Fraunhofer Institut Techno- und Wirtschaftsmathematik

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Berichte des Fraunhofer ITWM, Nr. 97 (2006)

ISSN 1434-9973

Bericht 97 (2006)

Die Veröffentlichungen in der Berichtsreihe des Fraunhofer ITWM können bezogen werden über:

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Vorwort

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Darüber hinaus bietet die Reihe ein Forum für die Berichterstattung über die zahlreichen Kooperationsprojekte des Instituts mit Partnern aus Industrie und Wirtschaft.

Berichterstattung heißt hier Dokumentation des Transfers aktueller Ergebnisse aus mathematischer Forschungs- und Entwicklungsarbeit in industrielle Anwendungen und Softwareprodukte – und umgekehrt, denn Probleme der Praxis generieren neue interessante mathematische Fragestellungen.

Prof. Dr. Dieter Prätzel-Wolters
Institutsleiter

Kaiserslautern, im Juni 2001

Fraunhofer Institut
Techno- und
Wirtschaftsmathematik

Interval Methods for Analog Circuits

Dr. Alexander Dreyer

Abstract

Reliable methods for the analysis of tolerance-affected analog circuits are of great importance in nowadays microelectronics. It is impossible to produce circuits with exactly those parameter specifications proposed in the design process. Such component tolerances will always lead to small variations of a circuit's properties, which may result in unexpected behaviour. If lower and upper bounds to parameter variations can be read off the manufacturing process, interval arithmetic naturally enters the circuit analysis area.

This paper focuses on the frequency-response analysis of linear analog circuits, typically consisting of current and voltage sources as well as resistors, capacitances, inductances, and several variants of controlled sources. These kind of circuits are still widely used in analog circuit design as equivalent circuit diagrams for representing in certain application tasks

Interval methods have been applied to analog circuits before. But yet this was restricted to circuit equations only, with no interdependencies between the matrix elements. But there also exist formulations of analog circuit equations containing dependent terms. Hence, for an efficient application of interval methods, it is crucial to regard possible dependencies in circuit equations. Part and parcel of this strategy is the handling of fill-in patterns for those parameters related to uncertain components. These patterns are used in linear circuit analysis for efficient equation setup. Such systems can efficiently be solved by successive application of the *Sherman-Morrison formula*.

The approach can also be extended to complex-valued systems from frequency domain analysis of more general linear circuits. Complex values result here from a Laplace transform of frequency-dependent components like capacitances and inductances. In order to apply interval techniques, a real representation of the linear system of equations can be used for separate treatment of real and imaginary part of the variables. In this representation each parameter corresponds to the superposition of two fill-in patterns. Crude bounds – obtained by treating both patterns independently – can be improved by consideration of the correlations to tighter enclosures of the solution.

The techniques described above have been implemented as an extension to the toolbox *Analog Insydes*, an add-on package to the computer algebra system *Mathematica* for modeling, analysis, and design of analog circuits.

Keywords: interval arithmetic, analog circuits, tolerance analysis, parametric linear systems, frequency response, symbolic analysis, CAD, computer algebra

1 Motivation

Numerical simulations of analog circuits can be used to analyze a circuit's behaviour without the need for a physical implementation. But actual circuit properties may differ from the results obtained by floating-point simulations, due to errors caused by rounding, component tolerances, and simplified models. Simulations based on interval arithmetic can be used as a unified framework to bound all these errors, but tend to be too conservative. In this paper a new approach for computing tight bounds to the frequency response of tolerance-affected analog circuits is described.

The behaviour of analog circuits can be described by a system of parameter-dependent linear or nonlinear equations. A symbolic setup of the equation system allows for assigning unique symbols to each circuit component parameter. The resulting circuit equations however often cannot be analyzed in a pure symbolic way: In the nonlinear case the system might not be solvable in a symbolic manner, but yet in the linear case the result may be of large complexity. In order to analyze the behaviour of such an analog circuit using a simulator or a numerical solver, the symbols representing netlist elements have to be replaced by the corresponding numerical values, according to a given design point.

The numerical approach has two major drawbacks: First of all, for an efficient numerical treatment of the equation system, all numerical values have to be converted into floating-point numbers. This may lead to growing of the overall error, due to rounding of the numerical values in each solving step. Another problem is caused by the fact that a design point may be defined in advance, but one cannot ensure a priori that the desired properties will exactly be met during manufacturing of the actual circuit. Component tolerances will always lead to small variations of a circuit's properties, which may result in effects not expected from the results of the numerical simulation. While rounding errors could be reduced or even completely avoided by a sophisticated treatment of the equation system, the latter problem cannot be overcome within a single numerical simulation. Using a statistical method, like the Monte Carlo approach, the parameter variations of the production process may be simulated [1]. But this results in a large number of simulations and does not yield guaranteed solutions. Simulation based on *interval arithmetic* can be used as a unified framework for both problems.

This paper starts with a brief introduction to the principles of interval computations and the relevance of interdependences to the accuracy of interval-valued results. Then mathematical methods especially tuned to handle systems arising from frequency-response analysis of linear analog circuits are described. This includes an approach, which is capable of computing tight bounds to variations due to a large number of tolerance-affected components. Finally, the results are illustrated on several an example applications.

2 Interval Arithmetic

2.1 Basics

If upper and lower bounds for the uncertain parameters can be determined, these can be interpreted as the endpoints $\underline{x}$, $\overline{x}$ of a closed interval $[\underline{x}, \overline{x}] \subseteq \mathbb{R}$. This interval is usually denoted by $[x]$. A vector of intervals – or *box* – is consequently written as $[\mathbf{x}]$. The principles of interval arithmetic are quite simple (e.g. [2, 3]): during evaluation any expression is constructed by subsequent calls of elementary binary operations $(+, -, *, /)$ and basic functions like *sin*, *cos*, *log*, e^x and x^n, where the *intervalization* of binary operators is

$$\begin{aligned} &[\underline{x}, \overline{x}] \diamond [\underline{y}, \overline{y}] = [\underline{z}, \overline{z}], \text{ for } \diamond \in \{+, -, *, /\}, \\ &\text{with } \underline{z} = \min\{\underline{x} \diamond \underline{y}, \underline{x} \diamond \overline{y}, \overline{x} \diamond \underline{y}, \overline{x} \diamond \overline{y}\}, \\ &\text{and } \overline{z} = \max\{\underline{x} \diamond \underline{y}, \underline{x} \diamond \overline{y}, \overline{x} \diamond \underline{y}, \overline{x} \diamond \overline{y}\}. \end{aligned} \tag{1}$$

Functions like $e^{[\underline{x}, \overline{x}]}$ and $[\underline{x}, \overline{x}]^n$ can be defined in an analogous manner. The intervalization of any monotonic or piecewise monotonic elementary function is computed by evaluating the function on a finite set of *special points*, consisting of the interval's endpoints and local extrema.

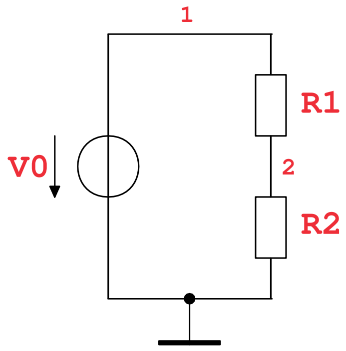
For bounding the range of a more complex expression we have to assign a corresponding *interval extension* to it. An interval-valued function $[f]$ is called an interval extension of the real-valued function f, if

$$[f]([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]) \supseteq \{f(y_1, \ldots, y_n) \,|\, y_i \in [\underline{x}_i, \overline{x}_i]\} \,. \tag{2}$$

The interval extension obtained by replacing real operations and elementary functions by their interval-valued equivalents is called *natural interval extension* [2].

2.2 The Dependence Problem and Symbolic Solutions

The major drawback in using interval computations is caused by the *dependence problem*. While computations using independent parameters will return tight bounds to the exact range of the function, two or more occurrences of the same parameter during the evaluation phase will result in too conservative estimations. For instance, consider the voltage divider circuit in Figure 1. Its behaviour can be



Voltage divider circuit.

Figure 1

described by the matrix equation

$$\begin{pmatrix} \frac{1}{R_1} & -\frac{1}{R_1} & 1 \\ -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_2} & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} V_1 \\ V_2 \\ I_{V_0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ V_0 \end{pmatrix} . \tag{3}$$

In this case, one can explicitely find a symbolic solution i. e. for each variable we obtain an expression in terms of V_0, R_1, and R_2

$$V_1 = V_0\,, V_2 = \frac{R_2 \cdot V_0}{R_1 + R_2}\,, \text{and } I_{V_0} = -\frac{V_0}{R_1 + R_2}\,.$$

Assuming $V_0 = 1\,\mathrm{V}$, as well as uncertain $R_1/1\,\Omega \in [9, 11]$, and $R_2/1\,\Omega \in [90, 110]$, interval arithmetic can be used to compute rough bounds to the range with respect to these settings,

$$V_1 = 1\,\mathrm{V}\,, V_2/1\,\mathrm{V} \in [0.743, 1.112]\,, \text{and } I_{V_0}/1\,\mathrm{mA} \in [-10.1, -8.27]\,.$$

But here the voltage V_2 can be bounded more accurately. Using the equivalent reformulation $V_2 = \frac{V_0}{1+R_1/R_2}$ to compute the interval results, the essentially tighter range $V_2/1\,\mathrm{V} \in [0.891, 0.925]$ is obtained. Since each uncertain parameter occurs only once, this is the best possible result. Therefore, simply replacing numerical solvers by an interval version would soon lead to rather useless results, consisting of large parts of the original search region, which had been obtained before. Hence, no further information would be generated in this case.

Several algorithms have been developed to solve linear and nonlinear interval-valued equation systems [2, 3]. These behave well if we are dealing with intervals of small width and little dependence of the interval-valued terms of the expressions involved, but real-life applications will require the treatment of wider intervals and parameters of multiple occurrences. For our purpose interval algorithms have to be tuned for solving equation systems resulting from industrial analog circuits.

3 Fill-in Patterns of Linear Circuits

In the case of linear analog circuits with uncertain parameters Kirchhoff's laws and element relations are summarized in a matrix equation of the following form:

$$\mathbf{A}(\mathbf{p}) \cdot \mathbf{x} = \mathbf{b}\,, \tag{4}$$

where $\mathbf{x}$ denotes the vector of internal currents and voltages, and the parameter vector $\mathbf{p} = (p_1, \ldots, p_{n_p})$ corresponds to tolerance-affected components, which are bounded by intervals $[p_i]$. Solving such an interval equation system means determining close bounds to the smallest box $[\mathbf{x}]$ with

$$[\mathbf{x}] \supseteq \left\{ (\mathbf{A}(\mathbf{p}))^{-1} \cdot \mathbf{b} \,\middle|\, p_i \in [p_i] \right\} . \tag{5}$$

Note that uncertain values of independent current and voltage sources can also be modeled as interval-valued parameters on the right-hand side $\mathbf{b}$. These kind of parameters can be moved to the matrix by the cost of introducing new rows and columns.

In order to apply interval arithmetic in linear circuit analysis, it is necessary to use a real formulation of the matrix equation. Hence, a complex-valued equation system used for AC analysis needs to be reformulated [4]. The result corresponding to each variable will be wrapped by a polygon in the complex plane (*wrapping effect*).

Earlier efforts to solve interval-valued linear circuit equations were restricted to formulations, in which each matrix element varies independently [4]. Therefore a new approach was developed to cope with multiple occurrences of parameters: First of all we will assume that the parameter dependence of $\mathbf{A}(\mathbf{p})$ can be written as a sequence of rank-one updates of a parameter-independent real-valued matrix $\mathbf{A}_0 \in \mathbb{R}^{n \times n}$:

$$\mathbf{A}(\mathbf{p}) = \mathbf{A}_0 + \sum_{i=1}^{n_\mathbf{p}} p_i \cdot (\mathbf{u}_i \cdot \mathbf{v}_i^\mathsf{T})\,, \tag{6}$$

with $\mathbf{u}_i, \mathbf{v}_i \in \mathbb{R}^n$, and $\mathbf{A}(\mathbf{p})$ is invertible for all $\mathbf{p} \in [\mathbf{p}]$. This is not a restriction at all, because this structure is already inherent to a linear circuit: Using the *sparse tableau formulation* (*STA*) [5] to generate the linear circuit equations, each p_i will occur only once and $\mathbf{u}_i, \mathbf{v}_i$ are just unit vectors, which define the corresponding matrix element. In the case of *modified nodal analysis* (*MNA*) [5] the matrices of the form $p_i \cdot (\mathbf{u}_i \cdot \mathbf{v}_i^\intercal)$ correspond to the well-known fill-in patterns used during equation setup.

4 Mathematical Methods

In this section we restrict ourselves to the treatment of linear circuit elements. Earlier efforts to solve interval-valued linear circuit equations were restricted to formulations, in which each matrix element varies independently [4]. Therefore a new approach was developed to cope with multiple occurrences of parameters: we have already seen in Section 3 that a resistive circuit may be represented by the parametric matrix equation $\mathbf{A}(\mathbf{p}) \cdot \mathbf{x} = \mathbf{b}$, whose parameter dependence can be written in the fill-in pattern form of Equation 5. It has already been pointed out independently by Dreyer [6] and Ganesan et al. [7], that the rank-one updates forming these systems emit useful monotonicity properties.

In case that capacitances and inductanctes are involved, the structure is likewise, but the corresponding p_i may vary on the imaginary axis instead. In any case, it can be utilized for efficient solving of interval-valued circuit equations [8].

4.1 Methods for Resistive Circuits

The *Sherman-Morrison formula* [9] allows for inverting a perturbed matrix for a change to a given matrix without inverting the whole matrix again.

Theorem 4.1 (Sherman-Morrison) *Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be invertible and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Then the matrix $\mathbf{A} + \mathbf{u} \cdot \mathbf{v}^\intercal$ is invertible if and only if $1 + \mathbf{v}^\intercal \mathbf{A}^{-1}\mathbf{u} \neq 0$. In this case we have:*

$$(\mathbf{A} + \mathbf{u} \cdot \mathbf{v}^\intercal)^{-1} = \mathbf{A}^{-1} - \frac{1}{1 + \mathbf{v}^\intercal \mathbf{A}^{-1}\mathbf{u}} \mathbf{A}^{-1}\mathbf{u}\mathbf{v}^\intercal \mathbf{A}^{-1} . \tag{7}$$

The relationship between determinants of original and changed matrix is exploited in the following.

Corollary 4.2 *Let* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *and* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$*. Then*

$$\det(\mathbf{A} + \mathbf{u} \cdot \mathbf{v}^\intercal) = \det(\mathbf{A}) \cdot \det(1 + \mathbf{v}^\intercal \mathbf{A}^{-1}\mathbf{u})\,.$$

There is also a slightly more general version of the theorem.

Theorem 4.3 (Sherman-Morrison-Woodbury) *Let* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *be an invertible matrix, and let* $\mathbf{U}, \mathbf{V} \in \mathbb{R}^{m\times n}$ *with* $m \leq n$*. Then* $\mathbf{A} + \mathbf{U} \cdot \mathbf{V}^\intercal$ *is invertible if and only if* $1 + \mathbf{V}^\intercal \mathbf{A}^{-1}\mathbf{U}$ *is invertible. In this case we have:*

$$(\mathbf{A} + \mathbf{U} \cdot \mathbf{V}^\intercal)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}\left(1 + \mathbf{V}^\intercal \mathbf{A}^{-1}\mathbf{U}\right)^{-1} \mathbf{V}^\intercal \mathbf{A}^{-1}\,.$$

For a detailed proof of the theorems see [9] or [11]. The latter also incorporates a proof of the corollary.

Equation 7 has already been used in the field of analog circuit analysis for calculating the influence of a single matrix entry to the solution [12, 13]. Linear equations in fill-in pattern form perfectly fit into the conditions of the Sherman-Morrison formula.

In order to show some properties of the fill-in pattern form Theorem 4.1 can be applied.

Theorem 4.4 (Convex-hull theorem for fill-in patterns)
Let $[\mathbf{p}] \in [\mathbb{R}]^{n_\mathbf{p}}$*, and let* $\mathbf{A}(\mathbf{p}) = \mathbf{A}_0 + \sum_{i=1}^{n_\mathbf{p}} p_i \cdot (\mathbf{u}_i \cdot \mathbf{v}_i^\intercal)$ *with* $\mathbf{u}_i, \mathbf{v}_i \in \mathbb{R}^n$ *be invertible for all* $\mathbf{p} \in [\mathbf{p}]$*.*

Then $\mathbf{A}(\mathbf{p})^{-1} \cdot \mathbf{b}$ *is (componentwise) monotonic and continuous in* p_i *for all* $i = 1, \ldots, n_\mathbf{p}$*. Furthermore, the following holds*

$$\operatorname{conv}\left\{\mathbf{A}(\mathbf{p})^{-1} \cdot \mathbf{b} \,\middle|\, \mathbf{p} \in [\mathbf{p}]\right\} = \operatorname{conv}\left\{\mathbf{A}(\mathbf{p})^{-1} \cdot \mathbf{b} \,\middle|\, \mathbf{p} \in \operatorname{corners}([\mathbf{p}])\right\}\,.$$

Proof: Since all parameters $p_1, \ldots, p_{n_\mathbf{p}}$ may vary independently, it sufficies to show the theorem for $n_\mathbf{p} = 1$. We may also assume, that $\mathbf{A}_0$ is invertible, because $\mathbf{A}_0$ can be replaced by $\mathbf{A}(\mathbf{p}_0)$ if exchanging $\mathbf{p}$ by $\mathbf{p} - \mathbf{p}_0$ for a fixed parameter setting $\mathbf{p}_0 \in [\mathbf{p}]$.

Then application of *Sherman-Morrison* leads to

$$(\mathbf{A}_0 + p \cdot (\mathbf{u}\,\mathbf{v}^\intercal))^{-1}\,\mathbf{b} = \mathbf{A}_0^{-1}\,\mathbf{b} - \frac{p}{1 + p \cdot \mathbf{v}^\intercal \mathbf{A}_0^{-1}\mathbf{u}} \mathbf{A}_0^{-1}\mathbf{u}\mathbf{v}^\intercal \mathbf{A}_0^{-1}\,\mathbf{b}\,,$$

whose continuity follows directly from the Theorem 4.1. The monotonic property with respect to p can easily be derived using elementary calculations on the right-hand side.

Finally, the last part of the theorem is a consequence of Farkas's Lemma [14]. Following, we will denote $\mathbf{x_p} := \mathbf{A}(\mathbf{p})^{-1} \cdot \mathbf{b}$ for all $\mathbf{p} \in \mathrm{corners}\,([\mathbf{p}])$. We shall prove that the system of linear equations

$$\sum_{\mathbf{p} \in \mathrm{corners}([\mathbf{p}])} \lambda_{\mathbf{p}}\, \mathbf{x_p} \quad = \quad \mathbf{x}_0 \tag{8}$$

$$\sum_{\mathbf{p} \in \mathrm{corners}([\mathbf{p}])} \lambda_{\mathbf{p}} \quad = \quad 1 \tag{9}$$

has a non-negative solution $(\lambda_{\mathbf{p}})_{\mathbf{p} \in \mathrm{corners}([\mathbf{p}])}$. By Farkas's Lemma it is sufficient to show that for each $\mathbf{q} \in \mathbb{R}^n$, $q_0 \in \mathbb{R}$, if $\mathbf{q}^\mathsf{T} \cdot \mathbf{x_p} + q_0 \geq 0,$ for all $\mathbf{p} \in \mathrm{corners}\,([\mathbf{p}])$. Then we have

$$\mathbf{q}^\mathsf{T} \cdot \mathbf{x}_0 + q_0 \geq 0\,.$$

Assume now, that $\mathbf{q}$ and q_0 are chosen such that

$$\mathbf{q}^\mathsf{T} \cdot \mathbf{x_p} + q_0 \geq 0, \text{ for all } \mathbf{p} \in \mathrm{corners}\,([\mathbf{p}])\ .$$

Since the monotonicity argument also holds for $f(\mathbf{p}) := \mathbf{q}^\mathsf{T} \cdot \mathbf{x_p}$, there exists a vector $\mathbf{p} \in \mathrm{corners}\,([\mathbf{p}])$ with $\mathbf{q}^\mathsf{T} \cdot \mathbf{x_p} \leq \mathbf{q}^\mathsf{T} \cdot \mathbf{x}_0$, and hence,

$$\mathbf{q}^\mathsf{T} \cdot \mathbf{x}_0 + q_0 \geq \mathbf{q}^\mathsf{T} \cdot \mathbf{x_p} + q_0 \geq 0$$

yields the desired relation. □

If regularity of $\mathbf{A}(\mathbf{p})$ can be established for all $\mathbf{p} \in [\mathbf{p}]$, Theorem 4.4 can be used to compute sharp bounds to the solution set of an uncertain linear system in fill-in pattern form. The following lemma yields an equivalent condition to the regularity of $\mathbf{A}(\mathbf{p})$, which can be computed efficiently.

Lemma 4.5 (Regularity test) *Let*

$$\mathbf{A}(\mathbf{p}) = \mathbf{A}_0 + \sum_{i=1}^{n_{\mathbf{p}}} p_i \cdot (\mathbf{u}_i \cdot \mathbf{v}_i^\mathsf{T})$$

with $\mathbf{u}_i, \mathbf{v}_i \in \mathbb{R}^n$*. Then* $\mathbf{A}(\mathbf{p})$ *is invertible for all* $\mathbf{p} \in [\mathbf{p}]$ *if and only if the value* $\mathrm{sign}\det(\mathbf{A}(\mathbf{p}))$ *is non-zero and constant for all* $\mathbf{p} \in \mathrm{corners}\,([\mathbf{p}])$ *.*

Proof: One only has to show, that a sign change in $\det(\mathbf{A}(\mathbf{p}))$ immediately yields a parameter setting $\mathbf{p}_0 \in [\mathbf{p}]$, such that $\mathbf{A}(\mathbf{p}_0)$ is singular. Therefore, suppose there were some corner points $\mathbf{p}_\nu$ and $\mathbf{p}_\mu$ such that the inequality $\det \mathbf{A}(\mathbf{p}_\nu) < 0 < \det \mathbf{A}(\mathbf{p}_\mu)$ holds. Without loss of generality both points only differ in one parameter component p_i. By Corollary 4.2 we know that $\det \mathbf{A}(\mathbf{p})$ is of the form

$$\det \mathbf{A}(\mathbf{p}) = f(p_i) := a + p_i \cdot b\,, \tag{10}$$

for real values a and b not depending on p_i. But a sign change of this function means that there exists a p_i^* in range with $f(p_i^*) = 0$, which is a contradiction to the fact, that the determinant must not be zero. $\square$

The regularity test can now be used in combination with Theorem 4.4 for computing sharp bounds to the range of $\mathbf{A}(\mathbf{p})^{-1} \cdot \mathbf{b}$ over $[\mathbf{p}]$, for $\mathbf{A}(\mathbf{p})$ in fill-in pattern form. This results in $2^{n_\mathbf{p}}$ linear systems to be processed. An interval hull can easily be obtained from a convex set by componentwise calculating minimum and maximum values. The whole procedure is exposed in Algorithm 1.

Algorithm 1 Real-valued linear system solver

Input: $\mathbf{A}(\mathbf{p}) \in \mathbb{R}^{n\times n}$ in the form of Theorem 4.4, $\mathbf{b} \in \mathbb{R}^n$, and $[\mathbf{p}] \in [\mathbb{R}]^{n_\mathbf{p}}$
Output: $S = \{\mathbf{x}_1, \ldots, \mathbf{x}_{2^{n_\mathbf{p}}}\}$ such that $\mathbf{A}(\mathbf{p})^{-1}\,\mathbf{b} \in \operatorname{conv}(S)$, for all $\mathbf{p} \in [\mathbf{p}]$

```
Set s := sign(det A(p_0)) for some p_0 ∈ [p].
if s = 0 then
  return failed
end if
Set S := ∅.
for p ∈ corners([p]) do
  /* Regularity test */
  if det A(p) ≠ s then
    return failed
  else
    S := S ∪ {A(p)^{-1} · b}
  end if
end for
```

The advantage of Algorithm 1 is, that it does not need interval computations. Of course, interval arithmetic can be used to bound rounding errors. In the case that these do not have to be tracked, already existing numerical solvers, like those of analog circuit simulators [15, 16], can be utilized for tolerance analysis.

Example As example, we reconsider the voltage divider circuit of Figure 1, whose circuit equations can be written as the following matrix equations

$$\begin{pmatrix} \frac{1}{R_1} & -\frac{1}{R_1} & 1 \\ -\frac{1}{R_1} & \frac{1}{R_1} + \frac{1}{R_2} & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} V_1 \\ V_2 \\ I_{V_0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ V_0 \end{pmatrix}, \tag{11}$$

for fixed $V_0 = 1\,\mathrm{V}$ and uncertain $R_1/1\,\Omega \in [9, 11]$, and $R_2/1\,\Omega \in [90, 110]$.

Since there is a one-to-one correspondence between the endpoints of R_i and $1/R_i$, one can immediately apply Algorithm 1 to the system.

Hence, it is enough to solve these equations for the four corner points

$$\text{corners}\begin{pmatrix} [9,11] \\ [90,110] \end{pmatrix} = \left\{ \begin{pmatrix} 9 \\ 90 \end{pmatrix}, \begin{pmatrix} 9 \\ 110 \end{pmatrix}, \begin{pmatrix} 11 \\ 90 \end{pmatrix}, \begin{pmatrix} 11 \\ 110 \end{pmatrix} \right\},$$

which results approximately in the convex solution set

$$\begin{pmatrix} V_1 \\ V_2 \\ I_{V_0} \end{pmatrix} \in \text{conv}\left\{ \begin{pmatrix} 1 \\ 0.909 \\ -0.0101 \end{pmatrix}, \begin{pmatrix} 1 \\ 0.925 \\ -0.0084 \end{pmatrix}, \begin{pmatrix} 1 \\ 0.891 \\ -0.0099 \end{pmatrix}, \begin{pmatrix} 1 \\ 0.909 \\ -0.00827 \end{pmatrix} \right\},$$

i. e. for the given tolerances we have $V_1 = 1\,\mathrm{V}$, and the remaining voltage and current are bounded as $V_2/1\,\mathrm{V} \in [0.891, 0.925]$, and $I_{V_0}/1\,\mathrm{mA} \in [-10.1, -8.27]$.

4.2 A faster approach

The approach described above is suitable for small parameter numbers $n_{\mathbf{p}}$ only, because the interval-valued problem is put down to the solution of $2^{n_{\mathbf{p}}}$ real-valued linear systems. In order to treat a large number of parameters, we use a kind of intervalization of the Sherman-Morrison formula to obtain a less accurate, but faster algorithm. As seen in the case of a single uncertain parameter $p \in [\underline{p}, \overline{p}]$, the solution is given as

$$\mathbf{A}(p)^{-1}\mathbf{b} = \mathbf{A}(p_0)^{-1}\,\mathbf{b} - \frac{p - p_0}{d(p)}\mathbf{A}(p_0)^{-1}\mathbf{u}\,\mathbf{v}^\top\mathbf{A}(p_0)^{-1}\,\mathbf{b}\,, \tag{12}$$

if we set $d(p) = 1 + (p - p_0)\cdot\mathbf{v}^\top\mathbf{A}(p_0)^{-1}\mathbf{u}$ for a fixed value $p_0 \in [\underline{p}, \overline{p}]$ under the conditions of the Sherman-Morrison theorem. But the latter can also be applied, in case one is able to show that $d(p) \neq 0$ for all $p \in [\underline{p}, \overline{p}]$. This can efficiently be calculated by evaluating

$$d([\underline{p}, \overline{p}]) = 1 + ([\underline{p}, \overline{p}] - p_0)\cdot(\mathbf{v}^\top\mathbf{A}(p_0)^{-1}\mathbf{u}) \tag{13}$$

using interval arithmetic. Hence, it can be verified, whether $d([\underline{p}, \overline{p}]) \not\ni 0$. If that is the case, the range of the factor $[\underline{m}, \overline{m}] := (p - p_0)\,/d(p)$ over all $p \in [\underline{p}, \overline{p}]$ is exactly bounded by

$$\underline{m} = ((\underline{p} - p_0)^{-1} + \mathbf{v}^\top\mathbf{A}_0^{-1}\mathbf{u})^{-1} \quad \text{and} \quad \overline{m} = ((\overline{p} - p_0)^{-1} + \mathbf{v}^\top\mathbf{A}_0^{-1}\mathbf{u})^{-1}. \tag{14}$$

At this point all essential parts for bounding Equation 13 are available. Indeed, this forms an interval-valued version of the Sherman-Morrison formula.

Theorem 4.6 (Interval-valued Sherman-Morrison) *Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be invertible and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Then the matrix $\mathbf{A} + p\cdot\mathbf{u}\,\mathbf{v}^\top$ is invertible for all $p \in [\underline{p}, \overline{p}]$ if and only if*

$$1 + ([p] - p_0)\cdot(\mathbf{v}^\top\mathbf{A}(p_0)^{-1}\mathbf{u}) \not\ni 0 \tag{15}$$

In this case, we can calculate for a right-hand side $\mathbf{b} \in \mathbb{R}^n$*:*

$$\left(\mathbf{A} + [\underline{p}, \overline{p}] \cdot \mathbf{u}\,\mathbf{v}^\intercal\right)^{-1} \mathbf{b} = \mathbf{A}^{-1}\mathbf{b} - [\underline{m}, \overline{m}] \cdot \left(\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^\intercal\mathbf{A}^{-1}\mathbf{b}\right) \tag{16}$$

with $[\underline{m}, \overline{m}] = [((\underline{p} - p_0)^{-1} + \mathbf{v}^\intercal\mathbf{A}_0^{-1}\mathbf{u})^{-1}, ((\overline{p} - p_0)^{-1} + \mathbf{v}^\intercal\mathbf{A}_0^{-1}\mathbf{u})^{-1}]$.

Proof: Due to the calculations above and the Sherman-Morrison formula, it only remains to motivate the computation of $\underline{m}$ and $\overline{m}$. The range of $[\underline{m}, \overline{m}]$ results from the valuable property of interval arithmetic, that certain kinds of divisions by zero are allowed. This includes $1/[0, b] = [1/b, \infty]$, and $1/[a, 0] = [-\infty, 1/a]$, which can be used to describe the range of

$$1/[a, b] = [-\infty, 1/a] \cup [1/b, \infty]\,, \text{ for } a < 0 < b\,. \tag{17}$$

Hence, for $p_0 \in [\overline{p}, \underline{p}]$ one can compute the range of $d(p) := 1/(p - p_0) + \mathbf{v}^\intercal\mathbf{A}_0^{-1}\mathbf{u}$ over $p \in [\underline{p}, \overline{p}]$ as

$$\begin{aligned} d([\overline{p}, \underline{p}]) &= 1/[\overline{p} - p_0, \underline{p} - p_0] + \mathbf{v}^\intercal\mathbf{A}_0^{-1}\mathbf{u} \\ &= [-\infty, (\underline{p} - p_0)^{-1} + \mathbf{v}^\intercal\mathbf{A}_0^{-1}\mathbf{u})] \cup [(\overline{p} - p_0)^{-1} + \mathbf{v}^\intercal\mathbf{A}_0^{-1}\mathbf{u}, \infty]\,. \end{aligned}$$

If Equation 15 holds, then both intervals obtained by these calculations do not include zero. Then inverting and reunifying the resulting intervals yields the regular interval $1/d([\overline{p}, \underline{p}]) = [\underline{m}, \overline{m}]$. □

For illustration, the voltage divider example of Section 4.1 is treated again, with fixed $R_2 = 100\,\Omega$ and $V_0 = 1\,\mathrm{V}$. Hence, we have

$$\begin{pmatrix} \frac{1}{R_1} & -\frac{1}{R_1} & 1 \\ -\frac{1}{R_1} & \frac{1}{R_1} + 0.01 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} V_1 \\ V_2 \\ I_{V_0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{18}$$

interpreting $\frac{1}{R_1}$ as parameter with $p \equiv \frac{1}{R_1} \in [\underline{p}, \overline{p}] = 1/[9, 11] = [0.0909, 0.1112]$, the fill-in pattern and the matrix $\mathbf{A}_0$ with respect to $p_0 = 0.1$ are given by

$$\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \text{ and } \mathbf{A}_0 = \begin{pmatrix} 0.1 & -0.1 & 1 \\ -0.1 & 0.11 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

One immediately computes

$$\mathbf{A}_0^{-1}\mathbf{b} = \begin{pmatrix} 1 \\ 0.909 \\ -0.009 \end{pmatrix} \text{ and } \mathbf{A}_0^{-1}\mathbf{u} = \begin{pmatrix} 1 \\ -9.0909 \\ 0.0909 \end{pmatrix}.$$

Because $\mathbf{v}^\mathsf{T}\mathbf{A}_0^{-1}\mathbf{u} = 9.0909$, $\frac{1}{\underline{p}-p_0} = -110$, and $\frac{1}{\overline{p}-p_0} = 90$, the system is invertible for all p. Hence, the factor

$$[x] = [1/(-110 + 9.0909), 1/(90 + 9.0909)] = [-0.0099, 0.0101]$$

may be calculated. Using Equation 16 we get tight bounds to the solution: $V_1 = 1\,\mathrm{V}$, as well as $V_2/1\,\mathrm{V} \in [0.900, 0.918]$ and $I_{V_0}/1\,\mathrm{mA} \in [-9.2, -9]$.

Reintroducing the second uncertain parameter $\frac{1}{R_2}$ from the example, one can clearly set $q \equiv \frac{1}{R_2} \in [q] = [0.009, 0.012]$, $q_0 = 0.01$ with fill-in patterns corresponding to the vectors

$$\mathbf{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{z} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

For tracking the variations of $\left(\mathbf{A}_0 + p \cdot \mathbf{u}\mathbf{v}^\mathsf{T} + q \cdot \mathbf{y}\mathbf{z}^\mathsf{T}\right)^{-1}\mathbf{b}$ due to both p and q the interval-valued Sherman-Morrison formula may be applied a second time. But for this purpose one cannot continue to use a real-valued matrix $\mathbf{A}_0$, instead it has to replaced by all possible matrices $\mathbf{A}(p) = \mathbf{A}_0 + p \cdot \mathbf{u}\mathbf{v}^\mathsf{T}$ in range. The necessary intermediate results $\mathbf{A}(p)^{-1}\mathbf{b}$ and $\mathbf{A}(p)^{-1}\mathbf{y}$ can be bounded for $p \in [p]$ by interval-vectors with the method described above. In particular,

$$\mathbf{A}([p])^{-1}\mathbf{y} = \begin{pmatrix} 0 \\ [8.181, 10] \\ [0.891, 0.9244] \end{pmatrix} \text{ and } \mathbf{A}([p])^{-1}\mathbf{b} = \begin{pmatrix} 1 \\ [0.900, 0.918] \\ [-0.0092, -0.009] \end{pmatrix},$$

whereas the latter was already computed above. Analogously, we obtain

$$-\mathbf{z}^\mathsf{T}\mathbf{A}([p])^{-1}\mathbf{y} = [-10, -8.181],$$

which indeed lies between $\frac{1}{\underline{q}-q_0} = -1100$, and $\frac{1}{\overline{q}-q_0} = 900$. Therefore, we can go on to calculate the factor $[x_2] = [-0.00092, 0.0012]$, and finally the value $\mathbf{z}^\mathsf{T}\mathbf{A}([p])^{-1}\mathbf{b} = [0.900, 0.918]$ is obtained.

The final solution can be computed by evaluating

$$\mathbf{A}([p])^{-1}\mathbf{b} - [x_2] \cdot \left(\mathbf{A}([p])^{-1}\mathbf{y}\right) \cdot \left(\mathbf{z}^\mathsf{T}\mathbf{A}([p])^{-1}\mathbf{b}\right) =$$
$$\begin{pmatrix} 1 \\ [0.900, 0.918] \\ [-0.0092, -0.009] \end{pmatrix} - [-0.0009, 0.0012] \cdot \begin{pmatrix} 0 \\ [8.181, 10] \\ [0.891, 0.9244] \end{pmatrix} \cdot [0.9, 0.918],$$

which leads to $V_1 = 1\,\mathrm{V}$, and

$$V_2/1\,\mathrm{V} \in [0.8908, 0.926], \text{ and } I_{V_0}/1\,\mathrm{A} \in [-0.0102, -0.00824].$$

This result contains slightly larger intervals than the one obtained in Section 4.1. The reason for this is, that the parameter interval $[p]$ occurs more than once during the evaluation procedure, because $\mathbf{A}([p])^{-1}\mathbf{b}$ as well as $\mathbf{A}([p])^{-1}\mathbf{y}$ depend on it.

The advantage of this approach is that it can easily be extented to $n_{\mathbf{p}}$ parameters $p_\nu \in [\underline{p}_\nu, \overline{p}_\nu]$ and corresponding fill-in patterns $\mathbf{u}_\nu\, \mathbf{v}_\nu^\mathsf{T}$ for $\nu = 1, \ldots, n_{\mathbf{p}}$. After fixing a vector $\mathbf{p}_0 = \big(p_{1,0}, \ldots, p_{n_{\mathbf{p}},0}\big)^\mathsf{T}$, with $p_{\nu,0} \in [\underline{p}_\nu, \overline{p}_\nu]$, the procedure is initialized by simultaneously computing

$$[\tilde{\mathbf{b}}] := \mathbf{A}(\mathbf{p}_0)^{-1} \cdot \mathbf{b} \text{ and } [\tilde{\mathbf{u}}_\nu] := \mathbf{A}(\mathbf{p}_0)^{-1} \cdot \mathbf{u}_\nu \text{ for all } \nu\,.$$

If in the i^{th} step the condition $(\underline{p}_i - p_{i,0})^{-1} < -\mathbf{v}_i^\mathsf{T} \cdot [\tilde{\mathbf{u}}_i] < (\overline{p}_i - p_{i,0})^{-1}$ holds, i interval vectors have to be updated in the following scheme

$$\begin{aligned}
[x] &\leftarrow [\Big((\underline{p}_i - p_{i,0})^{-1} + \max \mathbf{v}_i^\mathsf{T} \cdot [\tilde{\mathbf{u}}_i]\Big)^{-1}, \Big((\overline{p}_i - p_{i,0})^{-1} + \min \mathbf{v}_i^\mathsf{T} \cdot [\tilde{\mathbf{u}}_i]\Big)^{-1}] \\
[\tilde{\mathbf{b}}] &\leftarrow [\tilde{\mathbf{b}}] - [x] \cdot [\tilde{\mathbf{u}}_i] \cdot \Big(\mathbf{v}_i^\mathsf{T} \cdot [\tilde{\mathbf{b}}]\Big) \qquad (19) \\
[\tilde{\mathbf{u}}_j] &\leftarrow [\tilde{\mathbf{u}}_j] - [x] \cdot [\tilde{\mathbf{u}}_i] \cdot \Big(\mathbf{v}_i^\mathsf{T} \cdot [\tilde{\mathbf{u}}_j]\Big)\,, \text{ for } j > i\,.
\end{aligned}$$

The brackets in the last term of each line force computation of the vector products ${\mathbf{v}_i}^\mathsf{T} \cdot [\tilde{\mathbf{b}}]$ and ${\mathbf{v}_i}^\mathsf{T} \cdot [\tilde{\mathbf{u}}_j]$ prior to the multiplication with $[\tilde{\mathbf{u}}_i]$.

This is more efficient than the algebraic equivalent sequences

$$([\tilde{\mathbf{u}}_j] \cdot {\mathbf{v}_i}^\mathsf{T}) \cdot [\tilde{\mathbf{b}}] \text{ and } ([\tilde{\mathbf{u}}_j] \cdot {\mathbf{v}_i}^\mathsf{T}) \cdot [\tilde{\mathbf{u}}_j]\,,$$

respectively, which generate interval-valued matrices first. The latter would cause additional pessimism due to the dependence problem.

The procedure is summarized in Algorithm 2. The number of evaluations of formula 16 is of order $\mathcal{O}(n_{\mathbf{p}}^2)$. Hence, this approach is much more suited to the treatment of a large number of uncertain parameters than Algorithm 1. But the lower complexity is payed back by reduced accuracy, because interdependences are not removed completely.

Algorithm 2 Quick real-valued linear system solver

Input: $\mathbf{A}(\mathbf{p}) = \mathbf{A}_0 + \sum_{i=1}^{n_\mathbf{p}} p_i \cdot (\mathbf{u}_i\,\mathbf{v}_i^\mathsf{T}) \in \mathbb{R}^{n\times n}$ in the form of Theorem 4.4, $\mathbf{b} \in \mathbb{R}^n$, and $\mathbf{p} \in [\mathbf{p}] \in [\mathbb{R}]^{n_\mathbf{p}}$

Output: $[\mathbf{x}] \supseteq \boldsymbol{\Sigma}(\mathbf{A}(\mathbf{p}), \mathbf{b})$, for all $\mathbf{p} \in [\mathbf{p}]$

```
Select p_0 ∈ [p], set [p̃] := [p] − p_0.
for i := 1, ..., n_p do
   Set [ũ_i] := A_0^{-1} · u_i.
end for
Set [ũ_{n_p+1}] := A_0^{-1} · b, S := {[ũ_i] | i = 1, ..., n_p + 1}.
for i = 1, ..., n_p do
   Set S := S \ {[ũ_i]}
   /* Regularity test */
   if 1/min[p̃_i] < −v_i^T · [ũ_i] < 1/max[p̃_i] then
      [x] = [(1/min[p̃_i] + max v_i^T · [ũ_i])^{-1}, (1/max[p̃_i] + min v_i^T · [ũ_i])^{-1}]
      for [ũ] ∈ S do
         [ũ] := [ũ] − [x] · [ũ_i] · (v_i^T · [ũ])
      end for
   else
      return failed
   end if
end for
[x] := [ũ_{n_p+1}]
```

5 Frequency-Response Analysis

The small-signal analysis of analog circuits can be achieved by solving complex-valued linear systems, which are also in fill-in pattern form. Therefore, we will assume, that $\mathbf{C} \in \mathbb{C}^{n\times n}$ is invertible and the parameter dependence is given by

$$\mathbf{C}(\mathbf{p}) = \mathbf{C}_0 + \sum_{\nu=1}^{n_\mathbf{p}} p_\nu \cdot e^{i\varphi_\nu} \cdot (\mathbf{u}_\nu \cdot \mathbf{v}_\nu^\mathsf{T})$$

with $\mathbf{u}_\nu, \mathbf{v}_\nu \in \mathbb{R}^n$ and uncertain parameters $p_\nu \in \mathbb{R}$ with corresponding (but certain) complex phase φ_ν. The latter is not a restriction at all, because usually the parameters in analog circuit analysis vary either on the real or on the imaginary axis.

In order to apply (real) interval techniques it is necessary to reformulate the complex-valued equation system

$$\mathbf{C} \cdot \mathbf{x} = \mathbf{d}$$

by the following equivalent real representation $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ with

$$\mathbf{A} = \begin{pmatrix} \operatorname{Re}\mathbf{C} & -\operatorname{Im}\mathbf{C} \\ \operatorname{Im}\mathbf{C} & \operatorname{Re}\mathbf{C} \end{pmatrix}, \mathbf{x} = \begin{pmatrix} \operatorname{Re}\mathbf{x} \\ \operatorname{Im}\mathbf{x} \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} \operatorname{Re}\mathbf{d} \\ \operatorname{Im}\mathbf{d} \end{pmatrix}. \tag{20}$$

This follows immediately from the complex expansion. The parameter dependence can therefore be written as

$$\mathbf{A}(\mathbf{p}) = \mathbf{A}_0 + \sum_{\nu=1}^{n_\mathbf{p}} p_\nu \cdot (\mathbf{u}_{\mathrm{up},\nu} \cdot \mathbf{v}_{\mathrm{up},\nu}{}^\mathsf{T} + \mathbf{u}_{\mathrm{low},\nu} \cdot \mathbf{v}_{\mathrm{low},\nu}{}^\mathsf{T}), \tag{21}$$

where the vectors $\mathbf{u}_{\mathrm{up},\nu} = (\mathbf{u}_\nu, 0)^\mathsf{T}$ and $\mathbf{v}_{\mathrm{up},\nu} = (\cos\varphi_\nu \cdot \mathbf{v}_\nu, -\sin\varphi_\nu \cdot \mathbf{v}_\nu)^\mathsf{T}$ denote the dependences of the upper part of the matrix $\mathbf{A}(\mathbf{p})$, while the vectors $\mathbf{u}_{\mathrm{low},\nu} = (0, \mathbf{u}_\nu)^\mathsf{T}$ and $\mathbf{v}_{\mathrm{low},\nu} = (\sin\varphi_\nu \cdot \mathbf{v}_\nu, \cos\varphi_\nu \cdot \mathbf{v}_\nu)^\mathsf{T}$ corresponds to the lower part, respectively. By introducing new parameters $p_{\mathrm{up},\nu}, p_{\mathrm{low},\nu} := p_\nu$ the parameter dependence of Equation21 can be reformulated to the form used in Section 4:

$$\mathbf{A}(\mathbf{p}) = \mathbf{A}_0 + \sum_{\nu=1}^{n_\mathbf{p}} p_{\mathrm{up},\nu} \cdot (\mathbf{u}_{\mathrm{up},\nu} \cdot \mathbf{v}_{\mathrm{up},\nu}{}^\mathsf{T}) + \sum_{\nu=1}^{n_\mathbf{p}} p_{\mathrm{low},\nu} \cdot (\mathbf{u}_{\mathrm{low},\nu} \cdot \mathbf{v}_{\mathrm{low},\nu}{}^\mathsf{T}). \tag{22}$$

Note, that this approach will suffer from the dependence problem, because the parameters $p_{\mathrm{up},\nu}$ and $p_{\mathrm{low},\nu}$ are treated independently and hence, the dependence between lower and upper part of $\mathbf{A}(\mathbf{p})$ is lost. In Figure 2 the overestimation of the range is illustrated. In order to obtain a tight wrapping of the solution set (red triangle in Figure 2).

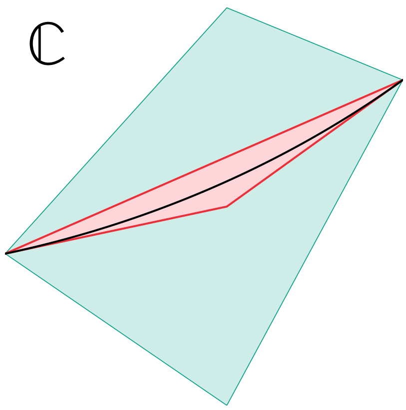


Figure 2 Single parameter variance. Real representation (quadrangle), and tight wrapping (red triangle).

The pair of vectors $\mathbf{u}_{\mathrm{low},\nu}, \mathbf{v}_{\mathrm{low},\nu}$ on the one hand, and $\mathbf{u}_{\mathrm{up},\nu}, \mathbf{v}_{\mathrm{up},\nu}$ on the other, share the valuable property, that

$$\mathbf{u}_{\mathrm{low},\nu} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \mathbf{u}_{\mathrm{up},\nu} \quad \text{and} \quad \mathbf{v}_{\mathrm{low},\nu} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \mathbf{v}_{\mathrm{up},\nu}. \tag{23}$$

In order to give explicit fomulae for the triangle's corners, we first have to show some lemmas. First, we start with a problem in $\mathbb{R}^{2\times 2}$, which can be shown using Cramer's rule.

Lemma 5.1 *Let $p_0 \in \mathbb{R}$, and let $\mathbf{A} \in \mathbb{R}^{2\times 2}$ be a matrix such that $\mathbf{A} + p \cdot 1$ be invertible for all $p \geq p_0 \in \mathbb{R}$. Furthermore, assume that*

$$\mathbf{A} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Then there exist $\lambda_1, \lambda_2 \geq 0$ such that $(\mathbf{A} + p \cdot 1)^{-1} = \lambda_1 1 + \lambda_2 \cdot (\mathbf{A} + p_0 \cdot 1)^{-1}$.

Proof: Since $\mathbf{A} + p \cdot 1$ is invertible, we have that

$$\det(\mathbf{A} + p \cdot 1) = (a+p)^2 + b^2 \neq 0\,.$$

Hence, matrix inversion can be done using Cramer's rule

$$(\mathbf{A} + p \cdot 1)^{-1} = \frac{1}{(a+p)^2 + b^2} \cdot (\mathbf{A}^v + p \cdot 1)\ , \text{ with } \mathbf{A}^v = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

On the other hand we have

$$\begin{aligned}
(\mathbf{A} + p \cdot 1)^{-1} &= \frac{1}{(a+p)^2 + b^2} (\mathbf{A}^v + p \cdot 1) \\
&= \frac{1}{(a+p)^2 + b^2} (\mathbf{A}^v + p_0 \cdot 1 + (p - p_0) \cdot 1) \\
&= \frac{(a+p_0)^2 + b^2}{(a+p)^2 + b^2} \cdot \frac{1}{(a+p_0)^2 + b^2} (\mathbf{A}^v + p_0 \cdot 1) + \frac{p - p_0}{(a+p)^2 + b^2} \cdot 1 \\
&= \frac{(a+p_0)^2 + b^2}{(a+p)^2 + b^2} \cdot (\mathbf{A} + p_0 \cdot 1)^{-1} + \frac{p - p_0}{(a+p)^2 + b^2} \cdot 1\,.
\end{aligned}$$

Finally, set

$$\lambda_1 = \frac{(a+p_0)^2 + b^2}{(a+p)^2 + b^2} \text{ and } \lambda_2 = \frac{p - p_0}{(a+p)^2 + b^2}\,,$$

each of which is greater than or equal to zero. □

A likewise, but multi-dimensional relationship, can be reduced to Lemma 5.1 by the Sherman-Morrison-Woodbury theorem (Theorem 4.3).

Lemma 5.2 *Let the matrix $\mathbf{A} \in \mathbb{R}^{2n\times 2n}$ be invertible, $\mathbf{U}, \mathbf{V} \in \mathbb{R}^{2\times 2n}$ and also let $0 \leq p_0 \in \mathbb{R}$ such that $\mathbf{A} + p \cdot \mathbf{U}\,\mathbf{V}^\intercal$ is invertible for all $0 \leq p \leq p_0$. Furthermore, assume that both $\mathbf{A}, \mathbf{U}$, and $\mathbf{V}$ are of the form*

$$\begin{pmatrix} \mathbf{A}_1 & -\mathbf{A}_2 \\ \mathbf{A}_2 & \mathbf{A}_1 \end{pmatrix},\ \begin{pmatrix} \mathbf{u}_1 & -\mathbf{u}_2 \\ \mathbf{u}_2 & \mathbf{u}_1 \end{pmatrix},\ \textit{and}\ \begin{pmatrix} \mathbf{v}_1 & -\mathbf{v}_2 \\ \mathbf{v}_2 & \mathbf{v}_1 \end{pmatrix},$$

respectively, for $\mathbf{A}_i \in \mathbb{R}^{n\times n}, \mathbf{u}_i, \mathbf{v}_i \in \mathbb{R}^n$. Then there exist $\lambda_1, \lambda_2 \geq 0$ such that

$$\begin{aligned}(\mathbf{A} + p \cdot \mathbf{U}\,\mathbf{V}^\intercal)^{-1} \quad &= \quad \mathbf{A}^{-1} - \lambda_1 \mathbf{A}^{-1}\mathbf{U}\,\mathbf{V}^\intercal\mathbf{A}^{-1} \\ &\qquad - \lambda_2 \cdot \left(\mathbf{A}^{-1} - (\mathbf{A} + p_0 \cdot \mathbf{U}\,\mathbf{V}^\intercal)^{-1}\right) .\end{aligned} \tag{24}$$

Proof: One may assume, that p, p_0 are strictly positive, because in either case, $p = 0$ or $p_0 = 0$, Equation 24 is valid for $\lambda_1, \lambda_2 = 0$.

From the Sherman-Morrison-Woodbury theorem (Theorem 4.3) follows

$$(\mathbf{A} + p \cdot \mathbf{U}\,\mathbf{V}^\intercal)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}\left((1/p)1 + \mathbf{V}^\intercal\mathbf{A}^{-1}\mathbf{U}\right)^{-1}\mathbf{V}^\intercal\mathbf{A}^{-1}. \tag{25}$$

Since the matrix $\mathbf{V}^\intercal\mathbf{A}^{-1}\mathbf{U} \in \mathbb{R}^{2\times 2}$, and $1/p < 1/p_0$ match the conditions for matrix and parameter of Lemma 5.1, there exist $\lambda_1, \lambda_2 \geq 0$ such that

$$\left((1/p)1 + \mathbf{V}^\intercal\mathbf{A}^{-1}\mathbf{U}\right)^{-1} = \lambda_1 1 + \lambda_2 \cdot \left((1/p_0)1 + \mathbf{V}^\intercal\mathbf{A}^{-1}\mathbf{U}\right)^{-1} ,$$

which can be applied to Equation 25. We obtain

$$\begin{aligned}&(\mathbf{A} + p \cdot \mathbf{U}\,\mathbf{V}^\intercal)^{-1} = \\ &\mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}\left(\lambda_1 1 + \lambda_2 \cdot \left((1/p_0)\,1 + \mathbf{V}^\intercal\mathbf{A}^{-1}\mathbf{U}\right)^{-1}\right)\mathbf{V}^\intercal\mathbf{A}^{-1} = \\ &\mathbf{A}^{-1} - \lambda_1\mathbf{A}^{-1}\mathbf{U}\,\mathbf{V}^\intercal\mathbf{A}^{-1} - \lambda_2\mathbf{A}^{-1}\mathbf{U}\left((1/p_0)\,1 + \mathbf{V}^\intercal\mathbf{A}^{-1}\mathbf{U}\right)^{-1}\mathbf{V}^\intercal\mathbf{A}^{-1} ,\end{aligned} \tag{26}$$

which is equal to Equation 24 again by Theorem 4.3. □

Lemma 5.3 *Let $\mathbf{u}_{\mathrm{low}}, \mathbf{v}_{\mathrm{low}}, \mathbf{u}_{\mathrm{up}}, \mathbf{v}_{\mathrm{up}} \in \mathbb{R}^{2\,n}$ be vectors, defining fill-in patterns as in Equation 22. Furthermore, let $\mathbf{A}_0 \in \mathbb{R}^{2\,n\times 2\,n}$ be an invertible matrix arising from a real representation.*

If at least one of $\mathbf{A}_0 + \overline{p} \cdot \mathbf{u}_{\mathrm{low}} \cdot \mathbf{v}_{\mathrm{low}}^\intercal$ or $\mathbf{A}_0 + \overline{p} \cdot \mathbf{u}_{\mathrm{up}} \cdot \mathbf{v}_{\mathrm{up}}^\intercal$ is invertible, then there exists $\lambda \in \mathbb{R}$ such that

$$\begin{aligned}\tfrac{1}{2}\left(\mathbf{A}_0 + \overline{p}\,\mathbf{u}_{\mathrm{low}}\mathbf{v}_{\mathrm{low}}^\intercal\right)^{-1} + \left(\mathbf{A}_0 + \overline{p}\,\mathbf{u}_{\mathrm{up}}\mathbf{v}_{\mathrm{up}}^\intercal\right)^{-1} = \\ \mathbf{A}_0^{-1} - \lambda\mathbf{A}_0^{-1}\left(\mathbf{u}_{\mathrm{low}}\mathbf{v}_{\mathrm{low}}^\intercal + \mathbf{u}_{\mathrm{up}}\mathbf{v}_{\mathrm{up}}^\intercal\right)\mathbf{A}_0^{-1} .\end{aligned} \tag{27}$$

In case that the regularity can be established for all $p \in [0, \overline{p}]$, λ is strictly positive.

Proof: First, note that Equation 23 yields $\mathbf{v}_{\mathrm{low}}^\intercal\mathbf{A}^{-1}\mathbf{u}_{\mathrm{low}} = \mathbf{v}_{\mathrm{up}}^\intercal\mathbf{A}^{-1}\mathbf{u}_{\mathrm{up}}$. Using

$$\lambda = \frac{1}{2}\frac{\overline{p}}{1 + \overline{p}\,\mathbf{v}_{\mathrm{low}}^\intercal\mathbf{A}^{-1}\mathbf{u}_{\mathrm{low}}} = \frac{1}{2}\frac{\overline{p}}{1 + \overline{p}\,\mathbf{v}_{\mathrm{up}}^\intercal\mathbf{A}^{-1}\mathbf{u}_{\mathrm{up}}} \tag{28}$$

Equation 27 follows from application of Sherman-Morrison to each summand of the left-hand side. The positivity of λ is a consequence of the fact that the denominator $1 + p\,\mathbf{v}_{\mathrm{up}}^{\intercal}\mathbf{A}^{-1}\mathbf{u}_{\mathrm{up}}$ must not be zero for all p between 0 and $\overline{p}$. □

Finally, we can prove a complex version of Theorem 4.4.

Theorem 5.4 *Let $\mathbf{u}_{\mathrm{low}}, \mathbf{v}_{\mathrm{low}}, \mathbf{u}_{\mathrm{up}}, \mathbf{v}_{\mathrm{up}} \in \mathbb{R}^{2\,n}$ be vectors, such that the conditions of Equation 23 hold. Furthermore, let $\mathbf{A}_0 \in \mathbb{R}^{2\,n\times 2\,n}$ be a matrix arising from a real representation.*

If $\mathbf{A}(p_{\mathrm{low}}, p_{\mathrm{up}}) = \mathbf{A}_0 + p_{\mathrm{low}}\cdot\mathbf{u}_{\mathrm{low}}\cdot\mathbf{v}_{\mathrm{low}}^{\intercal} + p_{\mathrm{up}}\cdot\mathbf{u}_{\mathrm{up}}\cdot\mathbf{v}_{\mathrm{up}}^{\intercal}$ is invertible for all $p_{\mathrm{low}} \in [\underline{p}, \overline{p}]$ and $p_{\mathrm{up}} \in [\underline{p}, \overline{p}]$, then the inverse $\mathbf{A}(p,p)^{-1}$ varies in

$$\mathrm{conv}\left(\mathbf{A}(\underline{p},\underline{p})^{-1};\quad \mathbf{A}(\overline{p},\overline{p})^{-1};\quad \frac{1}{2}\left(\mathbf{A}(\underline{p},\overline{p})^{-1}+\mathbf{A}(\overline{p},\underline{p})^{-1}\right)\right).$$

Proof: In order to simplify notation, one may assume that $\underline{p} = 0$ and define

$$\mathbf{A}(p)^{-1} = \mathbf{A}(p,p)^{-1} \quad \text{and} \quad \mathbf{A}_{1/2} = \frac{1}{2}\left(\mathbf{A}(\underline{p},\overline{p})^{-1}+\mathbf{A}(\overline{p},\underline{p})^{-1}\right). \tag{29}$$

Since one can always find $\mathbf{U}$ and $\mathbf{V}$ fitting into the conditions of Lemma 5.2, such that $\mathbf{U}\mathbf{V}^{\intercal} = \mathbf{u}_{\mathrm{low}}\mathbf{v}_{\mathrm{low}}^{\intercal} + \mathbf{u}_{\mathrm{up}}\mathbf{v}_{\mathrm{up}}^{\intercal}$, that lemma can be combined with Lemma 5.3. Hence, there exist $\lambda_1, \lambda_2 \geq 0$ such that

$$\mathbf{A}(p)^{-1} = \mathbf{A}_0^{-1} + \lambda_1\cdot\left(\mathbf{A}_{1/2}^{-1} - \mathbf{A}_0^{-1}\right) + \lambda_2\cdot\left(\mathbf{A}(\overline{p})^{-1} - \mathbf{A}_0^{-1}\right). \tag{30}$$

On the other hand, $\mathbf{A}(p) = \mathbf{A}(\overline{p}) - (\overline{p} - p)\cdot\mathbf{U}\mathbf{V}^{\intercal}$ can also be constructed. Thus, we have non-negative μ_1, μ_2 with

$$\mathbf{A}(p)^{-1} = \mathbf{A}(\overline{p})^{-1} + \mu_1\cdot\left(\mathbf{A}_{1/2}^{-1} - \mathbf{A}(\overline{p})^{-1}\right) + \mu_2\cdot\left(\mathbf{A}_0^{-1} - \mathbf{A}(\overline{p})^{-1}\right). \tag{31}$$

Inspecting linear combinations of Equation 30 and 31 show that $\mathbf{A}(p)^{-1}$ lies in the desired convex set. □

We can immediately generalize Theorem 5.4 for the case of several parameters. For illustration consider the real representation $\mathbf{A}(p_1, p_2, p_1, p_2)$ of a complex-valued linear matrix in fill-in pattern form, which depends on two uncertain parameters $p_1 \in [\underline{p}_1, \overline{p}_1]$ and $p_2 \in [\underline{p}_2, \overline{p}_2]$. If $\mathbf{A}(p_1, p_2, p_1, p_2)$ is invertible for all p_1, p_2, we have that

$$\mathbf{A}(p_1, p_2, p_1, p_2)^{-1}\cdot\mathbf{b} \in \mathrm{conv}(\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_{1/2})$$

with

$$\begin{aligned}
\mathbf{x}_1 &= \mathbf{A}(\underline{p}_1, p_2, \underline{p}_1, p_2)^{-1} \cdot \mathbf{b} \\
\mathbf{x}_2 &= \mathbf{A}(\overline{p}_1, p_2, \overline{p}_1, p_2)^{-1} \cdot \mathbf{b} \\
\mathbf{x}_{1/2} &= \frac{1}{2} \cdot \Big(\mathbf{A}(\underline{p}_1, p_2, \overline{p}_1, p_2)^{-1} + \mathbf{A}(\overline{p}_1, p_2, \underline{p}_1, p_2)^{-1} \Big) \cdot \mathbf{b}
\end{aligned}$$

by Theorem 5.4. Invoking the procedures again with respect to p_2 for each

$$\mathbf{x} \in \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_{\frac{1}{2}}\},$$

we obtain that

$$\mathbf{A}(p_1, p_2, p_1, p_2)^{-1} \cdot \mathbf{b} \in \operatorname{conv}\Big(\mathbf{x}_{1,1},\ \mathbf{x}_{1,2},\ \mathbf{x}_{1,\frac{1}{2}},\ \mathbf{x}_{2,1},\ \mathbf{x}_{2,2},\ \mathbf{x}_{2,\frac{1}{2}},\ \mathbf{x}_{1/2,1},\ \mathbf{x}_{\frac{1}{2},2},\ \mathbf{x}_{\frac{1}{2},\frac{1}{2}}\Big)$$

with

$$\begin{aligned}
\mathbf{x}_{1,1} &= \mathbf{A}(\underline{p}_1, \underline{p}_2, \underline{p}_1, \underline{p}_2)^{-1} \cdot \mathbf{b} \\
\mathbf{x}_{1,2} &= \mathbf{A}(\underline{p}_1, \overline{p}_2, \underline{p}_1, \overline{p}_2)^{-1} \cdot \mathbf{b} \\
\mathbf{x}_{1,\frac{1}{2}} &= \frac{1}{2} \cdot \Big(\mathbf{A}(\underline{p}_1, \underline{p}_2, \underline{p}_1, \overline{p}_2)^{-1} + \mathbf{A}(\underline{p}_1, \overline{p}_2, \underline{p}_1, \underline{p}_2)^{-1} \Big) \cdot \mathbf{b} \\
\\
\mathbf{x}_{2,1} &= \mathbf{A}(\overline{p}_1, \underline{p}_2, \overline{p}_1, \underline{p}_2)^{-1} \cdot \mathbf{b} \\
\mathbf{x}_{2,2} &= \mathbf{A}(\overline{p}_1, \overline{p}_2, \overline{p}_1, \overline{p}_2)^{-1} \cdot \mathbf{b} \\
\mathbf{x}_{2,\frac{1}{2}} &= \frac{1}{2} \cdot \Big(\mathbf{A}(\overline{p}_1, \underline{p}_2, \overline{p}_1, \overline{p}_2)^{-1} + \mathbf{A}(\overline{p}_1, \overline{p}_2, \overline{p}_1, \underline{p}_2)^{-1} \Big) \cdot \mathbf{b} \\
\\
\mathbf{x}_{\frac{1}{2},1} &= \frac{1}{2} \cdot \Big(\mathbf{A}(\underline{p}_1, \underline{p}_2, \overline{p}_1, \underline{p}_2)^{-1} + \mathbf{A}(\overline{p}_1, \underline{p}_2, \underline{p}_1, \underline{p}_2)^{-1} \Big) \cdot \mathbf{b} \\
\mathbf{x}_{\frac{1}{2},2} &= \frac{1}{2} \cdot \Big(\mathbf{A}(\underline{p}_1, \overline{p}_2, \overline{p}_1, \overline{p}_2)^{-1} + \mathbf{A}(\overline{p}_1, \overline{p}_2, \underline{p}_1, \overline{p}_2)^{-1} \Big) \cdot \mathbf{b} \\
\mathbf{x}_{\frac{1}{2},\frac{1}{2}} &= \frac{1}{4} \cdot \Big(\mathbf{A}(\underline{p}_1, \underline{p}_2, \overline{p}_1, \overline{p}_2)^{-1} + \mathbf{A}(\underline{p}_1, \overline{p}_2, \overline{p}_1, \underline{p}_2)^{-1} + \\
&\qquad \mathbf{A}(\overline{p}_1, \underline{p}_2, \underline{p}_1, \overline{p}_2)^{-1} + \mathbf{A}(\overline{p}_1, \overline{p}_2, \underline{p}_1, \underline{p}_2)^{-1} \Big) \cdot \mathbf{b}
\end{aligned}$$

The general case is illustrated in Algorithm 3.

Example For instance, Algorithm 3 is explained by analyzing AC equations for the serial circuit of Figure 3. It has the same topological structure as the simple voltage divider of Figure 1, but one of the resistors is replaced by a capacitor. The same is true for the system of equations,

$$\begin{pmatrix} \frac{1}{R_1} & -\frac{1}{R_1} & 1 \\ -\frac{1}{R_1} & \frac{1}{R_1} + C_1\, s & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} V_1 \\ V_2 \\ I_{V_0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ V_0 \end{pmatrix}, \qquad (32)$$

Algorithm 3 Complex-valued linear systems solver

Input: Real representation: $\mathbf{A}(\mathbf{p},\mathbf{p}) \in \mathbb{R}^{2n\times 2n}$, $\mathbf{b} \in \mathbb{R}^{2n}$, and $\mathbf{p} \in [\mathbf{p}] \in [\mathbb{R}]^{n_\mathbf{p}}$
Output: $R = \{\mathbf{x}_1, \ldots, \mathbf{x}_{2^{n_\mathbf{p}}}\}$ such that $\mathbf{A}(\mathbf{p},\mathbf{p})^{-1}\mathbf{b} \subseteq \mathrm{conv}(R)$, for all $\mathbf{p} \in [\mathbf{p}]$

```
Set s := sign(det A(p_0, p_0)) for some p_0 ∈ [p].
if s = 0 then
   return failed
end if
/* Initialize matching sets */
Set S_[p] := {{(p̲_1, p̲_1)}, {(p̄_1, p̲_1)}, {(p̲_1, p̄_1), (p̄_1, p̲_1)}}.
for i = 2, ..., n_p do
   Set T := ∅.
   for S ∈ S_[p] do
      T = T ∪ {{(p, p̲_i, q, p̲_i) | (p, q) ∈ S}, {(p, p̄_i, q, p̄_i) | (p, q) ∈ S}}
      T = T ∪ {⋃_{(p,q)∈S} {(p, p̲_i, q, p̄_i), (p, p̄_i, q, p̲_i)}}
   end for
   Set S_[p] = T.
end for
/* Start computations */
Set R := ∅.
for S ∈ S_[p] do
   Set T := ∅.
   for (p, q) ∈ S do
      /* Regularity test */
      if det A(p, q) ≠ s then
         return failed
      else
         Set T := T ∪ {(A(p, q))^{-1} · b}
      end if
   end for
   Set R := R ∪ {1/#(T) · Σ_{x∈T} x}
end for
return R
```

which is to be treated for a constant frequency of $1000\,\mathrm{Hz}$, such that we have complex-valued $s = 2\,\pi\,i \cdot 1000\,\mathrm{s}^{-1}$ with an exact independent voltage source of $V_0 = 1\,\mathrm{V}$, and two tolerance-affected parameters given as $C_1/1\,\mu\mathrm{F} \in [0.8, 1.2]$ and $R_1/1\,\Omega \in [80, 120]$. For analyzing the range using Algorithm 4, 16 linear systems have to be solved. The results can be used to make up the nine points defining the convex set. The range of the current I_{V_0} in the complex range is shown in Figure 4. The smallest rectangle in the complex plane containing all solutions is easily obtained by computing the interval hull of real and imaginary

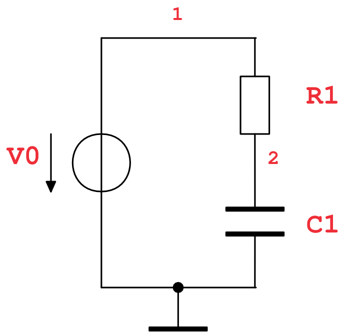


Figure 3 Serial circuit.

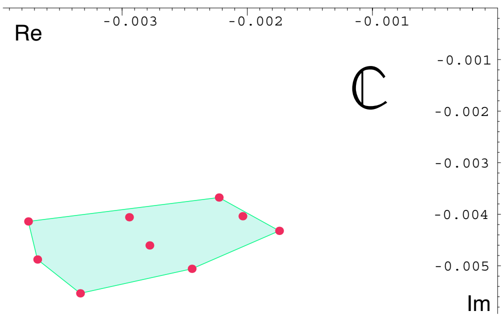


Figure 4 Range of the solution I_{V_0}.

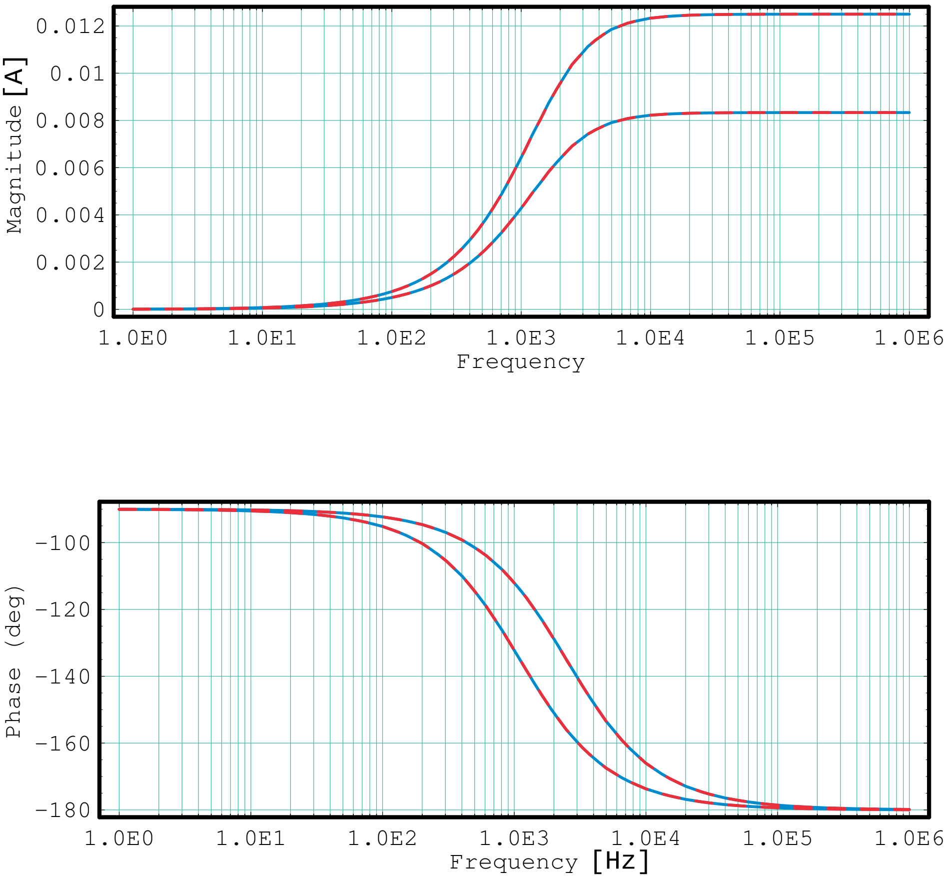
parts of all points such that

$$I_{V_0}/1\,\mathrm{mA} \in [-3.76, -1.73] + i\,[-5.53, -3.68]\,.$$

But we are sometimes interested in the range of the absolute and the phase value of the complex variable. These can be approximately determined computing the respective values over all points. In the example, we obtain

$$\left|I_{V_0}\right|/1\,\mathrm{mA} \in [4.30, 6.46] \text{ and phase } \varphi/1^\circ \in [-132.2, -111.9]\,.$$

Note, that the actual lower bound for the absolute value may be slightly smaller, if a normal vector of the nearest edge of the convex hull meets the origin. For a justification of this problem see [4, Section 3.3.2]. For most practical applications – like the example above – the approximate solution gives sufficiently accurate bounds (see Figure 4). Lower and upper bounds obtained from Algorithm 3 for the frequency response with respect to frequencies from $1\,\mathrm{Hz}$ up to $1\,\mathrm{MHz}$ together with a good (inner) approximation of the range are shown in Figure 5, which results from a sufficiently large number of parameter sweeps. The actual range is perfectly met by the results computed with the method described in this section.



Outer bounds to absolute value and phase of I_{V_0} obtained from Algorithm 3 (red) and parameter sweeps (blue). Figure 5

5.1 A faster approach

Although Theorem 5.4 can easily be generalized to the case of several parameters, it is not the method of choice for practical problems, because it would lead to $2^{2n_\mathbf{p}}$ linear systems to be solved. But in order to avoid such conservative bounds, as obtained by simply using the real representation, a combination with the ideas of the resistive case is suitable. In contrary, Algorithm 2, the quick variant for solving real-valued systems, can easily be applied to a real representation $\mathbf{A}(\mathbf{p}, \mathbf{p})$ with polynomial computational complexity. Unfortunately, this does not treat the dependence of the parameters accurately, because in this case the occurrences of the parameters in the matrix $\mathbf{A}(\mathbf{p}, \mathbf{p})$ do not fit in the scheme of the fill-in pattern form. Hence, the quick procedure actually bounds the solutions $\mathbf{A}(\mathbf{p}, \mathbf{q})^{-1}\mathbf{b}$ for all $\mathbf{p} \in [\mathbf{p}]$ and $\mathbf{q} \in [\mathbf{p}]$.

In order to avoid too conservative bounds, a combination of both approaches is presented in the following. The dependence of $\mathbf{A}(\mathbf{p}, \mathbf{p})$ with respect to a single parameter is given by Theorem 5.4. Therefore, the result $\mathbf{A}(p,p)^{-1} \cdot \mathbf{b}$ lies in

$$\operatorname{conv}\{\mathbf{A}(\underline{p},\underline{p})^{-1} \cdot \mathbf{b},\ \mathbf{A}(\overline{p},\overline{p})^{-1} \cdot \mathbf{b},\ \frac{1}{2}\left(\mathbf{A}(\overline{p},\underline{p})^{-1} + \mathbf{A}(\underline{p},\overline{p})^{-1}\right) \cdot \mathbf{b}\}\,, \tag{33}$$

for $p \in [\underline{p}, \overline{p}]$. As mentioned in above there exist vectors $\mathbf{u}_{\mathrm{up}}, \mathbf{v}_{\mathrm{up}}, \mathbf{u}_{\mathrm{low}}$, and $\mathbf{v}_{\mathrm{low}} \in \mathbb{R}^{2\,n}$, with

$$\mathbf{u}_{\mathrm{low}} = \mathbf{u}_{\mathrm{up}} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \mathbf{v}_{\mathrm{low}} = \mathbf{v}_{\mathrm{up}} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{34}$$

such that

$$\mathbf{A}(p,q) = \mathbf{A}(\underline{p},\underline{p}) + (p - \underline{p}) \cdot \mathbf{u}_{\mathrm{up}}\, {\mathbf{v}_{\mathrm{up}}}^{\mathsf{T}} + (q - \underline{p}) \cdot \mathbf{u}_{\mathrm{low}}\, {\mathbf{v}_{\mathrm{low}}}^{\mathsf{T}}\,. \tag{35}$$

Hence, the matrix $\mathbf{A}(p,q)$ is a matrix in fill-in pattern form with respect to the parameters p and q. Hence, we can try to use the Sherman-Morrison formula to compute the points

$$\mathbf{A}(\underline{p},\underline{p})^{-1} \cdot \mathbf{b},\ \mathbf{A}(\underline{p},\overline{p})^{-1} \cdot \mathbf{b},\ \mathbf{A}(\overline{p},\underline{p})^{-1} \cdot \mathbf{b},\ \mathbf{A}(\overline{p},\overline{p})^{-1} \cdot \mathbf{b}\,,$$

which are needed to bound the solution set accurately by Theorem 5.4. Starting at the point $(\underline{p},\underline{p})$, we compute

$$\tilde{\mathbf{b}} = \mathbf{A}(\underline{p},\underline{p})^{-1} \cdot \mathbf{b} \text{ and } \tilde{\mathbf{u}}_{\mathrm{up}} = \mathbf{A}(\underline{p},\underline{p})^{-1} \cdot \mathbf{u}\,.$$

As a consequence of Equation 34, the analogously defined vector $\tilde{\mathbf{u}}_{\mathrm{low}}$ is simply given by

$$\tilde{\mathbf{u}}_{\mathrm{low}} = \tilde{\mathbf{u}}_{\mathrm{up}} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now, let $\Delta p := \overline{p} - \underline{p}$, and set

$$d := \frac{1}{\Delta p} + \left(\mathbf{v}_{\mathrm{up}}^{\mathsf{T}} \cdot \tilde{\mathbf{u}}_{\mathrm{up}}\right)\,.$$

If $d > 0$, then the condition for the application of the Sherman-Morrison formula is fulfilled. Hence, we may calculate

$$\mathbf{b}^{*}_{\mathrm{up}} := \mathbf{A}(\overline{p},\underline{p})^{-1} \cdot \mathbf{b} \quad = \quad \tilde{\mathbf{b}} - \frac{1}{d}\,\tilde{\mathbf{u}}_{\mathrm{up}} \cdot \left(\tilde{\mathbf{v}}_{\mathrm{up}}^{\mathsf{T}} \cdot \tilde{\mathbf{b}}\right) \tag{36}$$

and analogously $\mathbf{b}^{*}_{\mathrm{low}}$ and $\mathbf{u}^{*}_{\mathrm{low}}$. Consequently, one can compute:

$$\frac{1}{2} \cdot \left(\mathbf{b}^{*}_{\mathrm{up}} + \mathbf{b}^{*}_{\mathrm{low}}\right) = \tilde{\mathbf{b}} - \frac{1}{2\,d} \cdot \left(\tilde{\mathbf{u}}_{\mathrm{up}} \cdot \left(\tilde{\mathbf{v}}_{\mathrm{up}}^{\mathsf{T}} \cdot \tilde{\mathbf{b}}\right) + \tilde{\mathbf{u}}_{\mathrm{low}} \cdot \left(\tilde{\mathbf{v}}_{\mathrm{low}}^{\mathsf{T}} \cdot \tilde{\mathbf{b}}\right)\right)\,. \tag{37}$$

Finally, we have to calculate $\mathbf{A}(\overline{p},\overline{p})^{-1} \cdot \mathbf{b}$. This is similar to the application of Algorithm 2 to a real-valued system with two parameters. For abbreviation define

$$d^{*} := \frac{1}{\Delta p} + \left(\mathbf{v}_{\mathrm{low}}^{\mathsf{T}} \cdot \mathbf{u}^{*}_{\mathrm{low}}\right)\,. \tag{38}$$

If $d^* > 0$ we can apply Sherman-Morrison a second time, such that

$$\mathbf{A}(\overline{p}, \overline{p})^{-1} \cdot \mathbf{b} = \mathbf{b}^*_{\mathrm{up}} - \frac{1}{d^*}\, \mathbf{u}^*_{\mathrm{low}} \cdot \left(\tilde{\mathbf{v}}^{\mathsf{T}}_{\mathrm{low}} \cdot \mathbf{b}^*_{\mathrm{up}} \right) . \tag{39}$$

Algorithm 4 Quick complex-valued linear system solver

Input: Real representation: $\mathbf{A}(\mathbf{p}) = \mathbf{A}_0 + \sum_{i=1}^{n_\mathbf{p}} p_i \left(\mathbf{u}_{\mathrm{up},i} \cdot \mathbf{v}^{\mathsf{T}}_{\mathrm{up},i} + \mathbf{u}_{\mathrm{low},i} \cdot \mathbf{v}^{\mathsf{T}}_{\mathrm{low},i} \right) \in \mathbb{R}^{2n \times 2n}$, $\mathbf{b} \in \mathbb{R}^n$, and $\mathbf{p} \in [\mathbf{p}] \in [\mathbb{R}]^{n_\mathbf{p}}$
Output: $[\mathbf{x}] \supseteq \mathbf{A}(\mathbf{p})^{-1}\mathbf{b}$, for all $\mathbf{p} \in [\mathbf{p}]$

- **for** $i := 1, \ldots, n_\mathbf{p}$ **do**
  - Set $\Delta p_i := \overline{p}_i - \underline{p}_i$.
  - Set $[\tilde{\mathbf{u}}_i] := \mathbf{A}_0^{-1} \cdot \mathbf{u}_{\mathrm{up},\mathbf{i}}$.
- **end for**
- Set $[\tilde{\mathbf{u}}_{n_\mathbf{p}+1}] := \mathbf{A}_0^{-1} \cdot \mathbf{b}$
- Set $S := \{[\tilde{\mathbf{u}}_i] \,|\, i = 1, \ldots, n_\mathbf{p} + 1\}$.
- **for** $i = 1, \ldots, n_\mathbf{p}$ **do**
  - $S := S \setminus \{[\tilde{\mathbf{u}}_i]\}$
  - Set $[\tilde{\mathbf{u}}_{\mathrm{low},i}] := [\tilde{\mathbf{u}}_i] \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
  - Set $[d] := \frac{1}{\Delta p_i} + \left(\mathbf{v}^{\mathsf{T}}_{\mathrm{up},i} \cdot [\tilde{\mathbf{u}}_i] \right)$.
  - /* Regularity test */
  - **if** $[d] > 0$ **then**
    - Set $[\mathbf{u}^*_i] = \left(1 - \frac{(\mathbf{v}^{\mathsf{T}}_{\mathrm{up},i} \cdot [\tilde{\mathbf{u}}_i])}{[d]} \right) [\tilde{\mathbf{u}}_{\mathrm{low},i}]$.
    - Set $[d^*] := \frac{1}{\Delta p_i} + \left(\mathbf{v}^{\mathsf{T}}_{\mathrm{low},i} \cdot [\mathbf{u}^*_i] \right)$.
    - **if** $[d^*] > 0$ **then**
      - **for** $[\tilde{\mathbf{u}}] \in S$ **do**
        - Set $[\mathbf{\Delta} \tilde{\mathbf{u}}] := [\tilde{\mathbf{u}}_i] \left(\mathbf{v}^{\mathsf{T}}_{\mathrm{up},i} \cdot [\tilde{\mathbf{u}}] \right)$.
        - $[\mathbf{u}^*] := [\tilde{\mathbf{u}}] - \frac{[\mathbf{\Delta} \tilde{\mathbf{u}}]}{[d]}$
        - $[\tilde{\mathbf{u}}] := \left[\{ [\tilde{\mathbf{u}}], [\tilde{\mathbf{u}}] - \frac{[\mathbf{\Delta} \tilde{\mathbf{u}}] + [\tilde{\mathbf{u}}_{\mathrm{low},i}] \left(\mathbf{v}^{\mathsf{T}}_{\mathrm{low},i} \cdot [\tilde{\mathbf{u}}] \right)}{2[d]}, [\mathbf{u}^*] - \frac{[\mathbf{u}^*{}_i] \left(\mathbf{v}^{\mathsf{T}}_{\mathrm{low},i} \cdot [\mathbf{u}^*] \right)}{[d^*]} \} \right]$
      - **end for**
    - **else**
      - **return** failed
    - **end if**
  - **else**
    - **return** failed
  - **end if**
- **end for**
- $[\mathbf{x}] := [\tilde{\mathbf{u}}_{n_\mathbf{p}+1}]$

In order to treat several parameters, we can now move from sets of real-valued vectors to boxes like in Algorithm 2. This is done by replacing the convex hull by the interval hull in Equation 33. As seen in the derivation of Algorithm 2, the vectors $[\tilde{\mathbf{u}}_i]$ have to be updated with respect to to each parameter in the same manner like the right-hand side vector $\mathbf{b}$. The procedure is summarized in Algorithm 4. Like for Algorithm 2 the loss of accuracy due to wrapping intermediate results by the interval hull pays off by reducing the computational complexity order to $n_{\mathbf{p}}^2$.

Example We now apply Algorithm 4 to the real representation of Equation 32 for the same data as above, i. e. for constant frequency of $1000\,\mathrm{Hz}$, such that we have $s = 2\,\pi\,i \cdot 1000\,\mathrm{s}^{-1}$, with an exact independent voltage source of $V_0 = 1\,\mathrm{V}$, and two tolerance affected parameters given as

$$C_1/1\,\mu\mathrm{F} \in [0.8, 1.2] \text{ and } R_1/1\,\Omega \in [80, 120]\,.$$

Again, we have to deal with a reciprocal parameter $\frac{1}{R_1}$, whose range is then the interval $[0.0083, 0.0125]$. Since we deal with constant frequency, we can interprete of the expression

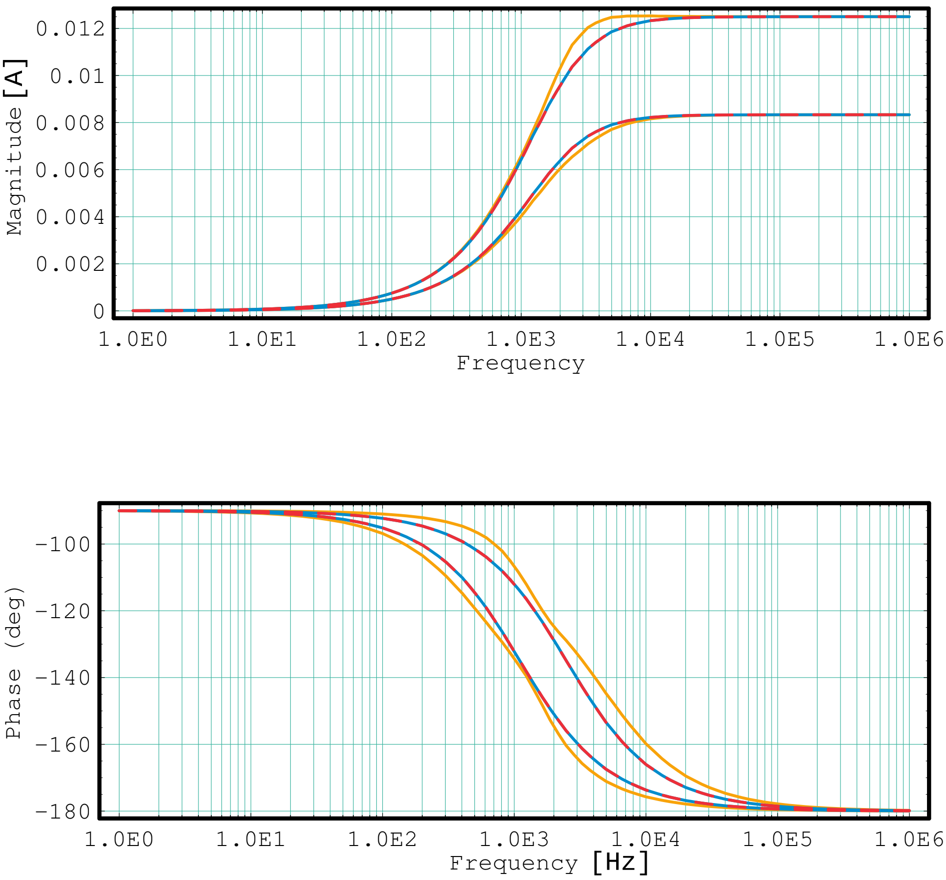
$$C_1\,2\boldsymbol{n}\pi f \in [0.8\,10^{-6}, 1.2\,10^{-6}] \cdot 2\,\pi\,1000 = [0.005, 0.0076]$$

as an parameter. Application of the algorithm leads to bounds for the current

$$I_{V_0}/1\,\mathrm{mA} \in [-4.27, -1.73] + i\,[-5.56, -3.68]\,,$$

which properly includes the rectangle $[-3.76, -1.73] + i\,[-5.53, -3.68]$, computed using Algorithm 2.

The response for the frequencies from $1\,\mathrm{Hz}$ up to $1\,\mathrm{MHz}$ is compared to the results from Algorithm 3 in Figure 6. The curves denoting lower and upper bounds for both computations, respectively, indicate similar qualitative behavior. Although the results obtained from Algorithm 4 are less accurate, they afford to draw conclusions about the performance of the circuit.



Outer bounds obtained using Algorithm 3 (solid orange), and Algorithm 4 (dashed). Figure 6

6 Implementations and Applications

In this section, we apply the implementations of the algorithms to practical examples. The main goal, besides the development of interval solvers, which are tuned to solve problems from analog circuit design, was to implement the resulting algorithms. This was done as an extension to the electronic design tool Analog Insydes on the basis of the computer algebra system Mathematica [17].

The selection of the example applications was motivated in the following way:

The analysis of a *tone controller* is the first example circuit. The output characteristic is computed with respect to uncertain resistances and capacitances. The result can be used for sizing the potentiometers, which control the amplification of low and high frequencies, because we can compute the worst case values of minimal and maximal amplification.

Secondly, the ability of the interval-valued AC analysis to treat real-world circuits is discussed with respect to an *operational amplifier*. It contains a large number of components and cannot be treated with classical interval methods.

The example calculations below have been computed under SuSE-Linux 9.0-i586 on an Intel Xeon™ $3\,\mathrm{GHz}$ dual-processor machine with four gigabytes of RAM, using Mathematica 4.2 and a developer version of Analog Insydes.

6.1 Implementations

Before discussing a few example applications in detail, we first give a brief description of the Analog Insydes commands, which are necessary to carry out the example calculations. For a more extensive treatment of Analog Insydes see [18]. We will concentrate on main issues, which are important for the treatment of the examples in this section. In particular, the procedures of the interval solvers are presented. These have been implemented as an extension to Analog Insydes on the basis of Mathematica. We will assume, that the reader is familiar with the basic Mathematica syntax. See [17] for an introduction to Mathematica.

The procedures for solving interval-valued problems are actually implementations of the newly introduced algorithms. The functions were set up in such a way, that they fit in the structure of the existing Analog Insydes routines. It was also attached importance to the usability and computation speed of the procedures. Obviously, interval algorithms still last longer than calculations using floating-point solvers, but at least for linear problems, these can be solved in reasonable time.

Data structures Analog Insydes uses a special data structure to store a symbolic system of equations. It contains the equations and variables together with additional information, like the design point. A **`DAEObject`** has the following structure:

`DAEObject[`*mode***`][{`***eqs***`,`** *vars***`}, {`***options***`}]`**

The value of *mode* shows, whether the **`DAEObject`** corresponds to a static, frequency domain, or transient analysis. The field *options* contains user-accessible information about the object.

In addition, the interval-solver extension carries a new object for marking terms in an expression. The **`TermMarkerObject`** includes the marked term, the variables of the term, and an additional symbol, which can be used to uniquely identify the object.

`TermMarkerObject[`*term***`,`***marker***`,{`***vars***`}][`***vars***`]`**

The **`TermMarkerObject`** respects derivation and evaluates to *term*, if *term* is a real or interval value. It is instantiated by the command

`MarkTerm[`*term***`,`***marker***`,`***symbs***`]`** ,

where *symbs* denotes a variable or a list of variables. It is currently used to mark nonlinear terms in interval models, which can easily be detected by the interval routines. Another type of object, which will occur during the calculations of this section is the **`DataObject`**, which is a container for the numerical results obtained from functions of Analog Insydes and the interval-solver extension.

Tolerance specifications Yet, the **`DAEObject`** does not contain information about the component tolerances. Hence, these have to be stated separately. A tolerance specification is usually given as a list, whose elements are rules of the form :

param $\rightarrow$ *val* .

The tolerance values *val* may be the symbol **`Automatic`**, intervals, or real numbers, where intervals directly state the parameter range, and a real number ε yields the interval

$$[1-\varepsilon, 1+\varepsilon]\cdot r\,,$$

where $r \in \mathbb{R}$ is given by the design point. Furthermore, if *val* is set to **`Automatic`** a standard value for ε is used. This value can be changed using the option setting **`Tolerance`** $\rightarrow$ *value* of the interval solvers. The right-hand side values are assigned to the symbolic parameters matching the expression *param*, which may be a symbol or a string pattern. For instance, **`R1`** matches only the element **`R1`**, but **`"R*"`** corresponds to all parameters, whose parameter names begin with the letter 'R'.

Linear interval solvers For an efficient frequency-response (AC) analysis of tolerance-affected analog circuits the algorithms of Section 4 are implemented. The interval-based tolerance analysis is invoked by **`IntervalACAnalysis`**. The syntax of the command is close to the one of the corresponding Analog Insydes command **`ACAnalysis`**, but with tolerance specifications added.

`IntervalACAnalysis[`_dae_**`,{`**_fvar, fstart, fend_**`},{`**_tolerance-specs_**`}`**$\langle$**`,`**_opts_$\rangle$**`]`**

The frequency variable is selected by *fvar*, and start and end frequencies are defined by *fstart* and *fend*, respectively. The underlying interval solver may be selected by adding optional arguments to *opts*.

The setting **`Method`** $\rightarrow$ **`MonotonicACAnalysis`** switches to Algorithm 3, while **`Method`** $\rightarrow$ **`QuickACAnalysis`** calls an implementation of Algorithm 4. Also, a reference implementation of Rohn's almost classical sign-accord method can be selected using **`Method`** $\rightarrow$ **`SignAccordACAnalysis`** [19]. The command returns a list of two objects, which are generated from the interval results. By default, these contain information about the curves corresponding to the values $a_{\min} \cdot e^{i\,\varphi_{\min}}$ and $a_{\max} \cdot e^{i\,\varphi_{\max}}$ for each frequency and variable. The values $a_{\min}$ and $a_{\max}$ denote the minimal and maximal absolute values of the complex-valued result, whereas $\varphi_{\min}$ and $\varphi_{\max}$ give the range of the phase. The latter are chosen, such that $\varphi_{\max} - \varphi_{\min}$ is minimal. The way how the curves are computed from the raw interval data, can be controlled by the user using the option **`PostProcessingFunction`**. The number of frequency values per decade to be treated can be controlled by adding

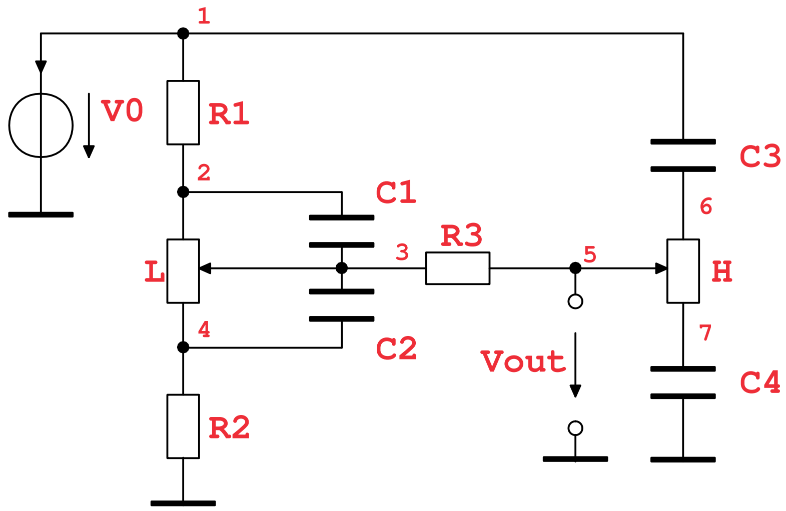
`PointsPerDecade` $\rightarrow$ *ppd*

to the option sequence. For **`MonotonicACAnalysis`** only, one can switch to Analog Insydes' upcoming fast numerical solvers using the setting

`UseExternals` $\rightarrow$ **`True`**.

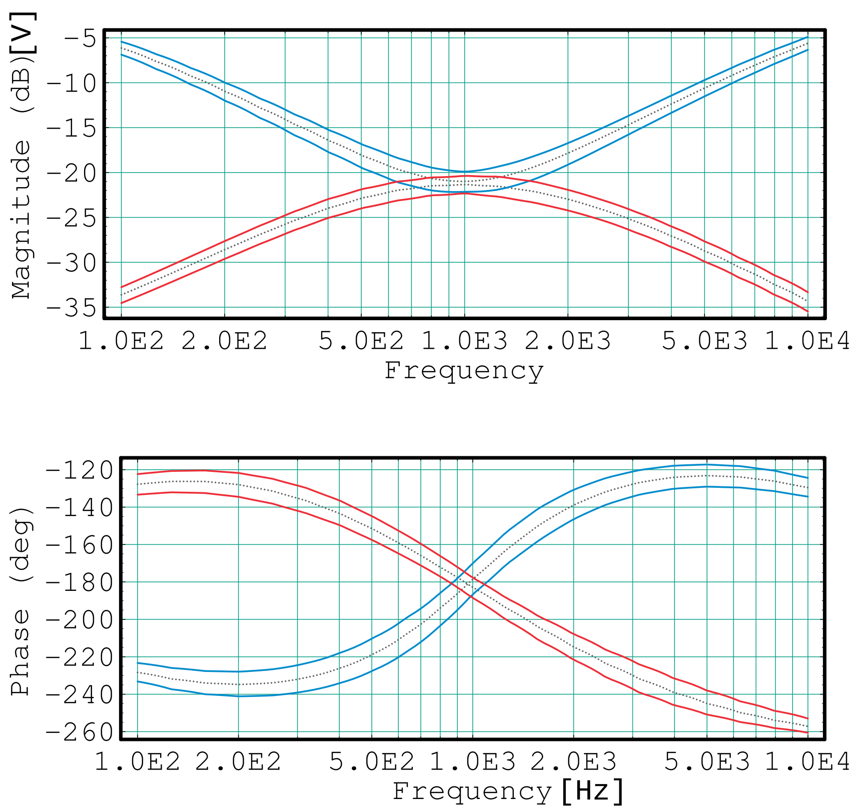
6.2 AC Analysis of a Tone-Control Circuit

Our next goal is to analyze a tone-control circuit. This filter can be used to control the amplification of low and high frequencies independently, whereas the mid-frequency stays constant [20]. A common realization is shown in Figure 7. The magnitude can be lowered or raised with two pre-set potentiometers. We are interested in the small-signal behavior for the extremal settings of the potentiometers, when both are set to maximal and minimal respectively. Now, Algorithm 2 can be used to compute sufficiently tight bounds for the frequency response. A Bode diagram of the results with respect to a tolerance of 5% for the output voltage is illustrated in Figure 8.



Schematics of a tone-control circuit.

Figure 7



Outer bounds for extremal potentiometer settings (blue/red), and original design point (dotted).

Figure 8

The computation for twenty points in the frequency domain from $100\,\mathrm{Hz}$ to $10\,\mathrm{kHz}$ needs 3.05 seconds. In contrary, a simple parameter sweep with three points per parameter yields a computation time of 35.22 seconds. It is also possible to obtain slightly sharper bounds using Algorithm 1, where the use of Analog Insydes' fast external routines is switched on. But for this approach $2^{14} = 16384$ corner points have to be treated, which lasts more than one hour and needs about two gigabytes of RAM.

Here, interval analysis delivers guaranteed lower bounds to the maximal and upper bounds to the lower amplification, respectively. Hence, one can determine with mathematical certainty, whether the ranges of the potentiometers are chosen large enough to obtain the desired output voltages.

6.3 AC Analysis of an Operational Amplifier

In this section we demonstrate the ability of Algorithm 2 to treat real-world examples. We consider the operational amplifier, whose schematics can be found in Figure 9. The small-signal behavior of the transistors is modeled by the linear circuit of Figure 10, which consists of resistors, capacitors, and controlled sources.

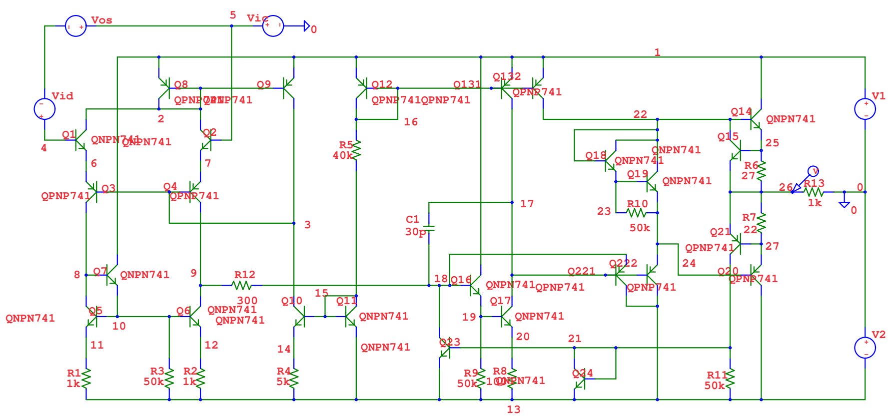
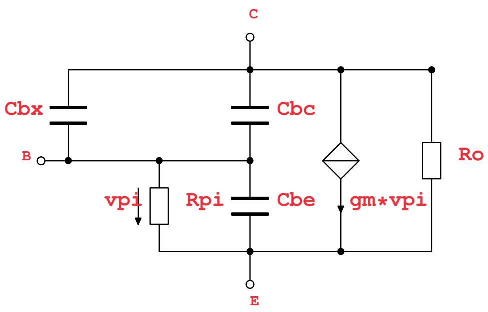


Figure 9 Schmematics of the operational amplifier $\mu A741$.

Since this equivalent circuit diagram is only valid for a constant operating point, we will only vary such circuit elements, that do not change the operating point. Hence, we assign tolerances of 20% to the capacitance values.

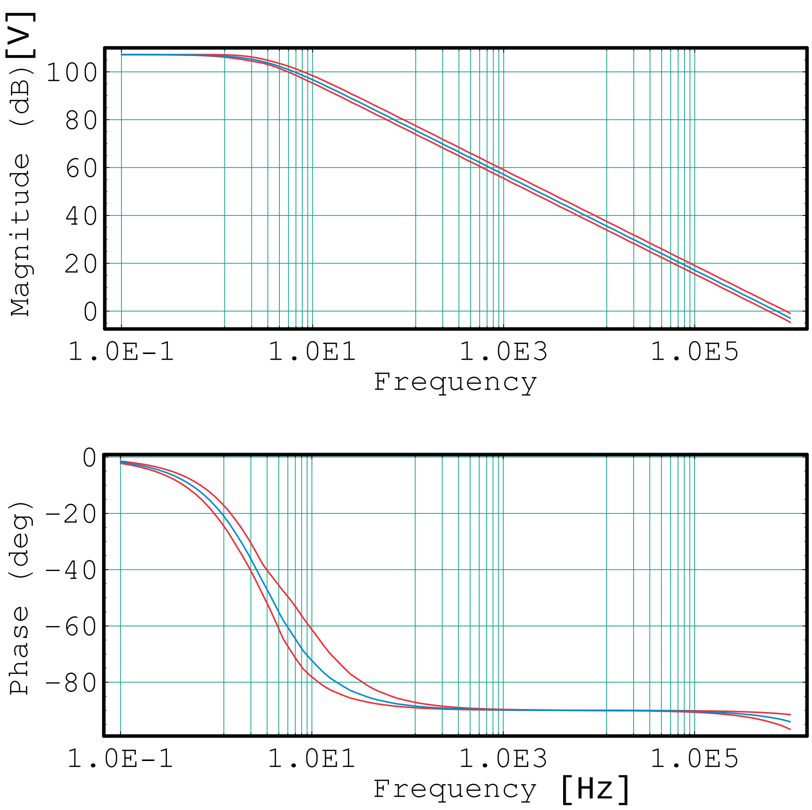
Then an interval-valued AC analysis is performed for 70 frequency values between $0.1\,\mathrm{Hz}$ and $1\,\mathrm{MHz}$ in eleven minutes and 47 seconds. The frequency response of the output voltage at node 26 is presented in Figure 11. We obtain



Simplified equivalent circuit schematics of a bipolar-junction transistor.

Figure 10

interpretable results: upper und lower bounds are very close to the curve, which corresponds to the original design point.



Outer bounds (red), and frequency response with respect to original design point (blue).

Figure 11

7 Conclusions

The methods described above were implemented as an extension to the toolbox *Analog Insydes* [16, 18], an add-on package to the computer algebra system *Mathematica* [17] for modeling, analysis, and design of analog circuits.

The techniques can be used to obtain meaningful bounds to the simulation results of analog circuits with uncertain parameters. As opposed to earlier attempts to use interval arithmetic in this area, which were restricted to the sparse tableau formulation, the dependence between parameters with multiple occurrences is treated explicitely.

For efficient implementation, it is crucial to regard possible dependencies in circuit equations. Part and parcel of of this strategy is the handling of fill-in patterns for those parameters related to uncertain components. These patterns arise from linear circuit analysis, where they are used for efficient equation setup. With a view towards the incorporation of the methods in the industrial design flow, procedures were presented, which are capable of the treatment of a large number of tolerance-affected components.

First of all, the application of interval analysis to analog circuits has been motivated. For this purpose, we have given a brief description of tolerance-related issues in circuit design. It has been mentioned, why intervals occur in a natural manner. Also, major drawbacks for the application of classical interval methods in this context have been pointed out. After stating preceding numerical and interval-based approaches for tolerance analysis, the main targets of this work have been specified.

Secondly, a short introduction to interval arithmetic and derived algorithms has been given. This includes the treatment of very basic notations and properties, as well as previous algorithms for solving interval-valued systems of equations and some implementation notes. Also, we have described the dependency problem, which causes very conservative bounds to the solution sets.

It has been proven, that for circuits consisting of linear elements these result in matrix equations in fill-in pattern form with respect to tolerance-affected component parameters. This special matrix structure can be used for utilizing monotonic properties of the solution set, which is crucial for efficient solving of the linear system.

But still, this restricts interval analysis to the treatment of small circuits. In order to tackle this problem, a special solver for linear systems in fill-in pattern form has been designed in Section 4. This involved the Sherman-Morrison formula in two ways: On the one hand, it has been used to prove monotony properties of the solutions, on the other hand repeated application of the formula gives a constructive way for computing sufficiently tight bounds. The latter is of quadratic complexity in the number of interval-valued parameters. This approach has also been extended to complex-valued systems emerging from small-signal analysis of linear circuits. In addition, an alternative algorithm has been presented, which can be used to compute close bounds to the variance of the results in the complex plane.

The algorithms shown in this paper have been implemented and integrated in the framework of the circuit analysis tool Analog Insydes. The implementation is concisely documented and tested in Section 6. The performance of the new methods has been demonstrated for a frequency response analysis of a tone-control circuit and a real-world operational amplifier. It illustrates that the linear methods are capable of providing interpretable results for analog circuits of considerable size in suitable time. Since equivalent circuit diagrams are widely used for standard small-signal applications, this finds practical utilization in analog circuit analysis of real-world devices.

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Published reports of the Fraunhofer ITWM

The PDF-files of the following reports are available under:
www.itwm.fraunhofer.de/de/zentral__berichte/berichte

1. D. Hietel, K. Steiner, J. Struckmeier

A Finite - Volume Particle Method for Compressible Flows

We derive a new class of particle methods for conservation laws, which are based on numerical flux functions to model the interactions between moving particles. The derivation is similar to that of classical Finite-Volume methods; except that the fixed grid structure in the Finite-Volume method is substituted by so-called mass packets of particles. We give some numerical results on a shock wave solution for Burgers equation as well as the well-known one-dimensional shock tube problem.
(19 pages, 1998)

2. M. Feldmann, S. Seibold

Damage Diagnosis of Rotors: Application of Hilbert Transform and Multi-Hypothesis Testing

In this paper, a combined approach to damage diagnosis of rotors is proposed. The intention is to employ signal-based as well as model-based procedures for an improved detection of size and location of the damage. In a first step, Hilbert transform signal processing techniques allow for a computation of the signal envelope and the instantaneous frequency, so that various types of non-linearities due to a damage may be identified and classified based on measured response data. In a second step, a multi-hypothesis bank of Kalman Filters is employed for the detection of the size and location of the damage based on the information of the type of damage provided by the results of the Hilbert transform.
Keywords: Hilbert transform, damage diagnosis, Kalman filtering, non-linear dynamics
(23 pages, 1998)

3. Y. Ben-Haim, S. Seibold

Robust Reliability of Diagnostic Multi-Hypothesis Algorithms: Application to Rotating Machinery

Damage diagnosis based on a bank of Kalman filters, each one conditioned on a specific hypothesized system condition, is a well recognized and powerful diagnostic tool. This multi-hypothesis approach can be applied to a wide range of damage conditions. In this paper, we will focus on the diagnosis of cracks in rotating machinery. The question we address is: how to optimize the multi-hypothesis algorithm with respect to the uncertainty of the spatial form and location of cracks and their resulting dynamic effects. First, we formulate a measure of the reliability of the diagnostic algorithm, and then we discuss modifications of the diagnostic algorithm for the maximization of the reliability. The reliability of a diagnostic algorithm is measured by the amount of uncertainty consistent with no-failure of the diagnosis. Uncertainty is quantitatively represented with convex models.
Keywords: Robust reliability, convex models, Kalman filtering, multi-hypothesis diagnosis, rotating machinery, crack diagnosis
(24 pages, 1998)

4. F.-Th. Lentes, N. Siedow

Three-dimensional Radiative Heat Transfer in Glass Cooling Processes

For the numerical simulation of 3D radiative heat transfer in glasses and glass melts, practically applicable mathematical methods are needed to handle such problems optimal using workstation class computers. Since the exact solution would require super-computer capabilities we concentrate on approximate solutions with a high degree of accuracy. The following approaches are studied: 3D diffusion approximations and 3D ray-tracing methods.
(23 pages, 1998)

5. A. Klar, R. Wegener

A hierarchy of models for multilane vehicular traffic
Part I: Modeling

In the present paper multilane models for vehicular traffic are considered. A microscopic multilane model based on reaction thresholds is developed. Based on this model an Enskog like kinetic model is developed. In particular, care is taken to incorporate the correlations between the vehicles. From the kinetic model a fluid dynamic model is derived. The macroscopic coefficients are deduced from the underlying kinetic model. Numerical simulations are presented for all three levels of description in [10]. Moreover, a comparison of the results is given there.
(23 pages, 1998)

Part II: Numerical and stochastic investigations

In this paper the work presented in [6] is continued. The present paper contains detailed numerical investigations of the models developed there. A numerical method to treat the kinetic equations obtained in [6] are presented and results of the simulations are shown. Moreover, the stochastic correlation model used in [6] is described and investigated in more detail.
(17 pages, 1998)

6. A. Klar, N. Siedow

Boundary Layers and Domain Decomposition for Radiative Heat Transfer and Diffusion Equations: Applications to Glass Manufacturing Processes

In this paper domain decomposition methods for radiative transfer problems including conductive heat transfer are treated. The paper focuses on semi-transparent materials, like glass, and the associated conditions at the interface between the materials. Using asymptotic analysis we derive conditions for the coupling of the radiative transfer equations and a diffusion approximation. Several test cases are treated and a problem appearing in glass manufacturing processes is computed. The results clearly show the advantages of a domain decomposition approach. Accuracy equivalent to the solution of the global radiative transfer solution is achieved, whereas computation time is strongly reduced.
(24 pages, 1998)

7. I. Choquet

Heterogeneous catalysis modelling and numerical simulation in rarified gas flows
Part I: Coverage locally at equilibrium

A new approach is proposed to model and simulate numerically heterogeneous catalysis in rarefied gas flows. It is developed to satisfy all together the following points:
1) describe the gas phase at the microscopic scale, as required in rarefied flows,
2) describe the wall at the macroscopic scale, to avoid prohibitive computational costs and consider not only crystalline but also amorphous surfaces,
3) reproduce on average macroscopic laws correlated with experimental results and
4) derive analytic models in a systematic and exact way.
The problem is stated in the general framework of a non static flow in the vicinity of a catalytic and non porous surface (without aging). It is shown that the exact and systematic resolution method based on the Laplace transform, introduced previously by the author to model collisions in the gas phase, can be extended to the present problem. The proposed approach is applied to the modelling of the EleyRideal and LangmuirHinshelwood recombinations, assuming that the coverage is locally at equilibrium. The models are developed considering one atomic species and extended to the general case of several atomic species. Numerical calculations show that the models derived in this way reproduce with accuracy behaviors observed experimentally.
(24 pages, 1998)

8. J. Ohser, B. Steinbach, C. Lang

Efficient Texture Analysis of Binary Images

A new method of determining some characteristics of binary images is proposed based on a special linear filtering. This technique enables the estimation of the area fraction, the specific line length, and the specific integral of curvature. Furthermore, the specific length of the total projection is obtained, which gives detailed information about the texture of the image. The influence of lateral and directional resolution depending on the size of the applied filter mask is discussed in detail. The technique includes a method of increasing directional resolution for texture analysis while keeping lateral resolution as high as possible.
(17 pages, 1998)

9. J. Orlik

Homogenization for viscoelasticity of the integral type with aging and shrinkage

A multiphase composite with periodic distributed inclusions with a smooth boundary is considered in this contribution. The composite component materials are supposed to be linear viscoelastic and aging (of the nonconvolution integral type, for which the Laplace transform with respect to time is not effectively applicable) and are subjected to isotropic shrinkage. The free shrinkage deformation can be considered as a fictitious temperature deformation in the behavior law. The procedure presented in this paper proposes a way to determine average (effective homogenized) viscoelastic and shrinkage (temperature) composite properties and the homogenized stressfield from known properties of the components. This is done by the extension of the asymptotic homogenization technique known for pure elastic nonhomogeneous bodies to the nonhomogeneous thermoviscoelasticity of the integral nonconvolution type. Up to now, the homogenization theory has not covered viscoelasticity of the integral type. SanchezPalencia (1980), Francfort & Suquet (1987) (see [2], [9]) have considered homogenization for viscoelasticity of the differential form and only up to the first derivative order. The integralmodeled viscoelasticity is more general then the differential one and includes almost all known differential models. The homogenization procedure is based on the construction of an asymptotic solution with respect to a period of the composite structure. This reduces the original problem to some auxiliary boundary value problems of elasticity and viscoelasticity on the unit periodic cell, of the same type as the original non-homogeneous problem. The existence and uniqueness results for such problems were obtained for kernels satisfying some constrain conditions. This is done by the extension of the Volterra integral operator theory to the Volterra operators with respect to the time, whose 1 kernels are space linear operators for any fixed time variables. Some ideas of such approach were proposed in [11] and [12], where the Volterra operators with kernels depending additionally on parameter were considered. This manuscript delivers results of the same nature for the case of the spaceoperator kernels.
(20 pages, 1998)

10. J. Mohring

Helmholtz Resonators with Large Aperture

The lowest resonant frequency of a cavity resonator is usually approximated by the classical Helmholtz formula. However, if the opening is rather large and the front wall is narrow this formula is no longer valid. Here we present a correction which is of third order in the ratio of the diameters of aperture and cavity. In addition to the high accuracy it allows to estimate the damping due to radiation. The result is found by applying the method of matched asymptotic expansions. The correction contains form factors describing the shapes of opening and cavity. They are computed for a number of standard geometries. Results are compared with numerical computations.
(21 pages, 1998)

11. H. W. Hamacher, A. Schöbel

On Center Cycles in Grid Graphs

Finding "good" cycles in graphs is a problem of great interest in graph theory as well as in locational analysis. We show that the center and median problems are NP hard in general graphs. This result holds both for the variable cardinality case (i.e. all cycles of the graph are considered) and the fixed cardinality case (i.e. only cycles with a given cardinality p are feasible). Hence it is of interest to investigate special cases where the problem is solvable in polynomial time. In grid graphs, the variable cardinality case is, for instance, trivially solvable if the shape of the cycle can be chosen freely.
If the shape is fixed to be a rectangle one can analyze rectangles in grid graphs with, in sequence, fixed dimension, fixed cardinality, and variable cardinality. In all cases a complete characterization of the optimal cycles and closed form expressions of the optimal objective values are given, yielding polynomial time algorithms for all cases of center rectangle problems.
Finally, it is shown that center cycles can be chosen as rectangles for small cardinalities such that the center cycle problem in grid graphs is in these cases completely solved.
(15 pages, 1998)

12. H. W. Hamacher, K.-H. Küfer

Inverse radiation therapy planning - a multiple objective optimisation approach

For some decades radiation therapy has been proved successful in cancer treatment. It is the major task of clinical radiation treatment planning to realize on the one hand a high level dose of radiation in the cancer tissue in order to obtain maximum tumor control. On the other hand it is obvious that it is absolutely necessary to keep in the tissue outside the tumor, particularly in organs at risk, the unavoidable radiation as low as possible.
No doubt, these two objectives of treatment planning - high level dose in the tumor, low radiation outside the tumor - have a basically contradictory nature. Therefore, it is no surprise that inverse mathematical models with dose distribution bounds tend to be infeasible in most cases. Thus, there is need for approximations compromising between overdosing the organs at risk and underdosing the target volume.
Differing from the currently used time consuming iterative approach, which measures deviation from an ideal (non-achievable) treatment plan using recursively trial-and-error weights for the organs of interest, we go a new way trying to avoid a priori weight choices and consider the treatment planning problem as a multiple objective linear programming problem: with each organ of interest, target tissue as well as organs at risk, we associate an objective function measuring the maximal deviation from the prescribed doses.
We build up a data base of relatively few efficient solutions representing and approximating the variety of Pareto solutions of the multiple objective linear programming problem. This data base can be easily scanned by physicians looking for an adequate treatment plan with the aid of an appropriate online tool.
(14 pages, 1999)

13. C. Lang, J. Ohser, R. Hilfer

On the Analysis of Spatial Binary Images

This paper deals with the characterization of microscopically heterogeneous, but macroscopically homogeneous spatial structures. A new method is presented which is strictly based on integral-geometric formulae such as Crofton's intersection formulae and Hadwiger's recursive definition of the Euler number. The corresponding algorithms have clear advantages over other techniques. As an example of application we consider the analysis of spatial digital images produced by means of Computer Assisted Tomography.
(20 pages, 1999)

14. M. Junk

On the Construction of Discrete Equilibrium Distributions for Kinetic Schemes

A general approach to the construction of discrete equilibrium distributions is presented. Such distribution functions can be used to set up Kinetic Schemes as well as Lattice Boltzmann methods. The general principles are also applied to the construction of Chapman Enskog distributions which are used in Kinetic Schemes for compressible Navier-Stokes equations.
(24 pages, 1999)

15. M. Junk, S. V. Raghurame Rao

A new discrete velocity method for Navier-Stokes equations

The relation between the Lattice Boltzmann Method, which has recently become popular, and the Kinetic Schemes, which are routinely used in Computational Fluid Dynamics, is explored. A new discrete velocity model for the numerical solution of Navier-Stokes equations for incompressible fluid flow is presented by combining both the approaches. The new scheme can be interpreted as a pseudo-compressibility method and, for a particular choice of parameters, this interpretation carries over to the Lattice Boltzmann Method.
(20 pages, 1999)

16. H. Neunzert

Mathematics as a Key to Key Technologies

The main part of this paper will consist of examples, how mathematics really helps to solve industrial problems; these examples are taken from our Institute for Industrial Mathematics, from research in the Technomathematics group at my university, but also from ECMI groups and a company called TecMath, which originated 10 years ago from my university group and has already a very successful history.
(39 pages (4 PDF-Files), 1999)

17. J. Ohser, K. Sandau

Considerations about the Estimation of the Size Distribution in Wicksell's Corpuscle Problem

Wicksell's corpuscle problem deals with the estimation of the size distribution of a population of particles, all having the same shape, using a lower dimensional sampling probe. This problem was originary formulated for particle systems occurring in life sciences but its solution is of actual and increasing interest in materials science. From a mathematical point of view, Wicksell's problem is an inverse problem where the interesting size distribution is the unknown part of a Volterra equation. The problem is often regarded ill-posed, because the structure of the integrand implies unstable numerical solutions. The accuracy of the numerical solutions is considered here using the condition number, which allows to compare different numerical methods with different (equidistant) class sizes and which indicates, as one result, that a finite section thickness of the probe reduces the numerical problems. Furthermore, the relative error of estimation is computed which can be split into two parts. One part consists of the relative discretization error that increases for increasing class size, and the second part is related to the relative statistical error which increases with decreasing class size. For both parts, upper bounds can be given and the sum of them indicates an optimal class width depending on some specific constants.
(18 pages, 1999)

18. E. Carrizosa, H. W. Hamacher, R. Klein, S. Nickel

Solving nonconvex planar location problems by finite dominating sets

It is well-known that some of the classical location problems with polyhedral gauges can be solved in polynomial time by finding a finite dominating set, i. e. a finite set of candidates guaranteed to contain at least one optimal location.
In this paper it is first established that this result holds for a much larger class of problems than currently considered in the literature. The model for which this result can be proven includes, for instance, location problems with attraction and repulsion, and location-allocation problems.
Next, it is shown that the approximation of general gauges by polyhedral ones in the objective function of our general model can be analyzed with regard to the subsequent error in the optimal objective value. For the approximation problem two different approaches are described, the sandwich procedure and the greedy algorithm. Both of these approaches lead - for fixed epsilon - to polynomial approximation algorithms with accuracy epsilon for solving the general model considered in this paper.
Keywords: Continuous Location, Polyhedral Gauges, Finite Dominating Sets, Approximation, Sandwich Algorithm, Greedy Algorithm
(19 pages, 2000)

19. A. Becker

A Review on Image Distortion Measures

Within this paper we review image distortion measures. A distortion measure is a criterion that assigns a "quality number" to an image. We distinguish between mathematical distortion measures and those distortion measures in-cooperating a priori knowledge about the imaging devices (e. g. satellite images), image processing algorithms or the human physiology. We will consider representative examples of different kinds of distortion measures and are going to discuss them.
Keywords: Distortion measure, human visual system
(26 pages, 2000)

20. H. W. Hamacher, M. Labbé, S. Nickel, T. Sonneborn

Polyhedral Properties of the Uncapacitated Multiple Allocation Hub Location Problem

We examine the feasibility polyhedron of the uncapacitated hub location problem (UHL) with multiple allocation, which has applications in the fields of air passenger and cargo transportation, telecommunication and postal delivery services. In particular we determine the dimension and derive some classes of facets of this polyhedron. We develop some general rules about lifting facets from the uncapacitated facility location (UFL) for UHL and projecting facets from UHL to UFL. By applying these rules we get a new class of facets for UHL which dominates the inequalities in the original formulation. Thus we get a new formulation of UHL whose constraints are all facet–defining. We show its superior computational performance by benchmarking it on a well known data set.
Keywords: integer programming, hub location, facility location, valid inequalities, facets, branch and cut
(21 pages, 2000)

21. H. W. Hamacher, A. Schöbel

Design of Zone Tariff Systems in Public Transportation

Given a public transportation system represented by its stops and direct connections between stops, we consider two problems dealing with the prices for the customers: The fare problem in which subsets of stops are already aggregated to zones and "good" tariffs have to be found in the existing zone system. Closed form solutions for the fare problem are presented for three objective functions. In the zone problem the design of the zones is part of the problem. This problem is NP hard and we therefore propose three heuristics which prove to be very successful in the redesign of one of Germany's transportation systems.
(30 pages, 2001)

22. D. Hietel, M. Junk, R. Keck, D. Teleaga

The Finite-Volume-Particle Method for Conservation Laws

In the Finite-Volume-Particle Method (FVPM), the weak formulation of a hyperbolic conservation law is discretized by restricting it to a discrete set of test functions. In contrast to the usual Finite-Volume approach, the test functions are not taken as characteristic functions of the control volumes in a spatial grid, but are chosen from a partition of unity with smooth and overlapping partition functions (the particles), which can even move along pre- scribed velocity fields. The information exchange between particles is based on standard numerical flux functions. Geometrical information, similar to the surface area of the cell faces in the Finite-Volume Method and the corresponding normal directions are given as integral quantities of the partition functions. After a brief derivation of the Finite-Volume-Particle Method, this work focuses on the role of the geometric coefficients in the scheme.
(16 pages, 2001)

23. T. Bender, H. Hennes, J. Kalcsics, M. T. Melo, S. Nickel

Location Software and Interface with GIS and Supply Chain Management

The objective of this paper is to bridge the gap between location theory and practice. To meet this objective focus is given to the development of software capable of addressing the different needs of a wide group of users. There is a very active community on location theory encompassing many research fields such as operations research, computer science, mathematics, engineering, geography, economics and marketing. As a result, people working on facility location problems have a very diverse background and also different needs regarding the software to solve these problems. For those interested in non-commercial applications (e. g. students and researchers), the library of location algorithms (LoLA can be of considerable assistance. LoLA contains a collection of efficient algorithms for solving planar, network and discrete facility location problems. In this paper, a detailed description of the functionality of LoLA is presented. In the fields of geography and marketing, for instance, solving facility location problems requires using large amounts of demographic data. Hence, members of these groups (e. g. urban planners and sales managers) often work with geographical information too s. To address the specific needs of these users, LoLA was inked to a geographical information system (GIS) and the details of the combined functionality are described in the paper. Finally, there is a wide group of practitioners who need to solve large problems and require special purpose software with a good data interface. Many of such users can be found, for example, in the area of supply chain management (SCM). Logistics activities involved in strategic SCM include, among others, facility location planning. In this paper, the development of a commercial location software tool is also described. The too is embedded in the Advanced Planner and Optimizer SCM software developed by SAP AG, Walldorf, Germany. The paper ends with some conclusions and an outlook to future activities.
Keywords: facility location, software development, geographical information systems, supply chain management
(48 pages, 2001)

24. H. W. Hamacher, S. A. Tjandra

Mathematical Modelling of Evacuation Problems: A State of Art

This paper details models and algorithms which can be applied to evacuation problems. While it concentrates on building evacuation many of the results are applicable also to regional evacuation. All models consider the time as main parameter, where the travel time between components of the building is part of the input and the overall evacuation time is the output. The paper distinguishes between macroscopic and microscopic evacuation models both of which are able to capture the evacuees' movement over time.
Macroscopic models are mainly used to produce good lower bounds for the evacuation time and do not consider any individual behavior during the emergency situation. These bounds can be used to analyze existing buildings or help in the design phase of planning a building. Macroscopic approaches which are based on dynamic network flow models (minimum cost dynamic flow, maximum dynamic flow, universal maximum flow, quickest path and quickest flow) are described. A special feature of the presented approach is the fact, that travel times of evacuees are not restricted to be constant, but may be density dependent. Using multicriteria optimization priority regions and blockage due to fire or smoke may be considered. It is shown how the modelling can be done using time parameter either as discrete or continuous parameter.
Microscopic models are able to model the individual evacuee's characteristics and the interaction among evacuees which influence their movement. Due to the corresponding huge amount of data one uses simulation approaches. Some probabilistic laws for individual evacuee's movement are presented. Moreover ideas to model the evacuee's movement using cellular automata (CA) and resulting software are presented.
In this paper we will focus on macroscopic models and only summarize some of the results of the microscopic approach. While most of the results are applicable to general evacuation situations, we concentrate on building evacuation.
(44 pages, 2001)

25. J. Kuhnert, S. Tiwari

Grid free method for solving the Poisson equation

A Grid free method for solving the Poisson equation is presented. This is an iterative method. The method is based on the weighted least squares approximation in which the Poisson equation is enforced to be satisfied in every iterations. The boundary conditions can also be enforced in the iteration process. This is a local approximation procedure. The Dirichlet, Neumann and mixed boundary value problems on a unit square are presented and the analytical solutions are compared with the exact solutions. Both solutions matched perfectly.
Keywords: Poisson equation, Least squares method, Grid free method
(19 pages, 2001)

26. T. Götz, H. Rave, D. Reinel-Bitzer, K. Steiner, H. Tiemeier

Simulation of the fiber spinning process

To simulate the influence of process parameters to the melt spinning process a fiber model is used and coupled with CFD calculations of the quench air flow. In the fiber model energy, momentum and mass balance are solved for the polymer mass flow. To calculate the quench air the Lattice Boltzmann method is used. Simulations and experiments for different process parameters and hole configurations are compared and show a good agreement.
Keywords: Melt spinning, fiber model, Lattice Boltzmann, CFD
(19 pages, 2001)

27. A. Zemitis

On interaction of a liquid film with an obstacle

In this paper mathematical models for liquid films generated by impinging jets are discussed. Attention is stressed to the interaction of the liquid film with some obstacle. S. G. Taylor [Proc. R. Soc. London Ser. A 253, 313 (1959)] found that the liquid film generated by impinging jets is very sensitive to properties of the wire which was used as an obstacle. The aim of this presentation is to propose a modification of the Taylor's model, which allows to simulate the film shape in cases, when the angle between jets is different from 180°. Numerical results obtained by discussed models give two different shapes of the liquid film similar as in Taylors experiments. These two shapes depend on the regime: either droplets are produced close to the obstacle or not. The difference between two regimes becomes larger if the angle between jets decreases. Existence of such two regimes can be very essential for some applications of impinging jets, if the generated liquid film can have a contact with obstacles.
Keywords: impinging jets, liquid film, models, numerical solution, shape
(22 pages, 2001)

28. I. Ginzburg, K. Steiner

Free surface lattice-Boltzmann method to model the filling of expanding cavities by Bingham Fluids

The filling process of viscoplastic metal alloys and plastics in expanding cavities is modelled using the lattice Boltzmann method in two and three dimensions. These models combine the regularized Bingham model for viscoplastic with a free-interface algorithm. The latter is based on a modified immiscible lattice Boltzmann model in which one species is the fluid and the other one is considered as vacuum. The boundary conditions at the curved liquid-vacuum interface are met without any geometrical front reconstruction from a first-order Chapman-Enskog expansion. The numerical results obtained with these models are found in good agreement with available theoretical and numerical analysis.
Keywords: Generalized LBE, free-surface phenomena, interface boundary conditions, filling processes, Bingham viscoplastic model, regularized models
(22 pages, 2001)

29. H. Neunzert

»Denn nichts ist für den Menschen als Menschen etwas wert, was er nicht mit Leidenschaft tun kann«
Vortrag anlässlich der Verleihung des Akademiepreises des Landes Rheinland-Pfalz am 21.11.2001

Was macht einen guten Hochschullehrer aus? Auf diese Frage gibt es sicher viele verschiedene, fachbezogene Antworten, aber auch ein paar allgemeine Gesichtspunkte: es bedarf der »Leidenschaft« für die Forschung (Max Weber), aus der dann auch die Begeisterung für die Lehre erwächst. Forschung und Lehre gehören zusammen, um die Wissenschaft als lebendiges Tun vermitteln zu können. Der Vortrag gibt Beispiele dafür, wie in angewandter Mathematik Forschungsaufgaben aus praktischen Alltagsproblemstellungen erwachsen, die in die Lehre auf verschiedenen Stufen (Gymnasium bis Graduiertenkolleg) einfließen; er leitet damit auch zu einem aktuellen Forschungsgebiet, der Mehrskalenanalyse mit ihren vielfältigen Anwendungen in Bildverarbeitung, Materialentwicklung und Strömungsmechanik über, was aber nur kurz gestreift wird. Mathematik erscheint hier als eine moderne Schlüsseltechnologie, die aber auch enge Beziehungen zu den Geistes- und Sozialwissenschaften hat.
Keywords: Lehre, Forschung, angewandte Mathematik, Mehrskalenanalyse, Strömungsmechanik
(18 pages, 2001)

30. J. Kuhnert, S. Tiwari

Finite pointset method based on the projection method for simulations of the incompressible Navier-Stokes equations

A Lagrangian particle scheme is applied to the projection method for the incompressible Navier-Stokes equations. The approximation of spatial derivatives is obtained by the weighted least squares method. The pressure Poisson equation is solved by a local iterative procedure with the help of the least squares method. Numerical tests are performed for two dimensional cases. The Couette flow, Poiseuelle flow, decaying shear flow and the driven cavity flow are presented. The numerical solutions are obtained for stationary as well as instationary cases and are compared with the analytical solutions for channel flows. Finally, the driven cavity in a unit square is considered and the stationary solution obtained from this scheme is compared with that from the finite element method.
Keywords: Incompressible Navier-Stokes equations, Meshfree method, Projection method, Particle scheme, Least squares approximation
AMS subject classification: 76D05, 76M28
(25 pages, 2001)

31. R. Korn, M. Krekel

Optimal Portfolios with Fixed Consumption or Income Streams

We consider some portfolio optimisation problems where either the investor has a desire for an a priori specified consumption stream or/and follows a deterministic pay in scheme while also trying to maximize expected utility from final wealth. We derive explicit closed form solutions for continuous and discrete monetary streams. The mathematical method used is classical stochastic control theory.
Keywords: Portfolio optimisation, stochastic control, HJB equation, discretisation of control problems.
(23 pages, 2002)

32. M. Krekel

Optimal portfolios with a loan dependent credit spread

If an investor borrows money he generally has to pay higher interest rates than he would have received, if he had put his funds on a savings account. The classical model of continuous time portfolio optimisation ignores this effect. Since there is obviously a connection between the default probability and the total per-

centage of wealth, which the investor is in debt, we study portfolio optimisation with a control dependent interest rate. Assuming a logarithmic and a power utility function, respectively, we prove explicit formulae of the optimal control.
Keywords: Portfolio optimisation, stochastic control, HJB equation, credit spread, log utility, power utility, non-linear wealth dynamics
(25 pages, 2002)

33. J. Ohser, W. Nagel, K. Schladitz

The Euler number of discretized sets - on the choice of adjacency in homogeneous lattices

Two approaches for determining the Euler-Poincaré characteristic of a set observed on lattice points are considered in the context of image analysis { the integral geometric and the polyhedral approach. Information about the set is assumed to be available on lattice points only. In order to retain properties of the Euler number and to provide a good approximation of the true Euler number of the original set in the Euclidean space, the appropriate choice of adjacency in the lattice for the set and its background is crucial. Adjacencies are defined using tessellations of the whole space into polyhedrons. In R 3 , two new 14 adjacencies are introduced additionally to the well known 6 and 26 adjacencies. For the Euler number of a set and its complement, a consistency relation holds. Each of the pairs of adjacencies (14:1; 14:1), (14:2; 14:2), (6; 26), and (26; 6) is shown to be a pair of complementary adjacencies with respect to this relation. That is, the approximations of the Euler numbers are consistent if the set and its background (complement) are equipped with this pair of adjacencies. Furthermore, sufficient conditions for the correctness of the approximations of the Euler number are given. The analysis of selected microstructures and a simulation study illustrate how the estimated Euler number depends on the chosen adjacency. It also shows that there is not a uniquely best pair of adjacencies with respect to the estimation of the Euler number of a set in Euclidean space.
Keywords: image analysis, Euler number, neighborhod relationships, cuboidal lattice
(32 pages, 2002)

34. I. Ginzburg, K. Steiner

Lattice Boltzmann Model for Free-Surface flow and Its Application to Filling Process in Casting

A generalized lattice Boltzmann model to simulate free-surface is constructed in both two and three dimensions. The proposed model satisfies the interfacial boundary conditions accurately. A distinctive feature of the model is that the collision processes is carried out only on the points occupied partially or fully by the fluid. To maintain a sharp interfacial front, the method includes an anti-diffusion algorithm. The unknown distribution functions at the interfacial region are constructed according to the first order Chapman-Enskog analysis. The interfacial boundary conditions are satisfied exactly by the coefficients in the Chapman-Enskog expansion. The distribution functions are naturally expressed in the local interfacial coordinates. The macroscopic quantities at the interface are extracted from the least-square solutions of a locally linearized system obtained from the known distribution functions. The proposed method does not require any geometric front construction and is robust for any interfacial topology. Simulation results of realistic filling process are presented: rectangular cavity in two dimensions and Hammer box, Campbell box, Sheffield box, and Motorblock in three dimensions. To enhance the stability at high Reynolds numbers, various upwind-type schemes are developed. Free-slip and no-slip boundary conditions are also discussed.
Keywords: Lattice Boltzmann models; free-surface phenomena; interface boundary conditions; filling processes; injection molding; volume of fluid method; interface boundary conditions; advection-schemes; upwind-schemes
(54 pages, 2002)

35. M. Günther, A. Klar, T. Materne, R. Wegener

Multivalued fundamental diagrams and stop and go waves for continuum traffic equations

In the present paper a kinetic model for vehicular traffic leading to multivalued fundamental diagrams is developed and investigated in detail. For this model phase transitions can appear depending on the local density and velocity of the flow. A derivation of associated macroscopic traffic equations from the kinetic equation is given. Moreover, numerical experiments show the appearance of stop and go waves for highway traffic with a bottleneck.
Keywords: traffic flow, macroscopic equations, kinetic derivation, multivalued fundamental diagram, stop and go waves, phase transitions
(25 pages, 2002)

36. S. Feldmann, P. Lang, D. Prätzel-Wolters

Parameter influence on the zeros of network determinants

To a network N(q) with determinant D(s;q) depending on a parameter vector q Î Rr via identification of some of its vertices, a network N^ (q) is assigned. The paper deals with procedures to find N^ (q), such that its determinant D^ (s;q) admits a factorization in the determinants of appropriate subnetworks, and with the estimation of the deviation of the zeros of D^ from the zeros of D. To solve the estimation problem state space methods are applied.
Keywords: Networks, Equicofactor matrix polynomials, Realization theory, Matrix perturbation theory
(30 pages, 2002)

37. K. Koch, J. Ohser, K. Schladitz

Spectral theory for random closed sets and estimating the covariance via frequency space

A spectral theory for stationary random closed sets is developed and provided with a sound mathematical basis. Definition and proof of existence of the Bartlett spectrum of a stationary random closed set as well as the proof of a Wiener-Khintchine theorem for the power spectrum are used to two ends: First, well known second order characteristics like the covariance can be estimated faster than usual via frequency space. Second, the Bartlett spectrum and the power spectrum can be used as second order characteristics in frequency space. Examples show, that in some cases information about the random closed set is easier to obtain from these characteristics in frequency space than from their real world counterparts.
Keywords: Random set, Bartlett spectrum, fast Fourier transform, power spectrum
(28 pages, 2002)

38. D. d'Humières, I. Ginzburg

Multi-reflection boundary conditions for lattice Boltzmann models

We present a unified approach of several boundary conditions for lattice Boltzmann models. Its general framework is a generalization of previously introduced schemes such as the bounce-back rule, linear or quadratic interpolations, etc. The objectives are two fold: first to give theoretical tools to study the existing boundary conditions and their corresponding accuracy; secondly to design formally third- order accurate boundary conditions for general flows. Using these boundary conditions, Couette and Poiseuille flows are exact solution of the lattice Boltzmann models for a Reynolds number Re = 0 (Stokes limit).
Numerical comparisons are given for Stokes flows in periodic arrays of spheres and cylinders, linear periodic array of cylinders between moving plates and for Navier-Stokes flows in periodic arrays of cylinders for Re < 200. These results show a significant improvement of the overall accuracy when using the linear interpolations instead of the bounce-back reflection (up to an order of magnitude on the hydrodynamics fields). Further improvement is achieved with the new multi-reflection boundary conditions, reaching a level of accuracy close to the quasi-analytical reference solutions, even for rather modest grid resolutions and few points in the narrowest channels. More important, the pressure and velocity fields in the vicinity of the obstacles are much smoother with multi-reflection than with the other boundary conditions.
Finally the good stability of these schemes is highlighted by some simulations of moving obstacles: a cylinder between flat walls and a sphere in a cylinder.
Keywords: lattice Boltzmann equation, boudary condistions, bounce-back rule, Navier-Stokes equation
(72 pages, 2002)

39. R. Korn

Elementare Finanzmathematik

Im Rahmen dieser Arbeit soll eine elementar gehaltene Einführung in die Aufgabenstellungen und Prinzipien der modernen Finanzmathematik gegeben werden. Insbesondere werden die Grundlagen der Modellierung von Aktienkursen, der Bewertung von Optionen und der Portfolio-Optimierung vorgestellt. Natürlich können die verwendeten Methoden und die entwickelte Theorie nicht in voller Allgemeinheit für den Schulunterricht verwendet werden, doch sollen einzelne Prinzipien so heraus gearbeitet werden, dass sie auch an einfachen Beispielen verstanden werden können.
Keywords: Finanzmathematik, Aktien, Optionen, Portfolio-Optimierung, Börse, Lehrerweiterbildung, Mathematikunterricht
(98 pages, 2002)

40. J. Kallrath, M. C. Müller, S. Nickel

Batch Presorting Problems: Models and Complexity Results

In this paper we consider short term storage systems. We analyze presorting strategies to improve the effiency of these storage systems. The presorting task is called Batch PreSorting Problem (BPSP). The BPSP is a variation of an assigment problem, i. e., it has an assigment problem kernel and some additional constraints. We present different types of these presorting problems, introduce mathematical programming formulations and prove the NP-completeness for one type of the BPSP. Experiments are carried out in order to compare the different model formulations and to investigate the behavior of these models.
Keywords: Complexity theory, Integer programming, Assigment, Logistics
(19 pages, 2002)

41. J. Linn

On the frame-invariant description of the phase space of the Folgar-Tucker equation

The Folgar-Tucker equation is used in flow simulations of fiber suspensions to predict fiber orientation depending on the local flow. In this paper, a complete, frame-invariant description of the phase space of this differential equation is presented for the first time.
Key words: fiber orientation, Folgar-Tucker equation, injection molding
(5 pages, 2003)

42. T. Hanne, S. Nickel

A Multi-Objective Evolutionary Algorithm for Scheduling and Inspection Planning in Software Development Projects

In this article, we consider the problem of planning inspections and other tasks within a software development (SD) project with respect to the objectives quality (no. of defects), project duration, and costs. Based on a discrete-event simulation model of SD processes comprising the phases coding, inspection, test, and rework, we present a simplified formulation of the problem as a multiobjective optimization problem. For solving the problem (i. e. finding an approximation of the efficient set) we develop a multiobjective evolutionary algorithm. Details of the algorithm are discussed as well as results of its application to sample problems.
Key words: multiple objective programming, project management and scheduling, software development, evolutionary algorithms, efficient set
(29 pages, 2003)

43. T. Bortfeld , K.-H. Küfer, M. Monz, A. Scherrer, C. Thieke, H. Trinkaus

Intensity-Modulated Radiotherapy - A Large Scale Multi-Criteria Programming Problem -

Radiation therapy planning is always a tight rope walk between dangerous insufficient dose in the target volume and life threatening overdosing of organs at risk. Finding ideal balances between these inherently contradictory goals challenges dosimetrists and physicians in their daily practice. Today's planning systems are typically based on a single evaluation function that measures the quality of a radiation treatment plan. Unfortunately, such a one dimensional approach cannot satisfactorily map the different backgrounds of physicians and the patient dependent necessities. So, too often a time consuming iteration process between evaluation of dose distribution and redefinition of the evaluation function is needed.
In this paper we propose a generic multi-criteria approach based on Pareto's solution concept. For each entity of interest - target volume or organ at risk a structure dependent evaluation function is defined measuring deviations from ideal doses that are calculated from statistical functions. A reasonable bunch of clinically meaningful Pareto optimal solutions are stored in a data base, which can be interactively searched by physicians. The system guarantees dynamical planning as well as the discussion of tradeoffs between different entities.
Mathematically, we model the upcoming inverse problem as a multi-criteria linear programming problem. Because of the large scale nature of the problem it is not possible to solve the problem in a 3D-setting without adaptive reduction by appropriate approximation schemes.
Our approach is twofold: First, the discretization of the continuous problem is based on an adaptive hierarchical clustering process which is used for a local refinement of constraints during the optimization procedure. Second, the set of Pareto optimal solutions is approximated by an adaptive grid of representatives that are found by a hybrid process of calculating extreme compromises and interpolation methods.
Keywords: multiple criteria optimization, representative systems of Pareto solutions, adaptive triangulation, clustering and disaggregation techniques, visualization of Pareto solutions, medical physics, external beam radiotherapy planning, intensity modulated radiotherapy
(31 pages, 2003)

44. T. Halfmann, T. Wichmann

Overview of Symbolic Methods in Industrial Analog Circuit Design

Industrial analog circuits are usually designed using numerical simulation tools. To obtain a deeper circuit understanding, symbolic analysis techniques can additionally be applied. Approximation methods which reduce the complexity of symbolic expressions are needed in order to handle industrial-sized problems.
This paper will give an overview to the field of symbolic analog circuit analysis. Starting with a motivation, the state-of-the-art simplification algorithms for linear as well as for nonlinear circuits are presented. The basic ideas behind the different techniques are described, whereas the technical details can be found in the cited references. Finally, the application of linear and nonlinear symbolic analysis will be shown on two example circuits.
Keywords: CAD, automated analog circuit design, symbolic analysis, computer algebra, behavioral modeling, system simulation, circuit sizing, macro modeling, differential-algebraic equations, index
(17 pages, 2003)

45. S. E. Mikhailov, J. Orlik

Asymptotic Homogenisation in Strength and Fatigue Durability Analysis of Composites

Asymptotic homogenisation technique and two-scale convergence is used for analysis of macro-strength and fatigue durability of composites with a periodic structure under cyclic loading. The linear damage accumulation rule is employed in the phenomenological micro-durability conditions (for each component of the composite) under varying cyclic loading. Both local and non-local strength and durability conditions are analysed. The strong convergence of the strength and fatigue damage measure as the structure period tends to zero is proved and their limiting values are estimated.
Keywords: multiscale structures, asymptotic homogenization, strength, fatigue, singularity, non-local conditions
(14 pages, 2003)

46. P. Domínguez-Marín, P. Hansen, N. Mladenovi ́c , S. Nickel

Heuristic Procedures for Solving the Discrete Ordered Median Problem

We present two heuristic methods for solving the Discrete Ordered Median Problem (DOMP), for which no such approaches have been developed so far. The DOMP generalizes classical discrete facility location problems, such as the p-median, p-center and Uncapacitated Facility Location problems. The first procedure proposed in this paper is based on a genetic algorithm developed by Moreno Vega [MV96] for p-median and p-center problems. Additionally, a second heuristic approach based on the Variable Neighborhood Search metaheuristic (VNS) proposed by Hansen & Mladenovic [HM97] for the p-median problem is described. An extensive numerical study is presented to show the efficiency of both heuristics and compare them.
Keywords: genetic algorithms, variable neighborhood search, discrete facility location
(31 pages, 2003)

47. N. Boland, P. Domínguez-Marín, S. Nickel, J. Puerto

Exact Procedures for Solving the Discrete Ordered Median Problem

The Discrete Ordered Median Problem (DOMP) generalizes classical discrete location problems, such as the N-median, N-center and Uncapacitated Facility Location problems. It was introduced by Nickel [16], who formulated it as both a nonlinear and a linear integer program. We propose an alternative integer linear programming formulation for the DOMP, discuss relationships between both integer linear programming formulations, and show how properties of optimal solutions can be used to strengthen these formulations. Moreover, we present a specific branch and bound procedure to solve the DOMP more efficiently. We test the integer linear programming formulations and this branch and bound method computationally on randomly generated test problems.
Keywords: discrete location, Integer programming
(41 pages, 2003)

48. S. Feldmann, P. Lang

Padé-like reduction of stable discrete linear systems preserving their stability

A new stability preserving model reduction algorithm for discrete linear SISO-systems based on their impulse response is proposed. Similar to the Padé approximation, an equation system for the Markov parameters involving the Hankel matrix is considered, that here however is chosen to be of very high dimension. Although this equation system therefore in general cannot be solved exactly, it is proved that the approximate solution, computed via the Moore-Penrose inverse, gives rise to a stability preserving reduction scheme, a property that cannot be guaranteed for the Padé approach. Furthermore, the proposed algorithm is compared to another stability preserving reduction approach, namely the balanced truncation method, showing comparable performance of the reduced systems. The balanced truncation method however starts from a state space description of the systems and in general is expected to be more computational demanding.
Keywords: Discrete linear systems, model reduction, stability, Hankel matrix, Stein equation
(16 pages, 2003)

49. J. Kallrath, S. Nickel

A Polynomial Case of the Batch Presorting Problem

This paper presents new theoretical results for a special case of the batch presorting problem (BPSP). We will show tht this case can be solved in polynomial time. Offline and online algorithms are presented for solving the BPSP. Competetive analysis is used for comparing the algorithms.
Keywords: batch presorting problem, online optimization, competetive analysis, polynomial algorithms, logistics
(17 pages, 2003)

50. T. Hanne, H. L. Trinkaus

knowCube for MCDM – Visual and Interactive Support for Multicriteria Decision Making

In this paper, we present a novel multicriteria decision support system (MCDSS), called knowCube, consisting of components for knowledge organization, generation, and navigation. Knowledge organization rests upon a database for managing qualitative and quantitative criteria, together with add-on information. Knowledge generation serves filling the database via e. g. identification, optimization, classification or simulation. For "finding needles in haycocks", the knowledge navigation component supports graphical database retrieval and interactive, goal-oriented problem solving. Navigation "helpers" are, for instance, cascading criteria aggregations, modifiable metrics, ergonomic interfaces, and customizable visualizations. Examples from real-life projects, e.g. in industrial engineering and in the life sciences, illustrate the application of our MCDSS.
Key words: Multicriteria decision making, knowledge management, decision support systems, visual interfaces, interactive navigation, real-life applications.
(26 pages, 2003)

51. O. Iliev, V. Laptev

On Numerical Simulation of Flow Through Oil Filters

This paper concerns numerical simulation of flow through oil filters. Oil filters consist of filter housing (filter box), and a porous filtering medium, which completely separates the inlet from the outlet. We discuss mathematical models, describing coupled flows in the pure liquid subregions and in the porous filter media, as well as interface conditions between them. Further, we reformulate the problem in fictitious regions method manner, and discuss peculiarities of the numerical algorithm in solving the coupled system. Next, we show numerical results, validating the model and the algorithm. Finally, we present results from simulation of 3-D oil flow through a real car filter.
Keywords: oil filters, coupled flow in plain and porous media, Navier-Stokes, Brinkman, numerical simulation
(8 pages, 2003)

52. W. Dörfler, O. Iliev, D. Stoyanov, D. Vassileva

On a Multigrid Adaptive Refinement Solver for Saturated Non-Newtonian Flow in Porous Media

A multigrid adaptive refinement algorithm for non-Newtonian flow in porous media is presented. The saturated flow of a non-Newtonian fluid is described by the continuity equation and the generalized Darcy law. The resulting second order nonlinear elliptic equation is discretized by a finite volume method on a cell-centered grid. A nonlinear full-multigrid, full-approximation-storage algorithm is implemented. As a smoother, a single grid solver based on Picard linearization and Gauss-Seidel relaxation is used. Further, a local refinement multigrid algorithm on a composite grid is developed. A residual based error indicator is used in the adaptive refinement criterion. A special implementation approach is used, which allows us to perform unstructured local refinement in conjunction with the finite volume discretization. Several results from numerical experiments are presented in order to examine the performance of the solver.
Keywords: Nonlinear multigrid, adaptive refinement, non-Newtonian flow in porous media
(17 pages, 2003)

53. S. Kruse

On the Pricing of Forward Starting Options under Stochastic Volatility

We consider the problem of pricing European forward starting options in the presence of stochastic volatility. By performing a change of measure using the asset

price at the time of strike determination as a numeraire, we derive a closed-form solution based on Heston's model of stochastic volatility.
Keywords: Option pricing, forward starting options, Heston model, stochastic volatility, cliquet options
(11 pages, 2003)

54. O. Iliev, D. Stoyanov
Multigrid – adaptive local refinement solver for incompressible flows
A non-linear multigrid solver for incompressible Navier-Stokes equations, exploiting finite volume discretization of the equations, is extended by adaptive local refinement. The multigrid is the outer iterative cycle, while the SIMPLE algorithm is used as a smoothing procedure. Error indicators are used to define the refinement subdomain. A special implementation approach is used, which allows to perform unstructured local refinement in conjunction with the finite volume discretization. The multigrid - adaptive local refinement algorithm is tested on 2D Poisson equation and further is applied to a lid-driven flows in a cavity (2D and 3D case), comparing the results with bench-mark data. The software design principles of the solver are also discussed.
Keywords: Navier-Stokes equations, incompressible flow, projection-type splitting, SIMPLE, multigrid methods, adaptive local refinement, lid-driven flow in a cavity
(37 pages, 2003)

55. V. Starikovicius
The multiphase flow and heat transfer in porous media
In first part of this work, summaries of traditional Multiphase Flow Model and more recent Multiphase Mixture Model are presented. Attention is being paid to attempts include various heterogeneous aspects into models. In second part, MMM based differential model for two-phase immiscible flow in porous media is considered. A numerical scheme based on the sequential solution procedure and control volume based finite difference schemes for the pressure and saturation-conservation equations is developed. A computer simulator is built, which exploits object-oriented programming techniques. Numerical result for several test problems are reported.
Keywords: Two-phase flow in porous media, various formulations, global pressure, multiphase mixture model, numerical simulation
(30 pages, 2003)

56. P. Lang, A. Sarishvili, A. Wirsen
Blocked neural networks for knowledge extraction in the software development process
One of the main goals of an organization developing software is to increase the quality of the software while at the same time to decrease the costs and the duration of the development process. To achieve this, various decisions e.ecting this goal before and during the development process have to be made by the managers. One appropriate tool for decision support are simulation models of the software life cycle, which also help to understand the dynamics of the software development process. Building up a simulation model requires a mathematical description of the interactions between di.erent objects involved in the development process. Based on experimental data, techniques from the .eld of knowledge discovery can be used to quantify these interactions and to generate new process knowledge based on the analysis of the determined relationships. In this paper blocked neuronal networks and related relevance measures will be presented as an appropriate tool for quanti.cation and validation of qualitatively known dependencies in the software development process.
Keywords: Blocked Neural Networks, Nonlinear Regression, Knowledge Extraction, Code Inspection
(21 pages, 2003)

57. H. Knaf, P. Lang, S. Zeiser
Diagnosis aiding in Regulation Thermography using Fuzzy Logic
The objective of the present article is to give an overview of an application of Fuzzy Logic in Regulation Thermography, a method of medical diagnosis support. An introduction to this method of the complementary medical science based on temperature measurements – so-called thermograms – is provided. The process of modelling the physician's thermogram evaluation rules using the calculus of Fuzzy Logic is explained.
Keywords: fuzzy logic,knowledge representation, expert system
(22 pages, 2003)

58. M.T. Melo, S. Nickel, F. Saldanha da Gama
Largescale models for dynamic multicommodity capacitated facility location
In this paper we focus on the strategic design of supply chain networks. We propose a mathematical modeling framework that captures many practical aspects of network design problems simultaneously but which have not received adequate attention in the literature. The aspects considered include: dynamic planning horizon, generic supply chain network structure, external supply of materials, inventory opportunities for goods, distribution of commodities, facility configuration, availability of capital for investments, and storage limitations. Moreover, network configuration decisions concerning the gradual relocation of facilities over the planning horizon are considered. To cope with fluctuating demands, capacity expansion and reduction scenarios are also analyzed as well as modular capacity shifts. The relation of the proposed modeling framework with existing models is discussed. For problems of reasonable size we report on our computational experience with standard mathematical programming software. In particular, useful insights on the impact of various factors on network design decisions are provided.
Keywords: supply chain management, strategic planning, dynamic location, modeling
(40 pages, 2003)

59. J. Orlik
Homogenization for contact problems with periodically rough surfaces
We consider the contact of two elastic bodies with rough surfaces at the interface. The size of the micropeaks and valleys is very small compared with the macrosize of the bodies' domains. This makes the direct application of the FEM for the calculation of the contact problem prohibitively costly. A method is developed that allows deriving a macrocontact condition on the interface. The method involves the twoscale asymptotic homogenization procedure that takes into account the microgeometry of the interface layer and the stiffnesses of materials of both domains. The macrocontact condition can then be used in a FEM model for the contact problem on the macrolevel. The averaged contact stiffness obtained allows the replacement of the interface layer in the macromodel by the macrocontact condition.
Keywords: asymptotic homogenization, contact problems
(28 pages, 2004)

60. A. Scherrer, K.-H. Küfer, M. Monz, F. Alonso, T. Bortfeld
IMRT planning on adaptive volume structures – a significant advance of computational complexity
In intensity-modulated radiotherapy (IMRT) planning the oncologist faces the challenging task of finding a treatment plan that he considers to be an ideal compromise of the inherently contradictive goals of delivering a sufficiently high dose to the target while widely sparing critical structures. The search for this a priori unknown compromise typically requires the computation of several plans, i.e. the solution of several optimization problems. This accumulates to a high computational expense due to the large scale of these problems - a consequence of the discrete problem formulation. This paper presents the adaptive clustering method as a new algorithmic concept to overcome these difficulties. The computations are performed on an individually adapted structure of voxel clusters rather than on the original voxels leading to a decisively reduced computational complexity as numerical examples on real clinical data demonstrate. In contrast to many other similar concepts, the typical trade-off between a reduction in computational complexity and a loss in exactness can be avoided: the adaptive clustering method produces the optimum of the original problem. This flexible method can be applied to both single- and multi-criteria optimization methods based on most of the convex evaluation functions used in practice.
Keywords: Intensity-modulated radiation therapy (IMRT), inverse treatment planning, adaptive volume structures, hierarchical clustering, local refinement, adaptive clustering, convex programming, mesh generation, multi-grid methods
(24 pages, 2004)

61. D. Kehrwald
Parallel lattice Boltzmann simulation of complex flows
After a short introduction to the basic ideas of lattice Boltzmann methods and a brief description of a modern parallel computer, it is shown how lattice Boltzmann schemes are successfully applied for simulating fluid flow in microstructures and calculating material properties of porous media. It is explained how lattice Boltzmann schemes compute the gradient of the velocity field without numerical differentiation. This feature is then utilised for the simulation of pseudo-plastic fluids, and numerical results are presented for a simple benchmark problem as well as for the simulation of liquid composite moulding.
Keywords: Lattice Boltzmann methods, parallel computing, microstructure simulation, virtual material design, pseudo-plastic fluids, liquid composite moulding
(12 pages, 2004)

62. O. Iliev, J. Linn, M. Moog, D. Niedziela, V. Starikovicius
On the Performance of Certain Iterative Solvers for Coupled Systems Arising in Discretization of Non-Newtonian Flow Equations
Iterative solution of large scale systems arising after discretization and linearization of the unsteady non-Newtonian Navier–Stokes equations is studied. cross WLF model is used to account for the non-Newtonian behavior of the fluid. Finite volume method is used to discretize the governing system of PDEs. Viscosity is treated explicitely (e.g., it is taken from the previous time step), while other terms are treated implicitly. Different preconditioners (block–diagonal, block–triangular, relaxed incomplete LU factorization, etc.) are used in conjunction with advanced iterative methods, namely, BiCGStab, CGS, GMRES. The action of the preconditioner in fact requires inverting different blocks. For this purpose, in addition to preconditioned BiCGStab, CGS, GMRES, we use also algebraic multigrid method (AMG). The performance of the iterative solvers is studied with respect to the number of unknowns, characteristic velocity in the basic flow, time step, deviation from Newtonian behavior, etc. Results from numerical experiments are presented and discussed.
Keywords: Performance of iterative solvers, Preconditioners, Non-Newtonian flow
(17 pages, 2004)

63. R. Ciegis, O. Iliev, S. Rief, K. Steiner
On Modelling and Simulation of Different Regimes for Liquid Polymer Moulding
In this paper we consider numerical algorithms for solving a system of nonlinear PDEs arising in modeling of liquid polymer injection. We investigate the particular case when a porous preform is located within the mould, so that the liquid polymer flows through a porous medium during the filling stage. The nonlinearity of the governing system of PDEs is due to the non-Newtonian behavior of the polymer, as well as due to the moving free boundary. The latter is related to the penetration front and a Stefan type problem is formulated to account for it. A finite-volume method is used to approximate the given differential problem. Results of numerical experiments are presented.
We also solve an inverse problem and present algorithms for the determination of the absolute preform permeability coefficient in the case when the velocity of the penetration front is known from measurements. In both cases (direct and inverse problems) we emphasize on the specifics related to the non-Newtonian behavior of the polymer. For completeness, we discuss also the Newtonian case. Results of some experimental

measurements are presented and discussed.
Keywords: Liquid Polymer Moulding, Modelling, Simulation, Infiltration, Front Propagation, non-Newtonian flow in porous media
(43 pages, 2004)

64. T. Hanne, H. Neu

Simulating Human Resources in Software Development Processes

In this paper, we discuss approaches related to the explicit modeling of human beings in software development processes. While in most older simulation models of software development processes, esp. those of the system dynamics type, humans are only represented as a labor pool, more recent models of the discrete-event simulation type require representations of individual humans. In that case, particularities regarding the person become more relevant. These individual effects are either considered as stochastic variations of productivity, or an explanation is sought based on individual characteristics, such as skills for instance. In this paper, we explore such possibilities by recurring to some basic results in psychology, sociology, and labor science. Various specific models for representing human effects in software process simulation are discussed.
Keywords: Human resource modeling, software process, productivity, human factors, learning curve
(14 pages, 2004)

65. O. Iliev, A. Mikelic, P. Popov

Fluid structure interaction problems in deformable porous media: Toward permeability of deformable porous media

In this work the problem of fluid flow in deformable porous media is studied. First, the stationary fluid-structure interaction (FSI) problem is formulated in terms of incompressible Newtonian fluid and a linearized elastic solid. The flow is assumed to be characterized by very low Reynolds number and is described by the Stokes equations. The strains in the solid are small allowing for the solid to be described by the Lame equations, but no restrictions are applied on the magnitude of the displacements leading to strongly coupled, nonlinear fluid-structure problem. The FSI problem is then solved numerically by an iterative procedure which solves sequentially fluid and solid subproblems. Each of the two subproblems is discretized by finite elements and the fluid-structure coupling is reduced to an interface boundary condition. Several numerical examples are presented and the results from the numerical computations are used to perform permeability computations for different geometries.
Keywords: fluid-structure interaction, deformable porous media, upscaling, linear elasticity, stokes, finite elements
(28 pages, 2004)

66. F. Gaspar, O. Iliev, F. Lisbona, A. Naumovich, P. Vabishchevich

On numerical solution of 1-D poroelasticity equations in a multilayered domain

Finite volume discretization of Biot system of poroelasticity in a multilayered domain is presented. Staggered grid is used in order to avoid nonphysical oscillations of the numerical solution, appearing when a collocated grid is used. Various numerical experiments are presented in order to illustrate the accuracy of the finite difference scheme. In the first group of experiments, problems having analytical solutions are solved, and the order of convergence for the velocity, the pressure, the displacements, and the stresses is analyzed. In the second group of experiments numerical solution of real problems is presented.
Keywords: poroelasticity, multilayered material, finite volume discretization, MAC type grid
(41 pages, 2004)

67. J. Ohser, K. Schladitz, K. Koch, M. Nöthe

Diffraction by image processing and its application in materials science

A spectral theory for constituents of macroscopically homogeneous random microstructures modeled as homogeneous random closed sets is developed and provided with a sound mathematical basis, where the spectrum obtained by Fourier methods corresponds to the angular intensity distribution of x-rays scattered by this constituent. It is shown that the fast Fourier transform applied to three-dimensional images of microstructures obtained by micro-tomography is a powerful tool of image processing. The applicability of this technique is is demonstrated in the analysis of images of porous media.
Keywords: porous microstructure, image analysis, random set, fast Fourier transform, power spectrum, Bartlett spectrum
(13 pages, 2004)

68. H. Neunzert

Mathematics as a Technology: Challenges for the next 10 Years

No doubt: Mathematics has become a technology in its own right, maybe even a key technology. Technology may be defined as the application of science to the problems of commerce and industry. And science? Science maybe defined as developing, testing and improving models for the prediction of system behavior; the language used to describe these models is mathematics and mathematics provides methods to evaluate these models. Here we are! Why has mathematics become a technology only recently? Since it got a tool, a tool to evaluate complex, "near to reality" models: Computer! The model may be quite old – Navier-Stokes equations describe flow behavior rather well, but to solve these equations for realistic geometry and higher Reynolds numbers with sufficient precision is even for powerful parallel computing a real challenge. Make the models as simple as possible, as complex as necessary – and then evaluate them with the help of efficient and reliable algorithms: These are genuine mathematical tasks.
Keywords: applied mathematics, technology, modelling, simulation, visualization, optimization, glass processing, spinning processes, fiber-fluid interaction, trubulence effects, topological optimization, multicriteria optimization, Uncertainty and Risk, financial mathematics, Malliavin calculus, Monte-Carlo methods, virtual material design, filtration, bio-informatics, system biology
(29 pages, 2004)

69. R. Ewing, O. Iliev, R. Lazarov, A. Naumovich

On convergence of certain finite difference discretizations for 1D poroelasticity interface problems

Finite difference discretizations of 1D poroelasticity equations with discontinuous coefficients are analyzed. A recently suggested FD discretization of poroelasticity equations with constant coefficients on staggered grid, [5], is used as a basis. A careful treatment of the interfaces leads to harmonic averaging of the discontinuous coefficients. Here, convergence for the pressure and for the displacement is proven in certain norms for the scheme with harmonic averaging (HA). Order of convergence 1.5 is proven for arbitrary located interface, and second order convergence is proven for the case when the interface coincides with a grid node. Furthermore, following the ideas from [3], modified HA discretization are suggested for particular cases. The velocity and the stress are approximated with second order on the interface in this case. It is shown that for wide class of problems, the modified discretization provides better accuracy. Second order convergence for modified scheme is proven for the case when the interface coincides with a displacement grid node. Numerical experiments are presented in order to illustrate our considerations.
Keywords: poroelasticity, multilayered material, finite volume discretizations, MAC type grid, error estimates
(26 pages,2004)

70. W. Dörfler, O. Iliev, D. Stoyanov, D. Vassileva

On Efficient Simulation of Non-Newtonian Flow in Saturated Porous Media with a Multigrid Adaptive Refinement Solver

Flow of non-Newtonian in saturated porous media can be described by the continuity equation and the generalized Darcy law. Efficient solution of the resulting second order nonlinear elliptic equation is discussed here. The equation is discretized by a finite volume method on a cell-centered grid. Local adaptive refinement of the grid is introduced in order to reduce the number of unknowns. A special implementation approach is used, which allows us to perform unstructured local refinement in conjunction with the finite volume discretization. Two residual based error indicators are exploited in the adaptive refinement criterion. Second order accurate discretization on the interfaces between refined and non-refined subdomains, as well as on the boundaries with Dirichlet boundary condition, are presented here, as an essential part of the accurate and efficient algorithm. A nonlinear full approximation storage multigrid algorithm is developed especially for the above described composite (coarse plus locally refined) grid approach. In particular, second order approximation around interfaces is a result of a quadratic approximation of slave nodes in the multigrid - adaptive refinement (MG-AR) algorithm. Results from numerical solution of various academic and practice-induced problems are presented and the performance of the solver is discussed.
Keywords: Nonlinear multigrid, adaptive renement, non-Newtonian in porous media
(25 pages, 2004)

71. J. Kalcsics, S. Nickel, M. Schröder

Towards a Unified Territory Design Approach – Applications, Algorithms and GIS Integration

Territory design may be viewed as the problem of grouping small geographic areas into larger geographic clusters called territories in such a way that the latter are acceptable according to relevant planning criteria. In this paper we review the existing literature for applications of territory design problems and solution approaches for solving these types of problems. After identifying features common to all applications we introduce a basic territory design model and present in detail two approaches for solving this model: a classical location–allocation approach combined with optimal split resolution techniques and a newly developed computational geometry based method. We present computational results indicating the efficiency and suitability of the latter method for solving large–scale practical problems in an interactive environment. Furthermore, we discuss extensions to the basic model and its integration into Geographic Information Systems.
Keywords: territory desgin, political districting, sales territory alignment, optimization algorithms, Geographical Information Systems
(40 pages, 2005)

72. K. Schladitz, S. Peters, D. Reinel-Bitzer, A. Wiegmann, J. Ohser

Design of acoustic trim based on geometric modeling and flow simulation for non-woven

In order to optimize the acoustic properties of a stacked fiber non-woven, the microstructure of the non-woven is modeled by a macroscopically homogeneous random system of straight cylinders (tubes). That is, the fibers are modeled by a spatially stationary random system of lines (Poisson line process), dilated by a sphere. Pressing the non-woven causes anisotropy. In our model, this anisotropy is described by a one parametric distribution of the direction of the fibers. In the present application, the anisotropy parameter has to be estimated from 2d reflected light microscopic images of microsections of the non-woven.
After fitting the model, the flow is computed in digitized realizations of the stochastic geometric model using the lattice Boltzmann method. Based on the flow resistivity, the formulas of Delany and Bazley predict the frequency-dependent acoustic absorption of the non-woven in the impedance tube.
Using the geometric model, the description of a non-woven with improved acoustic absorption properties is obtained in the following way: First, the fiber thicknesses, porosity and anisotropy of the fiber system are modified. Then the flow and acoustics simulations are performed in the new sample. These two steps are repeatedc for various sets of parameters. Finally, the set of parameters for the geometric model leading to the best acoustic absorption is chosen.
Keywords: random system of fibers, Poisson line process, flow resistivity, acoustic absorption, Lattice-Boltzmann method, non-woven
(21 pages, 2005)

73. V. Rutka, A. Wiegmann

Explicit Jump Immersed Interface Method for virtual material design of the effective elastic moduli of composite materials

Virtual material design is the microscopic variation of materials in the computer, followed by the numerical evaluation of the effect of this variation on the material's macroscopic properties. The goal of this procedure is an in some sense improved material. Here, we give examples regarding the dependence of the effective elastic moduli of a composite material on the geometry of the shape of an inclusion. A new approach on how to solve such interface problems avoids mesh generation and gives second order accurate results even in the vicinity of the interface.
The Explicit Jump Immersed Interface Method is a finite difference method for elliptic partial differential equations that works on an equidistant Cartesian grid in spite of non-grid aligned discontinuities in equation parameters and solution. Near discontinuities, the standard finite difference approximations are modified by adding correction terms that involve jumps in the function and its derivatives. This work derives the correction terms for two dimensional linear elasticity with piecewise constant coefficients, i. e. for composite materials. It demonstrates numerically convergence and approximation properties of the method.
Keywords: virtual material design, explicit jump immersed interface method, effective elastic moduli, composite materials
(22 pages, 2005)

74. T. Hanne

Eine Übersicht zum Scheduling von Baustellen

Im diesem Dokument werden Aspekte der formalen zeitlichen Planung bzw. des Scheduling für Bauprojekte anhand ausgewählter Literatur diskutiert. Auf allgemeine Aspekte des Scheduling soll dabei nicht eingegangen werden. Hierzu seien als Standard-Referenzen nur Brucker (2004) und Pinedo (1995) genannt. Zu allgemeinen Fragen des Projekt-Managements sei auf Kerzner (2003) verwiesen.
Im Abschnitt 1 werden einige Anforderungen und Besonderheiten der Planung von Baustellen diskutiert. Diese treten allerdings auch in zahlreichen anderen Bereichen der Produktionsplanung und des Projektmanagements auf. In Abschnitt 2 werden dann Aspekte zur Formalisierung von Scheduling-Problemen in der Bauwirtschaft diskutiert, insbesondere Ziele und zu berücksichtigende Restriktionen. Auf eine mathematische Formalisierung wird dabei allerdings verzichtet. Abschnitt 3 bietet eine Übersicht über Verfahren und grundlegende Techniken für die Berechnung von Schedules. In Abschnitt 4 wird ein Überblick über vorhandene Software, zum einen verbreitete Internationale Software, zum anderen deutschsprachige Branchenlösungen, gegeben. Anschließend werden Schlussfolgerungen gezogen und es erfolgt eine Auflistung der Literaturquellen.
Keywords: Projektplanung, Scheduling, Bauplanung, Bauindustrie
(32 pages, 2005)

75. J. Linn

The Folgar–Tucker Model as a Differential Algebraic System for Fiber Orientation Calculation

The Folgar–Tucker equation (FTE) is the model most frequently used for the prediction of fiber orientation (FO) in simulations of the injection molding process for short–fiber reinforced thermoplasts. In contrast to its widespread use in injection molding simulations, little is known about the mathematical properties of the FTE: an investigation of e. g. its phase space M_{FT} has been presented only recently [12]. The restriction of the dependent variable of the FTE to the set M_{FT} turns the FTE into a differential algebraic system (DAS), a fact which is commonly neglected when devising numerical schemes for the integration of the FTE. In this article we present some recent results on the problem of trace stability as well as some introductory material which complements our recent paper [12].
Keywords: fiber orientation, Folgar–Tucker model, invariants, algebraic constraints, phase space, trace stability
(15 pages, 2005)

76. M. Speckert, K. Dreßler, H. Mauch, A. Lion, G. J. Wierda

Simulation eines neuartigen Prüfsystems für Achserprobungen durch MKS-Modellierung einschließlich Regelung

Testing new suspensions based on real load data is performed on elaborate multi channel test rigs. Usually, wheel forces and moments measured during driving maneuvers are reproduced by the test rig. Because of the complicated interaction between test rig and suspension each new rig configuration has to prove its efficiency with respect to the requirements and the configuration might be subject to optimization.
This paper deals with mathematical and physical modeling of a new concept of a test rig which is based on two hexapods. The model contains the geometric configuration as well as the hydraulics and the controller. It is implemented as an ADAMS/Car template and can be combined with different suspension models to get a complete assembly representing the entire test rig. Using this model, all steps required for a real test run such as controller adaptation, drive file iteration and simulation can be performed. Geometric or hydraulic parameters can be modified easily to improve the setup and adapt the system to the suspension and the given load data.
The model supports and accompanies the introduction of the new rig concept and can be used to prepare real tests on a virtual basis. Using both a front and a rear suspension the approach is described and the potentials coming with the simulation are pointed out.
Keywords: virtual test rig, suspension testing, multibody simulation, modeling hexapod test rig, optimization of test rig configuration
(20 pages, 2005)
In deutscher Sprache; bereits erschienen in: VDI-Berichte Nr. 1900, VDI-Verlag GmbH Düsseldorf (2005), Seiten 227-246

77. K.-H. Küfer, M. Monz, A. Scherrer, P. Süss, F. Alonso, A. S. A. Sultan, Th. Bortfeld, D. Craft, Chr. Thieke

Multicriteria optimization in intensity modulated radiotherapy planning

Inverse treatment planning of intensity modulated radiothrapy is a multicriteria optimization problem: planners have to find optimal compromises between a sufficiently highdose intumor tissuethat garantuee a high tumor control, and, dangerous overdosing of critical structures, in order to avoid high normal tissue complcication problems.
The approach presented in this work demonstrates how to state a flexible generic multicriteria model of the IMRT planning problem and how to produce clinically highly relevant Pareto-solutions. The model is imbedded in a principal concept of Reverse Engineering, a general optimization paradigm for design problems. Relevant parts of the Pareto-set are approximated by using extreme compromises as cornerstone solutions, a concept that is always feasible if box constraints for objective funtions are available. A major practical drawback of generic multicriteria concepts trying to compute or approximate parts of the Pareto-set is the high computational effort. This problem can be overcome by exploitation of an inherent asymmetry of the IMRT planning problem and an adaptive approximation scheme for optimal solutions based on an adaptive clustering preprocessing technique. Finally, a coherent approach for calculating and selecting solutions in a real-timeinteractive decision-making process is presented. The paper is concluded with clinical examples and a discussion of ongoing research topics.
Keywords: multicriteria optimization, extreme solutions, real-time decision making, adaptive approximation schemes, clustering methods, IMRT planning, reverse engineering
(51 pages, 2005)

78. S. Amstutz, H. Andrä

A new algorithm for topology optimization using a level-set method

The levelset method has been recently introduced in the field of shape optimization, enabling a smooth representation of the boundaries on a fixed mesh and therefore leading to fast numerical algorithms. However, most of these algorithms use a HamiltonJacobi equation to connect the evolution of the levelset function with the deformation of the contours, and consequently they cannot create any new holes in the domain (at least in 2D). In this work, we propose an evolution equation for the levelset function based on a generalization of the concept of topological gradient. This results in a new algorithm allowing for all kinds of topology changes.
Keywords: shape optimization, topology optimization, topological sensitivity, level-set
(22 pages, 2005)

79. N. Ettrich

Generation of surface elevation models for urban drainage simulation

Traditional methods fail for the purpose of simulating the complete flow process in urban areas as a consequence of heavy rainfall and as required by the European Standard EN-752 since the bi-directional coupling between sewer and surface is not properly handled. The methodology, developed in the BMBF/EUREKA-project RisUrSim, solves this problem by carrying out the runoff on the basis of shallow water equations solved on high-resolution surface grids. Exchange nodes between the sewer and the surface, like inlets and manholes, are located in the computational grid and water leaving the sewer in case of surcharge is further distributed on the surface.
So far, it has been a problem to get the dense topographical information needed to build models suitable for hydrodynamic runoff calculation in urban areas. Recent airborne data collection methods like laser scanning, however, offer a great chance to economically gather densely sampled input data. This paper studies the potential of such laser-scan data sets for urban water hydrodynamics.
Keywords: Flooding, simulation, urban elevation models, laser scanning
(22 pages, 2005)

80. H. Andrä, J. Linn, I. Matei, I. Shklyar, K. Steiner, E. Teichmann

OPTCAST – Entwicklung adäquater Strukturoptimierungsverfahren für Gießereien Technischer Bericht (KURZFASSUNG)

Im vorliegenden Bericht werden die Erfahrungen und Ergebnisse aus dem Projekt OptCast zusammengestellt. Das Ziel dieses Projekts bestand (a) in der Anpassung der Methodik der automatischen Strukturoptimierung für Gussteile und (b) in der Entwicklung und Bereitstellung von gießereispezifischen Optimierungstools für Gießereien und Ingenieurbüros.
Gießtechnische Restriktionen lassen sich nicht auf geometrische Restriktionen reduzieren, sondern sind nur über eine Gießsimulation (Erstarrungssimulation und Eigenspannungsanalyse) adäquat erfassbar, da die lokalen Materialeigenschaften des Gussteils nicht nur von der geometrischen Form des Teils, sondern auch vom verwendeten Material abhängen. Wegen dieser Erkenntnis wurde ein neuartiges iteratives Topologieoptimierungsverfahren unter Verwendung der Level-Set-Technik entwickelt, bei dem keine variable Dichte des Materials eingeführt wird. In jeder Iteration wird ein scharfer Rand des Bauteils berechnet. Somit ist die Gießsimulation in den iterativen Optimierungsprozess integrierbar.
Der Bericht ist wie folgt aufgebaut: In Abschnitt 2 wird der Anforderungskatalog erläutert, der sich aus der Bearbeitung von Benchmark-Problemen in der ersten Projektphase ergab. In Abschnitt 3 werden die Benchmark-Probleme und deren Lösung mit den im Projekt entwickelten Tools beschrieben. Abschnitt 4 enthält die Beschreibung der neu entwickelten Schnittstellen und die mathematische Formulierung des Topologieoptimierungsproblems. Im letzten Abschnitt wird das neue Topologieoptimierungsverfahren, das die Simulation des Gießprozesses einschließt, erläutert.
Keywords: Topologieoptimierung, Level-Set-Methode, Gießprozesssimulation, Gießtechnische Restriktionen, CAE-Kette zur Strukturoptimierung
(77 pages, 2005)

81. N. Marheineke, R. Wegener

Fiber Dynamics in Turbulent Flows Part I: General Modeling Framework

The paper at hand deals with the modeling of turbulence effects on the dynamics of a long slender elastic fiber. Independent of the choice of the drag model, a general aerodynamic force concept is derived on the basis of the velocity field for the randomly fluctuating component of the flow. Its construction as centered differentiable Gaussian field complies thereby with the requirements of the stochastic k-ε turbulence model and Kolmogorov's universal equilibrium theory on local isotropy.

Keywords: fiber-fluid interaction; Cosserat rod; turbulence modeling; Kolmogorov's energy spectrum; double-velocity correlations; differentiable Gaussian fields

Part II: Specific Taylor Drag

In [12], an aerodynamic force concept for a general air drag model is derived on top of a stochastic k-ε description for a turbulent flow field. The turbulence effects on the dynamics of a long slender elastic fiber are particularly modeled by a correlated random Gaussian force and in its asymptotic limit on a macroscopic fiber scale by Gaussian white noise with flow-dependent amplitude. The paper at hand now presents quantitative similarity estimates and numerical comparisons for the concrete choice of a Taylor drag model in a given application.

Keywords: flexible fibers; k-ε turbulence model; fiber-turbulence interaction scales; air drag; random Gaussian aerodynamic force; white noise; stochastic differential equations; ARMA process

(38 pages, 2005)

82. C. H. Lampert, O. Wirjadi

An Optimal Non-Orthogonal Separation of the Anisotropic Gaussian Convolution Filter

We give ananalytical and geometrical treatment of what it means to separate a Gaussian kernel along arbitrary axes in R^n, and we present a separation scheme that allows to efficiently implement anisotropic Gaussian convolution filters in arbitrary dimension. Based on our previous analysis we show that this scheme is optimal with regard to the number of memory accesses and interpolation operations needed.

Our method relies on non-orthogonal convolution axes and works completely in image space. Thus, it avoids the need for an FFT-subroutine. Depending on the accuracy and speed requirements, different interpolation schemes and methods to implement the one-dimensional Gaussian (FIR, IIR) can be integrated. The algorithm is also feasible for hardware that does not contain a floating-point unit.

Special emphasis is laid on analyzing the performance and accuracy of our method. In particular, we show that without anyspecial optimization of the source code, our method can perform anisotropic Gaussian filtering faster than methods relying on the Fast Fourier Transform.

Keywords: Anisotropic Gaussian filter, linear filtering, orientation space, nD image processing, separable filters

(25 pages, 2005)

83. H. Andrä, D. Stoyanov

Error indicators in the parallel finite element solver for linear elasticity DDFEM

This report discusses two approaches for a posteriori error indication in the linear elasticity solver DDFEM: An indicator based on the Richardson extrapolation and Zienkiewicz-Zhu-type indicator.

The solver handles 3D linear elasticity steady-state problems. It uses own input language to de-scribe the mesh and the boundary conditions. Finite element discretization over tetrahedral meshes with first or second order shape functions (hierarchical basis) has been used to resolve the model. The parallelization of the numerical method is based on the domain decomposition approach. DDFEM is highly portable over a set of parallel computer architectures supporting the MPI-standard.

Keywords: linear elasticity, finite element method, hierarchical shape functions, domain decom-position, parallel implementation, a posteriori error estimates

(21 pages, 2006)

84. M. Schröder, I. Solchenbach

Optimization of Transfer Quality in Regional Public Transit

In this paper we address the improvement of transfer quality in public mass transit networks. Generally there are several transit operators offering service and our work is motivated by the question how their timetables can be altered to yield optimized transfer possibilities in the over-all network. To achieve this, only small changes to the timetables are allowed.

The set-up makes it possible to use a quadratic semi-assignment model to solve the optimization problem. We apply this model, equipped with a new way to assess transfer quality, to the solu-tion of four real-world examples. It turns out that improvements in overall transfer quality can be determined by such optimization-based techniques. Therefore they can serve as a first step to-wards a decision support tool for planners of regional transit networks.

Keywords: public transit, transfer quality, quadratic assignment problem

(16 pages, 2006)

85. A. Naumovich, F. J. Gaspar

On a multigrid solver for the three-dimensional Biot poroelasticity system in multilayered domains

In this paper, we present problem-dependent prolongation and problem-dependent restriction for a multigrid solver for the three-dimensional Biot poroelasticity system, which is solved in a multilayered domain. The system is discretized on a staggered grid using the finite volume method. During the discretization, special care is taken of the discontinuous coefficients. For the efficient multigrid solver, a need in operator-dependent restriction and/or prolongation arises. We derive these operators so that they are consistent with the discretization. They account for the discontinuities of the coefficients, as well as for the coupling of the unknowns within the Biot system. A set of numerical experiments shows necessity of use of the operator-dependent restriction and prolongation in the multigrid solver for the considered class of problems.

Keywords: poroelasticity, interface problem, multigrid, operator-dependent prolongation

(11 pages, 2006)

86. S. Panda, R. Wegener, N. Marheineke

Slender Body Theory for the Dynamics of Curved Viscous Fibers

The paper at hand presents a slender body theory for the dynamics of a curved inertial viscous Newtonian fiber. Neglecting surface tension and temperature dependence, the fiber flow is modeled as a three-dimensional free boundary value problem via instationary incompressible Navier-Stokes equations. From regular asymptotic expansions in powers of the slenderness parameter leading-order balance laws for mass (cross-section) and momentum are derived that combine the unrestricted motion of the fiber center-line with the inner viscous transport. The physically reasonable form of the one-dimensional fiber model results thereby from the introduction of the intrinsic velocity that characterizes the convective terms.

Keywords: curved viscous fibers; fluid dynamics; Navier-Stokes equations; free boundary value problem; asymptotic expansions; slender body theory

(14 pages, 2006)

87. E. Ivanov, H. Andrä, A. Kudryavtsev

Domain Decomposition Approach for Automatic Parallel Generation of Tetrahedral Grids

The desire to simulate more and more geometrical and physical features of technical structures and the availability of parallel computers and parallel numerical solvers which can exploit the power of these machines have lead to a steady increase in the number of grid elements used. Memory requirements and computational time are too large for usual serial PCs. An a priori partitioning algorithm for the parallel generation of 3D nonoverlapping compatible unstructured meshes based on a CAD surface description is presented in this paper. Emphasis is given to practical issues and implementation rather than to theoretical complexity. To achieve robustness of the algorithm with respect to the geometrical shape of the structure authors propose to have several or many but relatively simple algorithmic steps. The geometrical domain decomposition approach has been applied. It allows us to use classic 2D and 3D high-quality Delaunay mesh generators for independent and simultaneous volume meshing. Different aspects of load balancing methods are also explored in the paper. The MPI library and SPMD model are used for parallel grid generator implementation. Several 3D examples are shown.

Key words: Grid Generation, Unstructured Grid, Delaunay Triangulation, Parallel Programming, Domain Decomposition, Load Balancing

(18 pages, 2006)

88. S. Tiwari, S. Antonov, D. Hietel, J. Kuhnert, R. Wegener

A Meshfree Method for Simulations of Interactions between Fluids and Flexible Structures

We present the application of a meshfree method for simulations of interaction between fluids and flexible structures. As a flexible structure we consider a sheet of paper. In a twodimensional framework this sheet can be modeled as curve by the dynamical Kirchhoff-Love theory. The external forces taken into account are gravitation and the pressure difference between upper and lower surface of the sheet. This pressure difference is computed using the Finite Pointset Method (FPM) for the incompressible Navier-Stokes equations. FPM is a meshfree, Lagrangian particle method. The dynamics of the sheet are computed by a finite difference method. We show the suitability of the meshfree method for simulations of fluidstructure interaction in several applications

Key words: Meshfree Method, FPM, Fluid Structure Interaction, Sheet of Paper, Dynamical Coupling

(16 pages, 2006)

89. R. Ciegis , O. Iliev, V. Starikovicius, K. Steiner

Numerical Algorithms for Solving Problems of Multiphase Flows in Porous Media

In this paper we discuss numerical algorithms for solving the system of nonlinear PDEs, arising in modelling of two-phase ows in porous media, as well as the proper object oriented implementation of these algorithms. Global pressure model for isothermal two-phase immiscible flow in porous media is considered in this paper. Finite-volume method is used for the space discretization of the system of PDEs. Different time stepping discretizations and linearization approaches are discussed. The main concepts of the PDE software tool MfsolverC++ are given. Numerical results for one realistic problem are presented.

Keywords: nonlinear algorithms, finite-volume method, software tools, porous media, flows

(16 pages, 2006)

90. D. Niedziela, O. Iliev, A. Latz

On 3D Numerical Simulations of Viscoelastic Fluids

In this work we present and solve a class of non–Newtonian viscoelastic fluid flow problems. Models for non-Newtonian fluids can be classified into three large groups depending on the type of the constitutive relation used: algebraic, differential and integral. The first type of models are most simple one, the last are the most physically adequate ones. Here we consider some models from the first and the third groups, and present robust and efficient algorithms for their solution. We present new mathematical model, which belongs to the class of generalized Newtonian models and is able to account for the anisotropy of the viscosity tensor observed in many real liquids. In particular, we discuss a unified model that captures both shear thinning and extensional thickening for complex flows. The resulting large variations of the viscosity tensor in space and time are leading to a strong numerical coupling of the momentum equations due to the appearance of mixed derivatives in the discretization. To treat this strong coupling appropriately, we present two modifications of classical projection methods (like e.g. SIMPLE). In the first modification all momentum equations are solved coupled (i.e. mixed derivative are discretized implicitly) but still iterations are performed between the momen-

tum equations and the continuity equation. The second one is a fully coupled method, where momentum and continuity equation are solved together using a proper preconditioner. The models involving integral constitutive relation which accounts for the history of deformations, result in a system of integro-differential equations. To solve it, we suggest a proper splitting scheme, which treats the integral and the differential parts consecutively. Integral Oldroyd B and Doi Edwards models are used to simulate flows of dilute and concentrated polymer solutions, respectively.
Keywords: non–Newtonian fluids, anisotropic viscosity, integral constitutive equation
(18 pages, 2006)

91. A. Winterfeld

Application of general semi-infinite Programming to Lapidary Cutting Problems

We consider a volume maximization problemarising in gemstone cutting industry. The problem is formulated as a general semi-infnite program(GSIP)and solved using an interior point method developed by Stein. It is shown, that the convexity assumption needed for the convergence of the algorithm can be satisfied by appropriate model-ling. Clustering techniques are used to reduce the number of container constraints, which is necessary to make the subproblems practically tractable. An iterative process consisting of GSIP optimization and adaptive refinement steps is then employed to obtain an optimal solution which is also feasible for the original problem. Some numeri-cal results based on real world data are also presented.
Keywords: large scale optimization, nonlinear programming, general semi-infinite optimization, design centering, clustering
(26 pages, 2006)

92. J. Orlik, A. Ostrovska

Space-Time Finite Element Approximation and Numerical Solution of Hereditary Linear Viscoelasticity Problems

In this paper we suggest a fast numerical approach to treat problems of the hereditary linear viscoelasticity, which results in the system of elliptic partial differential equations in space variables, who's coefficients are Volterra integral operators of the second kind in time. We propose to approximate the relaxation kernels by the product of purely time- and space-dependent terms, which is achieved by their piecewisepolynomial space-interpolation. A priori error estimate was obtained and it was shown, that such approximation does not decrease the convergence order, when an interpolation polynomial is chosen of the same order as the shape functions for the spatial finite element approximation, while the computational effort is significantly reduced.
Keywords: hereditary viscoelasticity; kern approximation by interpolation; space-time finite element approximation, stability and a priori estimate
(24 pages, 2006)

93. V. Rutka, A. Wiegmann, H. Andrä

EJIIM for Calculation of effective Elastic Moduli in 3D Linear Elasticity

The Explicit Jump Immersed Interface Method solves boundary value problems by embedding arbitrary domains in rectangular parallelepipeds and extending the boundary value problem to such a domain. Now the boundary conditions become jump conditions, and separate equations for the discretized solution of the pde and for the jump conditions are written. In thefirst set of equations, the jumps correct standard difference formulas, in the second set of equations solution values from interior grid points are used to compute quantities on the interface. For constant coefficient pde, the first set can be solved very efficiently by direct Fast Fourier transform inversion, and this is used to efficiently evaluate the smaller Schur-complement matrix-vector multiplication in a conjugate gradient approach to find the jumps. Here this approch is applied to the three-dimensional equations of linear elasticity. The resulting method is second order convergent for the displacements in the maximum norm as the grid is refined. It is applied to calculate the effective elastic moduli of fibrous and porous microstructures in three dimensions using the strain energy equivalence principle. From these effective moduli, best estimates under various symmetry assumptions are calculated.
Keywords: Elliptic PDE, linear elasticity, irregular domain, finite differences, fast solvers, effective elastic moduli
(24 pages, 2006)

94. A. Wiegmann, A. Zemitis

EJ-HEAT: A Fast Explicit Jump Harmonic Averaging Solver for the Effective Heat Conductivity of Composite Materials

The stationary heat equation is solved with periodic boundary conditions in geometrically complex composite materials with high contrast in the thermal conductivities of the individual phases. This is achieved by harmonic averaging and explicitly introducing the jumps across the material interfaces as additional variables. The continuity of the heat flux yields the needed extra equations for these variables. A Schur-complent formulation for the new variables is derived that is solved using the FFT and BiCGStab methods.
The EJ-HEAT solver is given as a 3-page Matlab program in the Appendix. The C++ implementation is used for material design studies. It solves 3-dimensional problems with around 190 Mio variables on a 64-bit AMD Opteron desktop system in less than 6 GB memory and in minutes to hours, depending on the contrast and required accuracy.
The approach may also be used to compute effective electric conductivities because they are governed by the stationary heat equation.
Keywords: Stationary heat equation, effective thermal conductivity, explicit jump, discontinuous coefficients, virtual material design, microstructure simulation, EJ-HEAT
(21 pages, 2006)

95. A. Naumovich

On a finite volume discretization of the three-dimensional Biot poroelasticity system in multilayered domains

In this paper we propose a finite volume discretization for the threedimensional Biot poroelasticity system in multilayered domains. For the stability reasons, staggered grids are used. The discretization accounts for discontinuity of the coefficients across the interfaces between layers with different physical properties. Numerical experiments, based on the proposed discretization showed second order convergence in the maximum norm for the primary as well as flux unknowns of the system. A certain application example is presented as well.
Keywords: Biot poroelasticity system, interface problems, finite volume discretization, finite difference method.
(21 pages, 2006)

96. M. Krekel, J. Wenzel

A unified approach to Credit Default Swaption and Constant Maturity Credit Default Swap valuation

In this paper we examine the pricing of arbitrary credit derivatives with the Libor Market Model with Default Risk. We show, how to setup the Monte Carlo-Simulation efficiently and investigate the accuracy of closed-form solutions for Credit Default Swaps, Credit Default Swaptions and Constant Maturity Credit Default Swaps. In addition we derive a new closed-form solution for Credit Default Swaptions which allows for time-dependent volatility and abitrary correlation structure of default intensities.
Keywords: LIBOR market model, credit risk, Credit Default Swaption, Constant Maturity Credit Default Swapmethod.
(43 pages, 2006)

96. A. Dreyer

Interval Methods for Analog Circiuts

Reliable methods for the analysis of tolerance-affected analog circuits are of great importance in nowadays microelectronics. It is impossible to produce circuits with exactly those parameter specifications proposedin the design process. Such component tolerances will always lead to small variations of a circuit's properties, which may result in unexpected behaviour. If lower and upper bounds to parameter variations can be read off the manufacturing process, interval arithmetic naturally enters the circuit analysis area.
This paper focuses on the frequency-response analysis of linear analog circuits, typically consisting of current and voltage sources as well as resistors, capacitances, inductances, and several variants of controlled sources. These kind of circuits are still widely used in analog circuit design as equivalent circuit diagrams for representing in certain application tasks. Interval methods have been applied to analog circuits before. But yet this was restricted to circuit equations only, with no interdependencies between the matrix elements. But there also exist formulations of analog circuit equations containing dependent terms. Hence, for an efficient application of interval methods, it is crucial to regard possible dependencies in circuit equations. Part and parcel of this strategy is the handling of fill-in patterns for those parameters related to uncertain components. These patterns are used in linear circuit analysis for efficient equation setup. Such systems can efficiently be solved by successive application of the Sherman-Morrison formula.
The approach can also be extended to complex-valued systems from frequency domain analysis of more general linear circuits. Complex values result here from a Laplace transform of frequency-dependent components like capacitances and inductances. In order to apply interval techniques, a real representation of the linear systemof equations can be used for separate treatment of real and imaginary part of the variables. In this representation each parameter corresponds to the superposition of two fill-in patterns. Crude bounds – obtained by treating both patterns independently – can be improved by consideration of the correlations to tighter enclosures of the solution. The techniques described above have been implemented as an extension to the toolbox Analog Insydes, an add-on package to the computer algebra system Mathematica for modeling, analysis, and design of analog circuits.
Keywords: interval arithmetic, analog circuits, tolerance analysis, parametric linear systems, frequency response, symbolic analysis, CAD, computer algebra
(36 pages, 2006)

Status quo: October 2006