undirected network, $\lambda \leq 0 \leq w$, and f_i be concave and continuous for all $i \in V$. Then, $V \cup BN \cup EQ \cup U$ is a finite dominating set.*

Proof
Augment G by inserting the elements of $BN \cup EQ \cup U$ as new nodes. Denote the augmented graph $G' = (V', E')$.

Let $e \in E'$ be an edge of the augmented graph. As we added the equilibrium points to the finite dominating set and all coverage decay functions are continuous, the order of the

functions $\{c_i(x)\}_{i=1}^n$ will not change on the edge. Thus, the objective function reduces to a weighted sum of the coverage functions

$$g(x) = \sum_{i=1}^n \lambda_i\, c_{\sigma_e(i)}(x)$$

where σ_e denotes the corresponding permutation of the coverage functions on e. As the coverage functions $c_{\sigma_e(i)}$ are concave (Lemma 2.2) and $\lambda \leq 0$, $g(x)$ is convex as a negatively weighted sum of concave functions. Therefore, the objective function attains its maximum at one of the end nodes of an edge and the result follows.

$\square$

As $|U|, |BN| = O(nm)$, the size of the finite dominating set is of order $O(nm + |EQ|)$, where $|EQ|$ depends on the actual representations of the decay functions.

Now we turn to the problem with semi-obnoxious facilities where we allow the modeling weights to be real-valued. Unfortunately, the result of Theorem 2.3 does not carry over to this case. The problem is that for a mixture of negative and positive λ_i, the functions $\lambda_i\, c_{\sigma_e(i)}(x)$ we are summing up for the objective function are convex for negative lambdas and concave for $\lambda_i > 0$. Hence, their sum will not necessarily be concave or convex.

However, we can extend Theorem 2.2 for stepwise and continuous piecewise linear decay functions to real-valued node and modeling weights. But before, we need the following definition. For two adjacent elements x, y of the set $V \cup BP$ on an edge, we denote by $mp(x, y)$ the midpoint between the two elements and by MP the set of all midpoints between adjacent elements.

Theorem 2.4 *Let N be an undirected network and $w, \lambda \in \mathbb{R}^n$.*

1. *For continuous piecewise linear decay functions, $V \cup BN \cup BP \cup EQ$ is a finite dominating set.*

2. *For stepwise decay functions, $V \cup BP \cup MP$ is a finite dominating set.*

Proof
The proof is similar to the one of Theorem 2.2.

1. By inserting the bottleneck points in addition to the elements of $EQ \cup BP$, the distance functions are linear on each edge of the augmented graph. Hence, also the coverage functions are linear on each edge as a composition of two linear functions d_i and f_i. Therefore, $g(x)$ again attains its maximum at one of the end nodes of the edges of the augmented graph.

2. As for the nonnegative case, the objective function is constant in the interior of an edge. However, now a point in the interior can have a strictly larger objective value then one of the end nodes. See Figure 4 for an illustration; if we want to minimize coverage, the interior of the right most segment of the objective function would yield the minimum. Therefore, we also have to check an interior point, for example the midpoint, for optimality.

Hence, the result follows. □

The size of the finite dominating sets is again $O(mn^2K)$ and $O(mnK)$, respectively, as $|BN| = O(mn)$ and $|MP| = O(mnK)$. Moreover, also the complexity for solving the problems is the same as in the nonnegative case.

2.3 The Conditional Location Case

Assume that there are already some facilities in operation. Denote $C \in G$ the set of points where these facilities are sited. The task is then to optimally locate an additional facility on the network.

Define the conditional distance function $d_i^c(x)$

$$d_i^c(x) \; = \; d_i(C \cup \{x\}) \,.$$

The functions d_i^c are piecewise linear and concave with at most two breakpoints. Therefore, the results for the unconditional problem with nonnegative modeling vectors carry over to the conditional location case.

Now consider the conditional obnoxious case, i.e., $\lambda \leq 0$. For the obnoxious unconditional problem we had to add the bottleneck points to the FDS in order to have the distance functions being linear between consecutive elements of the FDS. Now, the conditional distance functions have breakpoints at bottleneck points or at points where

$$d_i(x) \; = \; d_i(C) \,.$$

Formally, these points are defined as follows.

Definition 2.2 Conditional extreme points
Let $v_i \in V$ and $e \in E$. A point $x \in G$ is called conditional extreme point of node v_i, if $d_i(x) = d_i(C)$. CEP_i denotes the set of all conditional extreme points of node v_i and $CEP = \bigcup_{i=1}^n CEP_i$ the set of all conditional extreme points of nodes.

Note that CEP has $O(nm)$ elements. Now we can prove analogous results to the unconditional obnoxious problem.

Lemma 2.3 *Let f_i be concave for all $i \in V$ and z_1, z_2 be two consecutive elements of the set $V \cup BN \cup CEP \cup U$ on an edge, i.e., $[z_1, z_2] \cap (V \cup BN \cup CEP \cup U) = \{z_1, z_2\}$. Then, $c_i(x)$ is concave for $x \in [z_1, z_2]$.*

Theorem 2.5 *Let N be an undirected network, $\lambda \leq 0 \leq w$, and f_i be concave and continuous for all $i \in V$. Then, $V \cup BN \cup CEP \cup EQ \cup U$ is a finite dominating set.*

Theorem 2.6 *Let N be an undirected network, $w, \lambda \in \mathbb{R}^n$, and f_i be continuous piecewise linear (stepwise) functions. Then, $V \cup BN \cup CEP \cup EQ \cup BP$ ($V \cup CEP \cup BP \cup MP$) is a finite dominating set.*

The finite dominating sets and the algorithms have the same size and complexity, respectively, as the ones for the unconditional problem.

3   Multi Facility

Let now $|S| = p > 1$ and $w, \lambda \geq 0$. First, we note that this problem is NP-hard (since it reduces to the gradual covering decay problem of Berman et al. [3] for $\lambda = (1, \ldots, 1)$).

Before turning to the general problem, we first discuss a special case. Namely, the problem with non-decreasing modeling weights, i.e., $\lambda_1 \leq \ldots \leq \lambda_n$. Here, we can prove a result analogous to the one in Berman et al. [3].

Theorem 3.1 *Let N be an undirected network, $w, \lambda \geq 0$, and f_i be convex for all $i \in V$. Moreover, let the modeling weights be non-decreasing, i.e., $\lambda_1 \leq \ldots \leq \lambda_n$. Then, $V \cup L$ is a finite dominating set.*

Proof
Augment G by inserting the elements of L as new nodes. Denote the augmented graph $G' = (V', E')$. From Lemma 2.1 we know that $c_i(S \cup \{x\})$, $S \subset G$, is convex for $x \in e'$, $e' \in E'$, i.e., on each edge of the augmented graph G'. Moreover, for $\lambda_1 \leq \ldots \leq \lambda_n$, we have

$$g(S \cup \{x\}) = \sum_{i=1}^{n} \lambda_i \, c_{\sigma(i)}(S \cup \{x\}) = \max_{\pi \in P(1\ldots n)} \sum_{i=1}^{n} \lambda_i \, c_{\pi(i)}(S \cup \{x\})$$

as the permutation $\sigma \in P(1 \ldots n)$ that sorts the coverage functions c_i in nondecreasing order for a given $x \in e'$ is identical to the permutation π^* for which the maximum on the right hand side is obtained (see, e.g., Hardy et al. [7]). As the coverage functions $c_{\pi(i)}(S \cup \{x\})$ are convex on each edge of the augmented graph, so is the right hand side expression as a maximum of a weighted sum of convex functions.
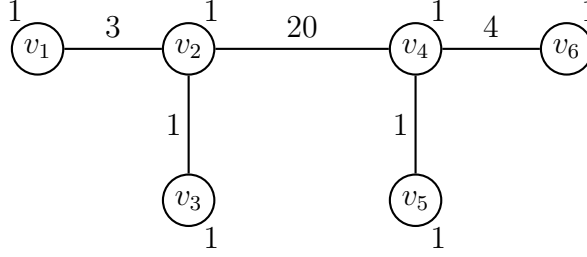
Figure 5: The tree network used in Example 3.1.

Suppose S^* is an optimal set of locations, and there exists $s^* \in S^* \setminus \{V \cup L\}$. Let $s^* \in e^*$, $e^* \in E'$. Replacing s^* by one of the end nodes of e^* will not decrease the objective function value. Thus, the result follows.

$\square$

Now, we turn to the general case. Unfortunately, the finite dominating set for the single facility problem, $V \cup L \cup EQ$, is no longer valid for $p > 1$, as the following counter example shows.

Example 3.1 *Consider the tree network in Figure 5 with $\ell_i = 0$, $i \in \{1, \ldots, 4, 6\}$, $\ell_5 = 1$, and $u_i = 3$, $i = 1, 4, 6$, $u_2 = 6$, $u_3 = 5$, $u_5 = 7$. Let $p = 2$ and $\lambda = (1, 0.2, \ldots, 0.2)$. Figure 6 shows the coverage functions on the edges $[v_1, v_2]$ and $[v_4, v_6]$.*

We will show that $V \cup L \cup EQ$ is no longer a finite dominating set for the multifacility problem. If we restrict X_2 to be a subset of $V \cup L \cup EQ$, the optimal solution is given by

$$S = \left\{ EQ_{13} = \left([v_1, v_2], \tfrac{1}{2}\right), \; EQ_{46} = \left([v_4, v_6], \tfrac{1}{2}\right) \right\},$$

with objective value $g(S) = 0.88\bar{3}$. However, if we drop this restriction we obtain a slightly better solution, namely

$$S^* = \left\{ x^* = \left([v_1, v_2], \tfrac{2}{3}\right), \; EQ_{46} = \left([v_4, v_6], \tfrac{1}{2}\right) \right\},$$

with an optimal objective function value of $0.88\bar{6}$. Note that x^* is neither a node, element of the set L, or equilibrium point.

Fortunately, for special classes of modeling vectors we can identify finite dominating sets.
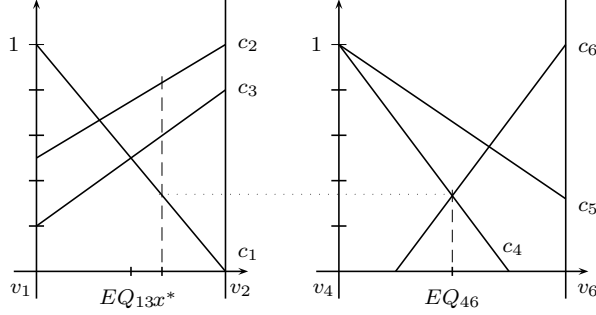
Figure 6: Coverage functions on $[v_1, v_2]$ and $[v_4, v_6]$ of the graph in Figure 5.

3.1 A Finite Dominating Set for a Special Classes of Modeling Vectors

For the ordered median location problem, for modeling vectors λ with

$$\lambda_1 = \ldots = \lambda_b < \lambda_{b+1} = \ldots = \lambda_n, \quad b \in \{1, \ldots, n-1\}$$

Kalcsics et al. [10] prove a finite dominating set. This result was later extended by Kalcsics [8] to modeling vectors λ with

$$\lambda_1 \geq \ldots \geq \lambda_b, \quad \lambda_{b+1} \geq \ldots \geq \lambda_n \quad \text{and} \quad \lambda_b < \lambda_{b+1}. \tag{4}$$

As the only requirement for the proofs was that the distance functions are concave and continuous, we can prove an analogous result for the gradual covering problem with convex and continuous coverage functions. Here, however, with modeling vectors λ of the form

$$\lambda_1 \leq \ldots \leq \lambda_b, \quad \lambda_{b+1} \leq \ldots \leq \lambda_n \quad \text{and} \quad \lambda_b > \lambda_{b+1}. \tag{5}$$

We denote by Λ^b the set of all modeling vectors that fulfill (5). Note that the modeling vectors λ for the median, k-Cent-cover, k-cover, and trimmed-cover problem belong to Λ^b.

In Theorem 3.1 we could observe that the problem is easy to solve if all modeling weights are non-decreasing. The modeling vectors $\lambda \in \Lambda^b$ have a similar representation. The only difference is, that there exist two consecutive modeling weights λ_b and λ_{b+1} where this property does not hold. The approach to identify a finite dominating set (FDS) is to consider situations where the order of the coverage functions changes at the positions b and $b + 1$, i.e., where $c_{\sigma(b)}(S) = c_{\sigma(b+1)}(S)$. Analyzing what happens in these situations will lead to the desired FDS.

From now on, we assume that $N = (G, \ell)$ is an undirected network, $p \geq 2$, w, $\lambda \geq 0$, and $\lambda \in \Lambda^b$. Moreover, let f_i be continuous and convex for all $i \in V$. The following discretization

21

result is equivalent to the one for the p-facility Ordered Median Problem (pOMP) with lambda vectors fulfilling (4), see Kalcsics et al. [10]. The only difference is, that the set V has to be replaced by $V \cup L$. We briefly describe the analogy. The Ordered Gradual Covering Location Problem

$$\max_{\substack{S \subset G \\ |S|=p}} \sum_{i=1}^{n} \lambda_i \, c_{\sigma(i)}(S)$$

can be equivalently formulated as

$$\min_{\substack{S \subset G \\ |S|=p}} \sum_{i=1}^{n} \tilde{\lambda}_i \, \tilde{c}_{\sigma(i)}(S) \,,$$

where $\tilde{\lambda}_i := \lambda_{n-i+1}$ and $\tilde{c}_i(S) := -c_i(S)$. If $\lambda \in \Lambda^b$, then $\tilde{\lambda}$ fulfills (4). As the function $c_i(S \cup \{x\})$ is continuous and convex between consecutive elements of $V \cup L$ on an edge, $\tilde{c}_i(S \cup \{x\})$ will continuous and concave.

Now, the p-facility Ordered Median Problem is defined as

$$\min_{\substack{S \subset G \\ |S|=p}} \sum_{i=1}^{n} \tilde{\lambda}_i \, d^w_{\sigma(i)}(S) \,,$$

where $d^w_i(S) = w_i d_i(S)$ and $\tilde{\lambda}$ fulfills (4). For proving the results for the pOMP, [10] only depend on the fact that the functions $d^w_i(S \cup \{x\})$ are continuous and concave. Therefore, if we restrict ourselves to subedges between consecutive elements of $V \cup L$ on an edge, we can simply replace $d^w_i(S)$ by $\tilde{c}_i(S)$ and use the same techniques as in [10] to prove the results.

Before we present the discretization result, we need the following two definitions.

Definition 3.1 Ranges
Let $S \subset G$. Define the set of ranges (canonical set of distances) by

$$R := \{\, r \in \mathbb{R} \mid \exists \, x \in EQ_{ij} : c_i(x) = r = c_j(x) \quad or$$
$$\exists \, v_i \in V, x \in V \cup L : r = c_i(x) \,\},$$

and the set of ranges with respect to S by

$$R(S) := \{\, r \in R \mid \exists \, x \in EQ_{ij} \cap S : c_i(x) = r = c_j(x) \quad or$$
$$\exists \, v_i \in V, \exists \, x \in (V \cup L) \cap S : r = c_i(x) \,\}.$$

The ranges correspond to coverage function values of equilibria or node to node distances.

Definition 3.2 Extreme points
Let $r \in \mathbb{R}$. A point $x \in G$ is called r-extreme point, if there exists a node $v_i \in V$ with $r = c_i(x)$. By

$$EP(r) = \{x \in G \mid \exists \, i \in \{1, \ldots, n\} : r = c_i(x)\}$$
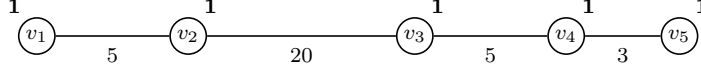
Figure 7: The path graph used in Example 3.2.

we denote the set of all r-extreme points on the network and by $EP(Q) = \bigcup_{r \in Q} EP(r)$ the set of r-extreme points with respect to a set $Q \subset \mathbb{R}$ of values.

Now, we can state the FDS.

Theorem 3.2 *There always exists an optimal solution $S \subset EP(R)$ such that $S \cap (V \cup L \cup EQ) \neq \emptyset$ and $S \subset EP(R(S))$.*

That is, there always exists an optimal solution S^* which contains a node, an element of L, or an equilibrium, and all other solution points are extreme points with respect to the nodes, the elements of L, and the equilibria of S^*. (Observe that each node, element of L, or equilibrium is an extreme point with respect to itself.)

Unfortunately, this result does not hold for arbitrary modeling vectors, as the following counterexample shows.

Example 3.2 *Consider the path graph in Figure 7 with $\ell_i = 0$, $i = 1, \ldots, 5$, $u_i = 10$, $i = 1, 2, 4, 5$, and $u_3 = 7$. Let $p = 2$ and $\lambda = (0, 1, 0, 1, 1)$. Figure 8 shows the coverage functions on the edges $[v_1, v_2]$ and $[v_3, v_4]$. As in an optimal solution nodes v_1 and v_2 will always be covered by a solution point on $[v_1, v_2]$ and the other nodes to a point on $[v_3, v_4]$ or $[v_4, v_5]$, the respective coverage functions are omitted. For this example, $X_2 = (x_1, x_2)$ with $x_1 = ([v_1, v_2], 0.3)$ and $x_2 = ([v_3, v_4], 0.7)$ is an optimal solution. Note that neither solution point is a node, element of the set L, equilibrium or extreme point.*

4   Solving the Discrete Multi-Facility OGCP

In this section we discuss exact solution approaches for discrete multi-facility OGCP, i.e., we assume that the set of potential locations for the new facilities is discrete. In view of the results in the previous section, a Finite Dominating Set can, under some conditions, be identified *a priori* in network location problems, which allows us to treat them as discrete location problems; alternatively, a discrete problem may arise in its own right in cases where potential facility locations have been pre-selected.

Without any loss of generality we assume that V is the set of potential facility sites (since
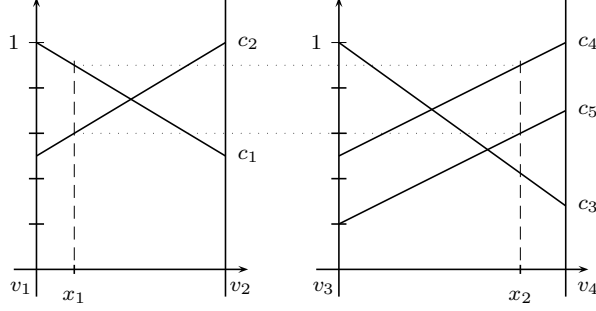
Figure 8: Coverage functions on $[v_1, v_2]$ and $[v_3, v_4]$ of the graph in Figure 7.

any non-nodal sites can be added to the set of nodes). The following formulation directly extends the standard UFLP formulation (e.g. see Berman and Krass [2]) and is similar to formulation (OMP[1]) presented in Nickel and Puerto [13].

Let us define the following quantity and set:

$$
c_{ij} = \begin{cases} w_i & \text{if} \quad d_i(j) \le l_i \\ w_i f_i(d_i(j)) & \text{if} \quad l_i \le d_i(j) \le u_i \quad \text{for } i, j \in V \\ 0 & \text{if} \quad d_i(j) \ge u_i \end{cases}
$$

$$
Q_{ij} = \{l \in \{1, \ldots, n\} | c_{il} \ge c_{ij}\}.
$$

The decision variables and the integer programming formulation can now be stated as follows:

$$
y_j = \begin{cases} 1 & \text{if a new facility is established at } v_j \in V \\ 0 & \text{else} \end{cases}
$$

$$
x_{ijk} = \begin{cases} 1 & \text{if node } v_i \text{ is covered by a facility at } v_j \\ & \text{and the corresponding coverage level} \\ & \text{is at position } k \text{ in the sorted vector} \\ 0 & \text{else} \end{cases}
$$

for $i, j, k = 1, \ldots, n$.

$$
\max \sum_{k=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_k \, c_{ij} x_{ijk}
$$

$$
s.t. \quad \sum_{j=1}^{n} y_j = p \tag{6}
$$

24

$$x_{ijk} \leq y_j \qquad \forall i,j,k = 1,\ldots,n \qquad (7)$$

$$\sum_{j=1}^{n}\sum_{k=1}^{n} x_{ijk} = 1 \qquad \forall i = 1,\ldots,n \qquad (8)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ijk} = 1 \qquad \forall k = 1,\ldots,n \qquad (9)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ijk} \leq \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\, x_{ij(k+1)} \qquad \forall k = 1,\ldots,n-1 \qquad (10)$$

$$\sum_{k=1}^{n}\sum_{l \in Q_{ij}} x_{ilk} \geq y_j \qquad \forall i,j = 1,\ldots,n \qquad (11)$$

$$y_j,\, x_{ijk} \in \{0,1\} \qquad \forall i,j,k = 1,\ldots,n$$

Constraints (6) ensure that p facilities are located. Constraints (7) guarantees that node v_i can only be covered from v_j if a facility has been established there. Constraints (8) and (9) make sure that node v_i is covered at exactly once (recall that $c_{ij} = 0$ is possible) and, that one coverage value has to be assigned to each position. Constraints (10) ensure that the coverage levels of the nodes are sorted in non-decreasing order for the objective function. Finally, constraints (11) guarantee that any node v_i is covered at its maximal level. The latter constraints are only required in case of negative modeling vectors λ - otherwise they will hold automatically due to the objective function.

This IP formulation requires n^3+n decision variables and n^3+n^2+3n constraints and thus the dimensionality grows very quickly with n. We now present an alternative IP formulation that takes advantage of the structure of the coverage functions and can yield significantly more compact and solvable formulations.

This formulation is based on the one in Berman and Krass [2] as well as Marin, Nickel, Puerto and Velten [11]. The main idea is that in order to compute the contribution of customer node $v_i \in V$ to the objective function, it is not necessary to know where the customer is covered from - we only need to know the coverage level the customer receives. For example, in the traditional coverage context, node v_i is either covered or not, and thus the coverage function can only take on two values: w_i or 0. In the gradual coverage framework, the number of possible values of the coverage function can be larger (theoretically, as large as n - if every possible facility location results in a different coverage), but may also be small in many applications. The formulation below exploits this feature by focusing on the coverage level received by each customer node.

Recall the definition of c_{ij}. For each $v_i \in V$ the distinct coverage values in the set $\{c_{ij} \mid v_j \in V\}$ are sorted as $c_i^{(0)} < \ldots < c_i^{(G_i)}$. We will call these values the *coverage levels* of node v_i. Note that the number of coverage levels $G_i \leq n$. We also define the following sets and decision variables (the meaning of decision variables y_j is the same as in the previous formulation) :

$J_i(r) = \{v_j \in V | c_{ij} = c_i^{(r)}\}$ for $v_i \in V$ and $r \in \{0, \ldots, G_i\}$—the set of nodes from which v_i can be covered at level r.

$x_{ik}^{(r)} = 1$ if the node with k^{th} smallest coverage level in the solution receives coverage level $c_i^{(r)}$; $x_{ik}^{(r)} = 0$ otherwise. Here $k \in \{1, \ldots, n\}$, $r \in \{1, \ldots, G_i\}$.

The "improved IP" formulation is:

$$\max \sum_{k=1}^{n} \sum_{i=1}^{n} \sum_{r=1}^{G_i} \lambda_k c_i^{(r)} x_{ik}^{(r)}$$

$$s.t \qquad \sum_{j=1}^{n} y_j = p \tag{12}$$

$$x_{ik}^{(r)} \leq \sum_{j \in J_i(r)} y_j, \quad \text{for } i, k = 1, \ldots, n, \ r = 1, \ldots, G_i \tag{13}$$

$$\sum_{i=1}^{n} \sum_{r=1}^{G_i} x_{ik}^{(r)} \leq 1, \quad \text{for } k = 1, \ldots, n \tag{14}$$

$$\sum_{k=1}^{n} \sum_{r=1}^{G_i} x_{ik}^{(r)} \leq 1, \quad \text{for } i = 1, \ldots, n \tag{15}$$

$$\sum_{i=1}^{n} \sum_{r=1}^{G_i} c_i^{(r)} x_{ik}^{(r)} \leq \sum_{i=1}^{n} \sum_{r=1}^{G_i} c_i^{(r)} x_{i(k+1)}^{(r)}, \quad \text{for } k = 1, \ldots, n-1 \tag{16}$$

$$p \sum_{k=1}^{n} \sum_{r=\rho}^{G_i} c_i^{(r)} x_{ik}^{(r)} \geq \sum_{j \in J_i(\rho)} y_j, \quad \text{for } i = 1, \ldots, n, \ \rho = 1, \ldots, G_i \tag{17}$$

$$x_{ik}^{(r)}, y_j \in \{0, 1\}$$

The objective function multiplies the $k - th$ smallest coverage level by the appropriate component of the modeling vector λ. Note that index k refers to the position of the node-coverage level combination in the ordered list used in the objective function. Constraint (12) require that p facilities be located, and constraints (13) ensure that node v_i can receive coverage level r only if one or more facilities are open in the set $J_i(r)$. Constraints (14) state there is at most one node-coverage level combination assigned to each position k. Note that we do not have decision variables corresponding to node-coverage level combinations receiving coverage zero, since zero coverage does not contribute to the objective function (such combinations would have index $r = 0$, but we always keep $r \geq 1$). Thus, some positions k may not be assigned to any node-coverage level combination if the number of positive combinations is less than n. That is why constraints (14) are inequalities. Similarly, constraints (15) require that each node i must be assigned to at most one coverage level and one position. Constraints (16) are sorting constraints, ensuring that the node-coverage

level combination assigned to position k does not have higher coverage than the combination assigned to position $k+1$. These constraints are not necessary when the components of the modeling vector are non-decreasing, since they will be automatically enforced by the objective function. Finally, constraints (17) specify that each node be assigned to the highest possible coverage level; these constraints are not necessary when the components of the modeling vector λ are non-negative.

The improved IP formulation has $n + n \sum_{i=1}^{n} G_i \leq n + n^3$ decision variables; when the number of distinct coverage levels is n for each node $v_i \in V$ it is equivalent to the previous formulation. However, as noted earlier, G_i may be much smaller than n for many applications: e.g., $G_i = 1$ for the standard cover model and $G_i = s - 1$ when the coverage decay function $f_i(t)$ is a $s-$level step function. In these cases, the improved IP formulation is significantly more compact than the original one. Moreover, as proved in [2] for the GCLP case, the LP relaxation for the improved formulation is just as tight as for the original formulation, thus using the improved formulation cannot hurt in terms of the problem solvability. The improved IP formulation is illustrated in the following example.

Example 4.1 *Consider a triangle network with link lengths $l(1,2) = 2$, $l(1,3) = 2$, $l(2,3) = 1$. The coverage decay function is a simple cover with radius 1 (i.e., only node 1 is covered from 1, while nodes 2 and 3 are covered from either 2 or 3). The node weights are $w_1 = 5$, $w_2 = 2$, $w_1 = 1$, one facility is to be located, and $\lambda = \{1,1,0\}$, indicating that we wish to maximize the total coverage of two worst-covered nodes .*

Here the coverage values are same as node weights, thus, $c_1^{(0)} = c_2^{(0)} = c_3^{(0)} = 0$, $c_1^{(1)} = 5$, $c_2^{(1)} = 2$, and $c_{(1)}^3 = 1$. It follows that $G_1 = G_2 = G_3 = 1$ and $J_1(1) = \{1\}$ and $J_2(1) = J_3(1) = \{2,3\}$.

The objective function is: $\max 5(x_{11}^{(1)} + x_{12}^{(1)}) + 2(x_{21}^{(1)} + x_{22}^{(1)}) + (x_{31}^{(1)} + x_{32}^{(1)})$ *(the values corresponding to $k = 3$ are skipped since $\lambda_3 = 0$).*
Constraint (12) is $y_1 + y_2 + y_3 = 1$.
The coverage constraints (13) are:

$$x_{1k}^{(1)} \leq y_1; \ k = 1, 2, 3$$

$$x_{ik}^{(1)} \leq y_2 + y_3 \text{ for } i = 2, 3 \ \text{ and } k = 1, 2, 3$$

Constraints (14) are:
$$x_{1k}^{(1)} + x_{2k}^{(1)} + x_{3k}^{(1)} \leq 1, \ \text{ for } k = 1, 2, 3$$

Similarly, constraints (15) are:

$$x_{i1}^{(1)} + x_{i2}^{(1)} + x_{i3}^{(1)} \leq 1, \ \text{ for } i = 1, 2, 3$$

Finally, the sorting constraint (16) are:

$$5x_{11}^{(1)} + 2x_{21}^{(1)} + x_{31}^{(1)} \leq 5x_{12}^{(1)} + 2x_{22}^{(1)} + x_{32}^{(1)}$$

$$5x_{12}^{(1)} + 2x_{22}^{(1)} + x_{32}^{(1)} \leq 5x_{13}^{(1)} + 2x_{23}^{(1)} + x_{33}^{(1)}.$$

Constraints (17) are not required since the modeling vector is non-negative. The solution can be obtained by inspection.

First suppose $y_1 = 1, y_2 = y_3 = 0$. Constraints (13) imply that $x_{ik}^{(1)} = 0$ for $i = 2, 3$ and $k = 1, 2, 3$. Constraints (16), together with (14, 15) imply that $x_{13}^{(1)} = 1, x_{12}^{(1)} = 0$, and $x_{11}^{(1)} = 0$, leading to the objective function value of 0. Note that no node-coverage level combination is assigned to positions 1 and 2 in this case, signifying that the corresponding coverage levels equal to 0.

On the other hand, if $y_2 = 1, y_1 = y_3 = 0$ then $x_{1k}^{(1)} = 0$ for $k = 1, 2, 3$. Checking the last constraint we see that the only feasible solution is $x_{23}^{(1)} = 1$, $x_{32}^{(1)} = 1$, with all other components of $x_{ik}^{(1)}$ equal to 0. This solution corresponds to the objective function value of 1.

The case of $y_3 = 1, y_1 = y_2 = 0$ leads to the same solution, which must be optimal. Thus, the optimal solution is to locate a facility either at nodes 2 or 3, obtaining the objective function value of 1.

4.1 Computational Results

To test the solvability of the improved IP formulation we conducted a series of computational experiments. Random networks were generated with $n = 5, 10, 15$ and 20, nodes. A step-function coverage decay function was used with 5 coverage levels; the corresponding coverage radii were set to 20%, 40%, 60%, 80%, and 100% of the average shortest distance between any two nodes on the network. The number of facilities p to be located was set to $p = 1, 3, 5$. Five types of modeling vectors λ were used, as described below:

Type 1: $\lambda = (1, \ldots, 1, 0, \ldots, 0)$

Type 2: $\lambda = (0, \ldots, 0, 1, \ldots, 1)$

Type 3: Components of λ are ascending random numbers in $[0, 1]$

Type 4: $\lambda = (1, \ldots, 1, -1, \ldots, -1)$

Type 5; Components of λ were randomly generated in $[0, 1]$.

In total, 300 problem instances were generated. The instances were solved using CPLEX solver on a Pentium 4 desktop with 3.2 Ghz CPU and 1MB of RAM. The time limit for each instance was set to 2 hrs of CPU time. The same formulation was used for all instances - i.e., constraints (16) and (17) were not dropped even when the structure of the λ vector indicated that these constraints were not necessary (this was done to facilitate comparisons across different problem instances). The overall results are presented on Tables 1 and 2 below. Table 1 summarizes the results for all λ types, while Table 2 contains results for Type 2 λ vectors only.

Overall, the OGCLP appears to be quite difficult to solve except for small instances. As expected, the number of variables and constraints grows rapidly with n - the problem with 20 nodes results in an IP with nearly 2000 variables and constraints. The majority of

N	p	Time (sec)	Opt. Gap	% Optimal	# Variables	# Constraints
5	1	0.05	57%	100%	70	93
	3	0.04	48%	100%	70	93
	5	0.01	46%	100%	70	93
5 Total		0.03	51%	100%	70	93
10	1	0.24	71%	100%	360	415
	3	5.03	43%	100%	360	415
	5	4.87	38%	100%	360	415
10 Total		3.38	50%	100%	360	415
15	1	41.01	46%	100%	915	1005
	3	2,444.98	25%	76%	915	1005
	5	3,173.87	11%	64%	915	1005
15 Total		1,886.62	30%	80%	915	1005
20	1	2,537.44	62%	76%	1800	1929
	3	5,482.61	16%	28%	1800	1929
	5	5,201.48	8%	32%	1800	1929
20 Total		4,407.18	40%	45%	1800	1929
Grand Total		1,574.30	44%	81%		

Table 1: Computational results for the Improved IP for OGCLP for all modeling vector types.

20-node problems with more than 1 facility could not be solved within 2 hours of CPU time. Moreover, the optimality gap (relative difference between IP solution and the LP relaxation) is large in most cases, indicating that the formulation is not tight.

The IP difficulty appears to be largely due to the sorting and largest-level constraints (16-17). Recall that when the components of λ are non-negative and non-decreasing, these constraints can be removed from the formulation; Type 2 λ vectors satisfy this requirement. The resulting formulation is then quite close to the GCLP formulation in [2], which is known to be very integer-friendly - with LP relaxation often having an integer solution or IP achieving integrality after just a few iterations of the solver. Table 2 shows that these properties appear to translate to the $OGCLP$ case as well. Even though constraints (16-17) were retained in the formulation, they are redundant (for the IP). It can be seen that optimality gaps are much smaller than for the general case, all instances with less than 20 nodes were solved in a few seconds, and the solution failures for 20-node instances with $p = 3$ and 5 were mostly due to memory issues (i.e., problem size) rather than to the tightness of the formulation. Similar improvements were observed for Type 3 λ vectors and, to a lesser extent for Type 4 vectors - in the latter case, only constraints (16) can be removed. On the other hand, the worst results were observed for decreasing λ vectors of Type 1 - the lack of tightness of the formulation appears to be particularly severe in this case.

In summary, our results indicate that IP-based approach to general OGCLP instances can only deal with relatively small problem instances; as noted in [13], the sorting constraints appear to cause particular problems for integer programming approaches in OMP-type models. The situation is better when the components of the modeling vector are non-decreasing

N	p	Time (sec)	Opt. Gap	% Optimal	# Variables	# Constraints
5	1	0.01	29%	100%	70	93
	3	0.02	11%	100%	70	93
	5	0.01	16%	100%	70	93
5 Total		0.01	19%	100%	70	93
10	1	0.03	43%	100%	360	415
	3	0.97	16%	100%	360	415
	5	0.39	15%	100%	360	415
10 Total		0.46	25%	100%	360	415
15	1	1.38	9%	100%	915	1005
	3	6.26	5%	100%	915	1005
	5	2.72	4%	100%	915	1005
15 Total		3.45	6%	100%	915	1005
20	1	6.42	33%	100%	1800	1929
	3	4,423.51	14%	40%	1800	1929
	5	3,757.57	9%	60%	1800	1929
20 Total		2,729.16	22%	67%	1800	1929
Grand Total		683.27	17%	92%	786.25	860.5

Table 2: Computational results for the Improved IP for OGCLP for Type 2 types only.

and non-negative, however the large dimensionality of the formulation is a concern in that case as well. On one hand, this lack of solvability is hardly surprising - after all, as shown earlier, OGCLP is a very general model including most standard location models as special cases; no easily-exploitable special structure is present for this case. On the other hand, further work on exact and approximate solution techniques for OGCLP is clearly in order, as the potential payoff (having a general method for most types of location problems) is large.

5 Summary

In this paper we formulated a new network location model - Ordered Gradual Cover Location Problem - which generalizes location problems with median, center and cover objectives, as well as their extensions, such as Ordered Median and Gradual Cover. Finite Dominating Set results were obtained for many types of modeling vectors for the 1-facility case; more restrictive results were derived for the multi-facility case. We also investigated exact solutions of discrete version of OGCLP via Integer Programming.

Clearly, much work remains to be done, particularly in the area of efficient solutions techniques. Sorting constraints needed to capture the effects of modeling vectors lead to formulations that are large and not very tight. Constraint programming approaches (which may have an easier time of representing such constraints) may be particularly effective here. Heuristic algorithms should be investigated in future work as well - after all, such algorithms are quite effective for gradual cover problems. Of course, having an effective algorithm (either

exact or approximate) for such a general problem would be extremely useful.

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