Tobias Seidel

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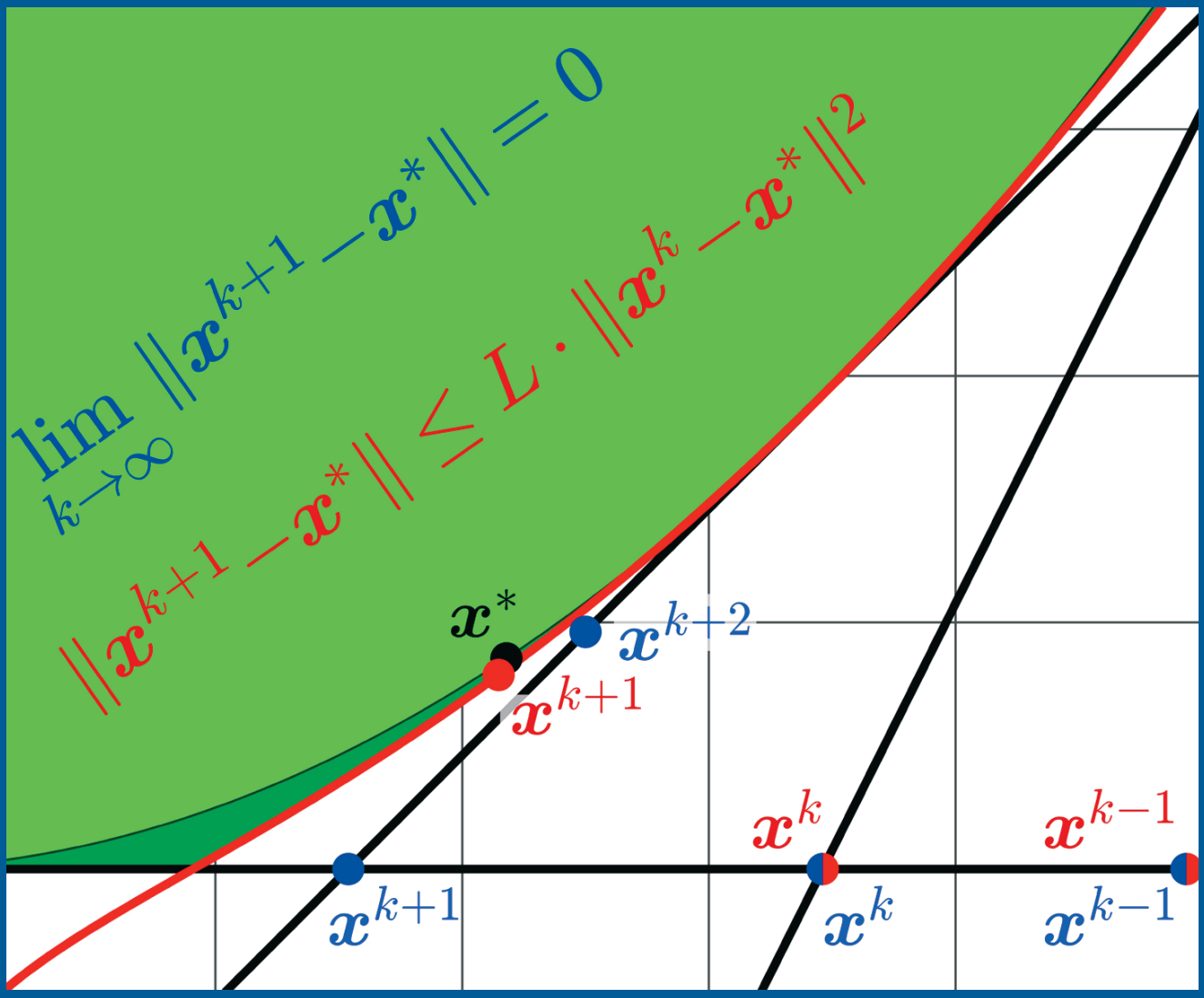

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Solving semi-infinite optimization problems with quadratic rate of convergence

Vom Fachbereich Mathematik der Technischen Universität Kaiserslautern zur Verleihung des akademischen Grades Doktor der Naturwissenschaften (Doctor rerum naturalium, Dr. rer. nat.) genehmigte Dissertation

von

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Zusammenfassung

Mittels semi-infiniter Optimierung kann eine Vielzahl von komplexen Problemen modelliert werden. Dazu zählen Zerlege-, Packungs- und Abdeckungsprobleme, die mit geometrischen Bedingungen wie Teilmengen- und Trennungsbedingungen formuliert werden.

Die einfache Beschreibung hat aber ihren Preis: Ein semi-infinites Optimierungsproblem ist häufig schwieriger zu lösen als ein endliches nicht lineares Problem. Ein klassischer, leicht zu implementierender Lösungsansatz besteht in der Wahl einer Diskretisierung der semi-infiniten Indexmenge. Ein Algorithmus, der die Diskretisierungspunkte adaptiv wählt und ihre Anzahl klein hält, ist der Blankenship- und Falk-Algorithmus. In jeder Iteration wird ein approximiertes Problem, welches auf der bisherigen Diskretisierung basiert, gelöst. In einem zweiten Schritt wird die am stärksten verletzte Restriktion bestimmt und zur Diskretisierung hinzugefügt. Viele Aussagen über die Eigenschaften eines Grenzwertes der Iterierten sind bekannt. Es gibt in der Literatur jedoch keine Aussagen, die die Geschwindigkeit der Konvergenz beschreiben. Diese Lücke ist der Ansatzpunkt für diese Arbeit.

Ziel der Arbeit ist es, semi-infinite Optimierungsprobleme mit Hilfe einer adaptiven Diskretisierungsmethode und einer quadratischen Rate der Konvergenz zu lösen. Der Fokus liegt zunächst auf dem Blankenship- und Falk-Algorithmus. Es wird ein Beispiel mit einer beliebig langsamen Konvergenz eingeführt. Mittels einer Regularitätsbedingung wird dieses Beispiel ausgeschlossen und Schranken in Abhängigkeit der Ordnung des Grenzwertes entwickelt. Diese Schranken liefern eine quadratische Rate der Konvergenz für den Fall eines Minimums der Ordnung 1. Anhand eines Beispiels wird gezeigt, dass dies nicht für Minima von höherer Ordnung gilt.

Dieses Beispiel ist Motivation für eine neue adaptive Diskretisierungsmethode, die eine quadratische Konvergenz auch für Minima von höherer Ordnung garantiert. Anstatt einer strikten Trennung, wie im Blankenship- und Falk-Algorithmus, werden weitere Informationen über die untere Stufe zu den diskretisierten Problemen hinzugefügt. Es werden drei Hauptresultate entwickelt: Zunächst wird formal gezeigt, dass das neue Verfahren unter wenigen Regularitätsannahmen eine quadratische Konvergenz garantiert. Weiter wird gezeigt, dass ein Grenzwert ein

stationärer Punkt ist, wenn die Iterierten stationäre Punkte sind. Schließlich werden Bedingungen eingeführt, unter denen lokale Lösungen gegen ein lokales Minimum konvergieren.

Die neue Methode wird auf den Fall einer variablen Indexmenge erweitert und es wird gezeigt, dass die vorherigen Resultate übertragen werden können. Die Gemeinsamkeiten und die Unterschiede zwischen dem Verfahren von Blankenship und Falk und der neuen Methode werden anhand von numerischen Beispielen untersucht. Die Beispiele zeigen, dass die Anzahl der Iterationen und die benötigte Zeit zum Lösen dieser reduziert werden kann.

Abstract

Semi-infinite programming can be used to model a large variety of complex optimization problems. Interesting applications include cutting and packing, or coverage problems, formulated with geometric constraints such as subset and separation conditions.

The simple description of such problems comes at a price: semi-infinite problems are often harder to solve than finite nonlinear problems. A classical solution approach, which is easy to implement, is based on discretizing the semi-infinite index set. The Blankenship and Falk algorithm adaptively chooses a small set of discretization points. On every iteration, a solution of an approximate problem, based on the current discretization, is calculated. In a second step, the most violated constraint is determined and added to the discretization. Many statements about the properties of a limit of the iterates are known. We are not aware of any results in the literature concerning the convergence speed. This gap is our starting point.

In this thesis we solve semi-infinite optimization problems using adaptive discretization methods having a quadratic rate of convergence. First, we investigate the classical Blankenship and Falk algorithm. We present a worst case example for which the method suffers from arbitrarily slow convergence. We exclude pathologies of this type by making a mild regularity assumption. We develop bounds on the distance from each iterate to the limit that depend on the order of the limit. These bounds show quadratic convergence for the special case of a minimum of order one. We present an example which shows that quadratic convergence is not possible for a minimum of order higher than one.

Motivated by this example we suggest a new adaptive discretization algorithm with guaranteed quadratic convergence. This rate holds even for minima of order higher than one. The key idea is to break the separated scheme of the Blankenship and Falk algorithm and add more information about the lower-level problems to the discretized problems. We develop three main results. First, we prove the Quadratic Convergence Theorem, which rigorously establishes quadratic convergence under mild regularity conditions. Second, we show that a limit point is a stationary point, if the iterates are stationary points. Finally, we establish conditions under which, for iterates that are local minima, their limit is a local minimum.

We extend our new method to the case of a variable index-set, and show that the previous results also hold for this case. We compare the Blankenship and Falk algorithm to our new method by considering a series of numerical examples. In these examples, our new method outperforms the Blankenship and Falk algorithm in terms of number of iterations and runtime.

Contents

List of symbols and abbreviations

General Notation

$\mathbb{N}$	natural numbers $\{1, 2, 3, \dots\}$
$\mathbb{R}$	real numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
$\|\boldsymbol{x}\|$	Euclidean norm, $\|\boldsymbol{x}\| := \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$
$B_\varepsilon(\boldsymbol{x}^*)$	open ball around $\boldsymbol{x}^*$ with radius ε
$\overline{B}_\varepsilon(\boldsymbol{x}^*)$	closed ball around $\boldsymbol{x}^*$ with radius ε
$D\boldsymbol{f}(\boldsymbol{x})$	derivative/Jacobi matrix
$D^2\boldsymbol{f}(\boldsymbol{x})$	second derivative / Hessian
$D_1\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})$	partial derivative of $\boldsymbol{f}$ with respect to the first variables $\boldsymbol{x}$
$D_{1,2}\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})$	$\Big(D_1\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) \Big\vert D_2\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})\Big)$
$D_1^2\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})$	Hessian with respect to the first variables $\boldsymbol{x}$
$D_x\Big[\boldsymbol{f}\big(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x})\big)\Big]_{\boldsymbol{x}=\boldsymbol{x}^*}$	$D\boldsymbol{g}(\boldsymbol{x}^*)$ for $\boldsymbol{g}(\boldsymbol{x}) := \boldsymbol{f}\big(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x})\big)$

Optimization problem

M	feasible set
f	objective Function
r	radius of local minimum
ρ	order of minimum
P	finite non-linear optimization problem
LICQ	linear independence constraint qualification
MFCQ	Mangasarian-Fromovitz constraint qualification
SOSC	second-order sufficient condition

Semi-infinite optimization

SIP	semi-infinite optimization problem
$\boldsymbol{x}$	optimization variables
$\boldsymbol{y}$	index variables/lower-level variables
Y	semi-infinite index set
$\boldsymbol{g}$	semi-infinite constraints
I	indices of semi-infinite constraints

$Y_0^i(\boldsymbol{x})$	active indices, $Y_0^i(\boldsymbol{x}) := \{\boldsymbol{y} \in Y \mid g_i(\boldsymbol{x}, \boldsymbol{y}) = 0\}$
$\boldsymbol{\lambda}$	Lagrange multipliers for the semi-infinite problem
$Q_i(\boldsymbol{x})$	i-th lower-level problem
φ_i	i-th optimal value function
$\boldsymbol{v}$	constraints describing the semi-infinite index set
J	indices of the lower-level constraints
$J_0(\boldsymbol{y})$	active indices in the lower level problem
$\mathcal{L}_i$	lower-level Lagrange function
$\boldsymbol{\mu}^i$	lower-level Lagrange multipliers
ELICQ	extended linear independence constraint qualification
EMFCQ	extended Mangasarian-Fromovitz constraint qualification
ESOSC	extended second-order sufficient condition

Generalized semi-infinite optimization

GSIP	generalized semi-infinite optimization problem
$Y(\boldsymbol{x})$	semi-infinite index set of GSIP
$\boldsymbol{t}$	transformation function
$\widetilde{\mathsf{SIP}}$	by transformation induced SIP
$\tilde{\boldsymbol{g}}$	semi-infinite constraints of $\widetilde{\mathsf{SIP}}$
Z	semi-infinite index set of $\widetilde{\mathsf{SIP}}$
$\boldsymbol{z}$	index variable of $\widetilde{\mathsf{SIP}}$
$\tilde{Q}_i(\boldsymbol{x})$	i-th lower-level problem of $\widetilde{\mathsf{SIP}}$

1 Introduction

The first step in the optimization of an application consists of the transfer of the application into a problem given in mathematical terms: the modeling phase. Examples, which demonstrate this step, can be found in [NPWU15]. The modeling phase exhibits two major challenges. For example, one needs to make sure that a solution found in a mathematical formulation also performs well in the true application. At worst the mathematical problem does not reflect the reality well enough at the found solution. What might look like a good solution in the mathematical problem can be a poor one for the true application.

We will mainly focus on a second important aspect of the modeling phase: it must be possible to solve the constructed problem numerically in a robust manner and within a reasonable time. To achieve this goal it is beneficial to maintain as "nice" a structure as possible of the original problem. A mathematical problem with an easy description (for example by polynomials) is often easier to treat than a problem with complex implicit formulations. This means that our goal is to find an easy formulation of complex situations. The complexity of the situation is then handled by a specialized algorithm. A class of optimization problems which is able to model many complex applications in an easy fashion are *semi-infinite optimization problems*.

In Chapter 2 we will formally introduce semi-infinite optimization problems. For a compact non-empty set $Y \subseteq \mathbb{R}^m$, the so-called *semi-infinite index set*, we consider the following type of problems in this work:

$$\begin{aligned} \mathsf{SIP}: \min_{\boldsymbol{x}\in\mathbb{R}^n} \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & g(\boldsymbol{x},\boldsymbol{y}) \leq 0 \text{ for all } \boldsymbol{y} \in Y \ . \end{aligned}$$

The special feature and the reason why the problem is called semi-infinite, is that the index set Y is allowed to have an infinite cardinality (while the number of optimization variables is finite). If the index set depends on $\boldsymbol{x}$, the problem is called *generalized semi-infinite optimization problem*. The possibility to demand infinitely many constraints is the power of semi-infinite optimization. It allows to model complex applications in such a way that the functions f and g are simple.

The complexity of the situation is covered by the infinite cardinality of Y. To demonstrate the discussed properties we briefly present two examples and show how they can be modeled as semi-infinite optimization problems.

A field of optimization with an increasing interest is so-called *robust optimization* (see for example [BTEN09]). Here one typically starts with an ordinary nonlinear optimization problem parameterized by $\boldsymbol{\xi}$:

$$\begin{aligned} \mathrm{P}(\boldsymbol{\xi}) : \min_{\boldsymbol{x}\in\mathbb{R}^n} \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & g(\boldsymbol{x},\boldsymbol{\xi}) \leq 0 \ . \end{aligned}$$

We have already mentioned at the beginning of this introduction that optimization problems can be used to model an application. That is why, often in a first step, parameters $\boldsymbol{\xi}$ have to be adjusted to obtain a model function. These parameters are determined using experiments. The obtained experimental data usually has measurements errors. This is why in many applications the true parameters are not known. If $\mathrm{P}(\boldsymbol{\xi})$ is solved for a single $\boldsymbol{\xi}$, the solution can be infeasible for a second choice of parameters. To avoid infeasiblility one can demand that the constraint is fulfilled for a complete range of parameters $\boldsymbol{\xi} \in U$. One obtains the following (worst-case) *robust optimization problem*:

$$\begin{aligned} \mathsf{SIP}_{\mathrm{rob}} : \min_{\boldsymbol{x}\in\mathbb{R}^n} \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & g(\boldsymbol{x},\boldsymbol{\xi}) \leq 0 \text{ for all } \boldsymbol{\xi} \in U \ . \end{aligned}$$

This problem is a semi-infinite optimization problem. The describing functions are exactly the same as before. This means that if the original functions have nice properties, the same properties are found again in the robust problem.

A geometric application of semi-infinite optimization are so-called *design-centering problems*. There, one considers a variable set $D(\boldsymbol{x}) \subseteq \mathbb{R}^m$, the so-called *design* and a fixed set $C \subseteq \mathbb{R}^m$, the so-called *container*. The task is to maximize the volume of $D(\boldsymbol{x})$ such that the design is contained within the container C. This means one considers the following optimization problem:

$$\begin{aligned} \mathrm{DC} : \max_{\boldsymbol{x}\in\mathbb{R}^n} \quad & \mathrm{vol}\big(D(\boldsymbol{x})\big) \\ \text{s.t.} \quad & D(\boldsymbol{x}) \subseteq C \ . \end{aligned}$$

If we assume that the container can be described by inequalities (for the sake of simplicity only one):

$$C = \{\boldsymbol{y} \in \mathbb{R}^m \mid g(\boldsymbol{y}) \leq 0\} \ ,$$

we can reformulate the containment condition and again end up with a semi-infinite problem:

$$\begin{aligned} \mathsf{SIP}_{\mathrm{DC}} : \min_{\boldsymbol{x}\in\mathbb{R}^n} \quad & -\operatorname{vol}\big(D(\boldsymbol{x})\big) \\ \text{s.t.} \quad & g(\boldsymbol{y}) \leq 0 \text{ for all } \boldsymbol{y} \in D(\boldsymbol{x}) \ . \end{aligned}$$

Whenever the container and the design have a nice description by easy functions, the same holds for the semi-infinite problem. Design-centering problems have an interesting application in the maximal utilization of gemstones [Win08, KSW08]. There the container is the raw stone and the design is the precious stone that has to be found. Not only containment conditions, but also other set-type conditions such as non-overlapping conditions, can be modeled easily by semi-infinite optimization problems (see for example [Sch13]).

Beside those two applications described in more detail, there is a whole variety of applications and related problems that can be modeled and solved by semi-infinite optimization problems. Classical applications are approximation problems ([HZ82]). More applications can be found in [HK93, Ste03, LS07]. In the works [MBB09, BBM09] the authors show that semi-infinite optimization can also be applied to model complicated constraints in thermodynamics.

The two examples as well as the further applications show that semi-infinite optimization is a great tool to find nice problem descriptions which maintain a lot of the original problem properties. However, these nice structural properties come at a price. There are infinitely many constraints, which automatically makes the problem harder to solve. Simply to check a given point for feasibility, an optimization problem (the so called *lower-level problem*) has to be solved.

This is why resolution approaches have been developed in the literature that try to reformulate the semi-infinite problems into a finite nonlinear problem. One possibility is to replace the lower-level problem by the KKT conditions. This approach results in an optimization problem with complementarity constraints. This reformulation was for example used in the works [SS03] and [SW10] to develop algorithms. Another possibility is to replace the lower-level problem with the help of the Wolfe dual (see [DHSS13]). As a main benefit no complementarity constraints have to be added.

Although only finite nonlinear optimization problems have then to be solved, these approaches have two drawbacks. First, the lower-level problems have to be convex. Otherwise these are not equivalent reformulations and a solution might be infeasible. A second drawback lies in the structure of the resulting problem. Due to the KKT conditions a complementarity condition is added. The Wolfe duality overcomes this, but still adds an equality constraint. This change in the structure

can make it harder to solve the problems in practice. In particular, the goal we formulated at the beginning is then not achieved. The nice structure and easy description may be lost.

Another well known possibility to solve semi-infinite optimization problems and to keep the structure of the original functions, are discretization methods (see for example [Ree91, RR98, LS07]). The idea is rather simple. We cannot consider infinitely many constraints at the same time, so we choose a finite subset $\dot{Y} \subseteq Y$. Instead of all constraints we only consider the ones in $\dot{Y} \subseteq Y$. We then consider the so called *discretized problem*:

$$\begin{aligned} \mathsf{SIP}(\dot{Y}): \min_{\boldsymbol{x}\in\mathbb{R}^n} \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & g(\boldsymbol{x},\boldsymbol{y}) \leq 0 \text{ for all } \boldsymbol{y} \in \dot{Y} \ . \end{aligned}$$

As there are only finitely many constraints, this problem can be solved by methods developed for finite nonlinear optimization such as SQP methods (for example [NW06]) or interior-point methods (for example [WB06]). The describing functions do not change, which means that the structure of the problems directly depends on the model. If we find an easy description of the application also the discretized problems will directly benefit from it. The question is how to choose a discretization $\dot{Y}$. We want to discuss here two different possibilities.

The first possibility is to choose a fine grid. This means that one chooses $\dot{Y}$ in such a way that the Hausdorff distance:

$$h = \max_{\boldsymbol{y}\in Y} \min_{\dot{\boldsymbol{y}}\in\dot{Y}} \|\boldsymbol{y} - \dot{\boldsymbol{y}}\|$$

becomes small (see for example [RR98]). One of the benefits is that the distance between a local solution $\boldsymbol{x}^*$ of the original problem and the solution of a discretized problem $\dot{\boldsymbol{x}}$ can be bounded (see [Sti01]). Unfortunately the results show that in the best case there is a $L > 0$ such that

$$\|\boldsymbol{x}^* - \dot{\boldsymbol{x}}\| \leq L \cdot h^2 \ .$$

In many cases only weaker bounds hold. This means that, to guarantee an accuracy of 10^{-6}, we need the Hausdorff distance to be around 10^{-3}. In a two-dimensional example we would need already around one million discretization points, if the best case bound holds. This means we need numerous discretization points which leads to an optimization problem with numerous constraints. Solving this optimization problem can be very time-consuming.

Another possibility is not to choose a fixed discretization set but to adapt a discretization set to a solution found so far. Blankenship and Falk already suggested in

[BF76] to start with a small discretization Y^1, then to solve the discretized problem $\mathsf{SIP}(Y^1)$ and obtain a first approximate solution $\boldsymbol{x}^1$. This first approximate solution only satisfies the constraints induced by the first discretization but is usually infeasible for the semi-infinite problem. In [BF76] it is suggested to add the most violated constraint, i.e. a $\boldsymbol{y}^1 \in Y$ with

$$g(\boldsymbol{x}^1, \boldsymbol{y}^1) = \max_{\boldsymbol{y} \in Y} g(\boldsymbol{x}^1, \boldsymbol{y}) \ .$$

The old discretization together with $\boldsymbol{y}^1$ form a new discretization Y^2, which is used to calculate the next approximate solution $\boldsymbol{x}^2$.

In the literature this simple scheme has been revisited multiple times and modified versions have been published. Reemtsen used in [Ree91] a fine discretization and solved the lower-level on this fine discretization. In [TR11] the authors modified the algorithm by Blankenship and Falk to obtain feasible points after finitely many steps. Mitsos then used these ideas in [Mit11] to obtain an outer and an inner approximation for global optimization. In [MT15] the ideas are extended to generalized semi-infinite optimization using a reformulation with disjunctive constraints. In his PhD-thesis [Sch13] Schwientek used a transformation function to solve generalized semi-infinite optimization problems more directly with the Blankenship and Falk algorithm.

Beside the nice structural properties described before, one of the reasons for the great success of the method by Blankenship and Falk are the nice convergence properties. The following results are well known to hold under mild assumptions (see [BF76], [Ree94]):

- If every $\boldsymbol{x}^k$ is feasible for $\mathsf{SIP}(Y^k)$, then any accumulation point is feasible for SIP.

- If every $\boldsymbol{x}^k$ is a global solution of $\mathsf{SIP}(Y^k)$, then any accumulation point is a global solution of SIP.

- If every $\boldsymbol{x}^k$ is a local solution of $\mathsf{SIP}(Y^k)$ and the radius of these solutions does not vanish in the limit, then any accumulation point is a local solution of SIP.

This means, that the guarantees for the solution of the discretized problem will under mild assumption also be true for a limit point. Most of the results also easily carry over to the aforementioned modifications.

As we have described it before, one of the motivations to use such an adaptive scheme is the hope to need fewer discretization points and to observe a fast con-

vergence. Unfortunately, to the best of our knowledge, there are no results known in the literature that describe a rate of convergence for adaptive discretization methods.

This is the question which is the main topic of this work: under which conditions does the algorithm by Blankenship and Falk provide a quadratic rate of convergence. For the case that the algorithm by Blankenship and Falk does not have a quadratic rate of convergence, we develop a new adaptive discretization method that converges quadratically under mild assumptions. The scope of the thesis is described in more detail in the next paragraph.

Scope of this work

The iterates $\boldsymbol{x}^k$ of the adaptive discretization method by Blankenship and Falk are in general not feasible for the original semi-infinite problem. The violation can be measured by

$$\alpha_k := \max_{\boldsymbol{y} \in Y} g(\boldsymbol{x}^k, \boldsymbol{y}) \ .$$

If $\alpha_k \leq 0$, then the current iterate will also be a solution of the original problem SIP. We can in some sense measure by α_k how close the approximate problem is to the original problem locally around $\boldsymbol{x}^k$. An important question to a quadratic rate of convergence is to bound the distance of the iterates $\boldsymbol{x}^k$ to a limit point $\boldsymbol{x}^*$ in terms of α_k.

This question is closely related to the question mentioned above: how to bound the same distance in terms of the mesh size h. Indeed, the statements provided in [Sti01] could be used to develop a bound in the case of adaptive discretization. However, we construct in this work a counterexample for the bounds provided in [Sti01] and develop new assumptions for a similar situation. Under these assumptions we show: if for every $k \in \mathbb{N}$ the iterate $\boldsymbol{x}^k$ is a local solution of the approximate problem SIP(Y^k) and they converge towards a local solution $\boldsymbol{x}^*$ of order ρ, then there is an $L_1 > 0$ such that

$$\|\boldsymbol{x}^k - \boldsymbol{x}^*\| \leq L_1 {\alpha_k}^{\frac{1}{\rho}} \ .$$

The next step towards a quadratic rate of convergence is to investigate the maximal violation α_k. Therefore the lower-level problems have to be considered. We show that under the Reduction Ansatz and some further regularity assumptions there is an $L_2 > 0$ such that

$$\alpha_k \leq L_2 \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^2 \ .$$

If the limit $\boldsymbol{x}^*$ is a local solution of order $\rho = 1$, then combining those two inequalities shows a quadratic rate of convergence.

Not every local solution is of order one. An easy example for a local minimum of order two is shown in Figure 1.1. We demonstrate that the iterates constructed according to the Blankenship and Falk algorithm do not converge with a quadratic rate. One of the reasons is the strict separated nature of the algorithm by Blankenship and Falk. In the approximate problem $\mathsf{SIP}(Y^k)$, all points in the discretization are considered, while all other points in Y are completely ignored. In the worst case we find a solution which maximizes the violation between two discretization points. In this work we aim at breaking this strictly separated scheme.

We have already seen that the Reduction Ansatz is needed to bound the maximal violation α_k. If we assume that the Reduction Ansatz holds at a given iterate $\boldsymbol{x}^k$ with $\alpha_k > 0$ we know that we can find a differentiable function $\boldsymbol{y}(\cdot)$ such that for every $\boldsymbol{x}$ close to $\boldsymbol{x}^k$ the point $\boldsymbol{y}(\boldsymbol{x})$ is a local solution of

$$\max_{\boldsymbol{y} \in Y} g(\boldsymbol{x}, \boldsymbol{y}) \ .$$

This means that we can develop the local solution linearly:

$$\overline{\boldsymbol{y}}^k(\boldsymbol{x}) := \boldsymbol{y}^k + D\boldsymbol{y}(\boldsymbol{x}) \cdot (\boldsymbol{x} - \boldsymbol{x}^k) \ .$$

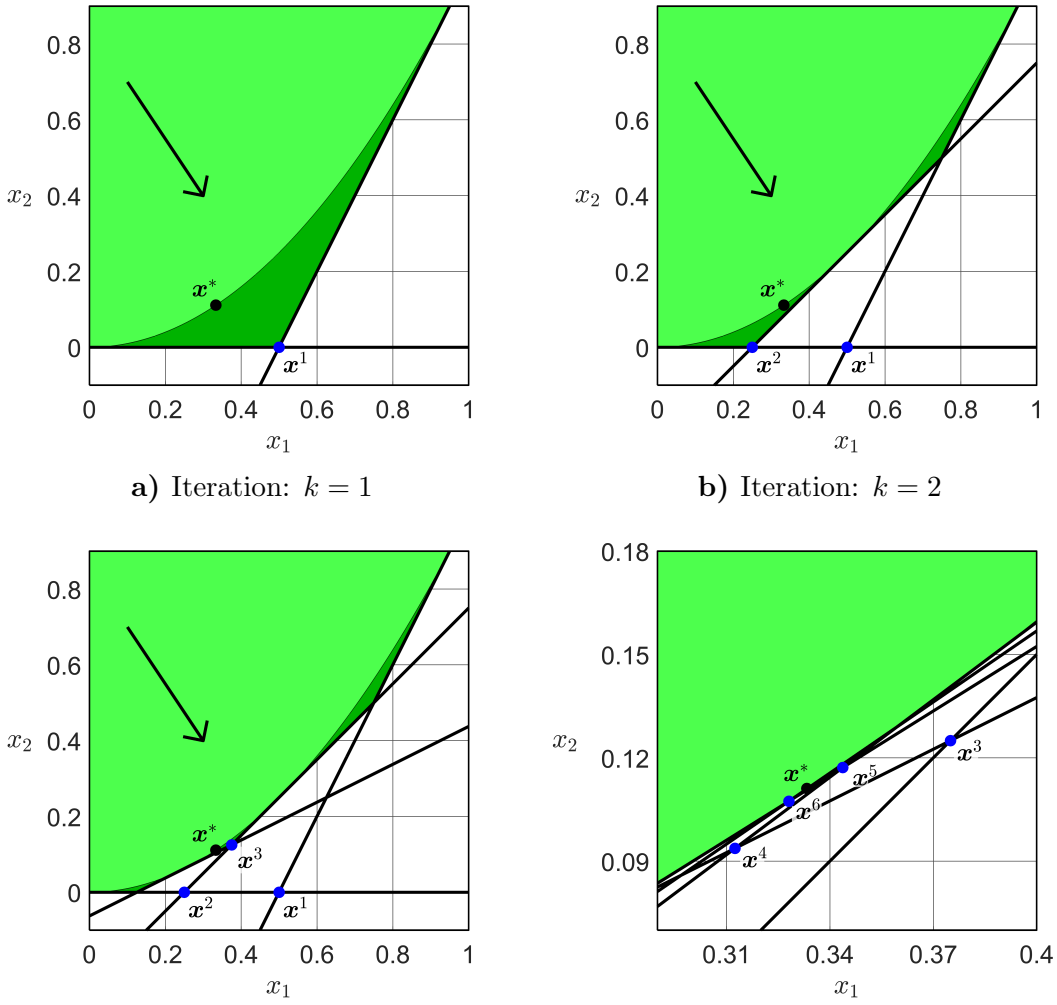
We use this linear information to construct an additional constraint that also accounts for the points in Y which are not yet in the discretization. The new approximate problem including the additional constraint and the constraints induced by the discretization is denoted by $\overline{\mathsf{SIP}}^k(Y^{k+1})$. The iterates of the new algorithm are shown in Figure 1.2.

The additional constraint has to be chosen carefully. The Blankenship and Falk algorithm has very nice convergence properties and one wants to keep as many of these properties as possible. With the chosen constraint we show in the following that one of the most important properties is directly inherited, namely that any accumulation point of the iterates $\boldsymbol{x}^k$ is feasible. For the Blankenship and Falk algorithm the type of solution of a limit point depends on the type of solution obtained for the approximate problems. As we add an additional constraint that changes for every iteration we do not have the following property anymore:

$$M^1 \supseteq M^2 \supseteq \cdots \supseteq M \ .$$

where M^k denotes the feasible set of the approximate problem $\overline{\mathsf{SIP}}^{k-1}(Y^k)$ in the k-th iteration and M the feasible set of problem SIP. We may even have

$$M^k \subseteq M \ .$$



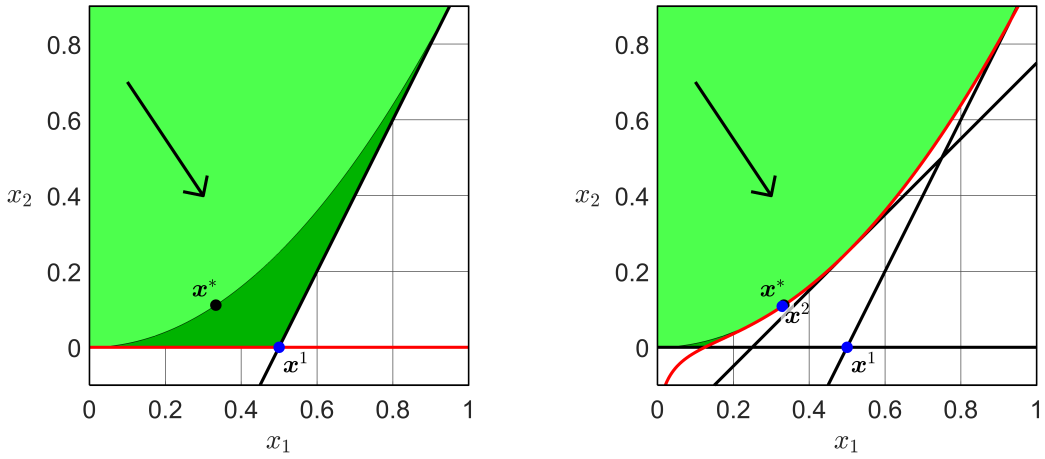
a) Iteration: $k = 1$ **b)** Iteration: $k = 2$

c) Iteration: $k = 3$ **d)** Iteration: $k = 6$

Figure 1.1: Example solved by the Blankenship and Falk algorithm with a linear rate of convergence [*light green* - feasible set of SIP, *dark green* - feasible set of approximate problem SIP(Y^k), *black arrow* - descent direction of objective, *black lines* - constraints induced by discretization, *blue points* - iterates, *black point* - limit solution]

As we construct the linearization at a point locally, we may cut away complete parts of the feasible set. This is why there is no hope that a limit point of global solutions is always a global solution of the original problem.

This is somehow expected as a linearization uses the derivative to approximate the solution, which is a local property. More interesting is the question about local minima. It plays an important role that $M^k \supseteq M$ does not hold in general. We use, compared to the analysis of the Blankenship and Falk algorithm, different



a) Iteration: $k = 1$ **b)** Iteration: $k = 2$

Figure 1.2: Adaptive discretization algorithm with a quadratic rate of convergence [*light green* - feasible set of SIP, *dark green* - feasible set of approximate problem $\overline{\mathsf{SIP}}^{k-1}(Y^k)$, *black arrow* - descent direction of objective, *black lines* - constraints induced by discretization, *red line* - additional constraint, *blue points* - iterates, *black point* - limit solution]

techniques to investigate this question. In a first step we consider stationary points and prove that under mild assumptions the limit is again a stationary point. After we have investigated stationary points we assume that an iterate $\boldsymbol{x}^k$, which is a local minimum of the approximate problem, satisfies a second-order sufficient condition. If the curvature in this second-order sufficient condition does not vanish and some further regularity assumptions hold, the limit will be a local solution of SIP.

After we have shown that many nice properties carry over from the original algorithm to the new algorithm, we turn to the reason why we introduced an additional constraint: the quadratic rate of convergence. Compared to our analysis of the Blankenship and Falk algorithm, we use a different tool: the so-called *strong stability* of stationary points. This concept was introduced by Kojima in [Koj80]. We assume that every iterate $\boldsymbol{x}^k$ is a stationary point of $\overline{\mathsf{SIP}}^{k-1}(Y^k)$ and that they converge towards a limit $\boldsymbol{x}^*$, which is a strongly stable stationary point. We show that $\boldsymbol{x}^k$ is a stationary point of

$$\begin{aligned} \mathsf{SIP}(\alpha_k, \boldsymbol{\beta}_k): \min_{\boldsymbol{x}\in\mathbb{R}^n} \quad & f(\boldsymbol{x}) + \boldsymbol{\beta}_k(\boldsymbol{x} - \boldsymbol{x}^k) \\ \text{s.t.} \quad & g(\boldsymbol{x}, \boldsymbol{y}) \leq \alpha_k \text{ for all } \boldsymbol{y} \in \dot{Y} \end{aligned}$$

for appropriate $\alpha_k, \boldsymbol{\beta}_k$. We show that the parameters α_k and $\boldsymbol{\beta}_k$ can be bounded under mild regularity assumptions, i.e. there are $L_1, L_2 > 0$ such that

$$|\alpha_k| \leq L_1 \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^4 ,$$
$$\|\boldsymbol{\beta}_k\| \leq L_2 \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^2 .$$

Strong stability of the limit point then leads to the Quadratic Convergence Theorem.

After we have described the algorithm and the analysis in the case of a fixed index set Y, we show how the algorithm can be extended to the case of a variable index set. In his PhD-thesis, Schwientek [Sch13] showed that the adaptive discretization can be extended to the case of variable index sets with the help of a transformation. The main assumption is the existence of a compact fixed set Z and a function $\boldsymbol{t}$ such that for every $\boldsymbol{x} \in \mathbb{R}^n$, we have:

$$\boldsymbol{t}(\boldsymbol{x}, Z) = Y(\boldsymbol{x}) .$$

The same can be used for the new adaptive discretization algorithm. There are two possibilities to construct a linearization. One can linearize the solution either in the space of $Y(\boldsymbol{x})$ or in the space of Z. We show that, for both possibilities, the statements for the case of a fixed index set can be transferred to the generalized case.

To show that all the theoretical statements can actually be observed, we implemented the examples in MATLAB®([MAT16]). We demonstrate that the new algorithms need for the final steps (from accuracy 10^{-1} to 10^{-5}), severely fewer iterations, while for the first steps both algorithms behave similarly. We discuss different further numerical aspects of the developed algorithms.

Structure of the thesis

The thesis is structured as follows. We formally introduce the needed theory of nonlinear and semi-infinite optimization in Chapter 2. We cover for both types of optimization problems the first and second-order optimality conditions (in Sections 2.1.1 and 2.2.1), as well as the concept of strong stability of stationary points (in Sections 2.1.2 and 2.2.2).

In Chapter 3 we investigate the question of a quadratic rate of convergence for the adaptive discretization proposed by Blankenship and Falk. We first introduce the

algorithm and summarize important results from the literature. We continue by investigating bounds based on the maximal current violation. We first transfer the results obtained in [Sti01] to the situation of adaptive discretization. In Section 3.1.1 we then construct a counterexample for the results by Still. In the next Section we give a new statement with a correct proof. In the second part of Chapter 3 we prove a quadratic rate of convergence for the special case of local minima of order one. We end this chapter with an example that shows that for minima of a higher order no quadratic rate can be expected.

Motivated by this example we introduce the new algorithm in Chapter 4. We show that the same example can now be solved with a quadratic rate of convergence. In Section 4.1 we investigate the properties of a limit point. We begin by proving that again every accumulation point of a sequence constructed by the algorithm is feasible. Next we show that the limit of stationary points is a stationary point in Section 4.1.1. For the convergence of local solutions we provide an example that shows that the results cannot be directly transferred and then give two different positive results in Section 4.1.2. In Section 4.2 we present and prove the Quadratic Convergence Theorem for the introduced algorithm.

In Chapter 5 we show how the developed ideas can be transferred to the case of a variable index set. This means we show how the newly developed algorithm can be adapted to handle generalized semi-infinite optimization problems. We show in Section 5.2 that the results previously presented also hold for this case.

We present numerical examples in Chapter 6. We show that the developed theoretical statements can be observed for these examples. We summarize our observations in Section 6.3.

The thesis closes with a summary of the results and a suggestion of future work.

2 Foundations of nonlinear and semi-infinite optimization

This chapter gives a short introduction into the theory of nonlinear and semi-infinite optimization, which is needed for this work. A more detailed introduction into nonlinear optimization can be found in [GK02, JS04], or in the English books [Ber99, BSS06, NW06]. Semi-infinite optimization is covered extensively in the books [HZ82, Pol97] and the overview articles [HK93, LS07]. We first introduce an optimization problem in general terms. Then in two sections we introduce first nonlinear optimization problems and then semi-infinite optimization problems. Both sections are structured similarly. In the first part of these sections we introduce conditions for optimality and stationary points. In the second part we summarize results about the strong stability of stationary points, which has been first introduced in [Koj80] and [Rob80]. While the concept is well studied for nonlinear optimization, there is less literature about this concept in semi-infinite optimization. We mainly follow the line presented in [Rüc99] but adapt the results to the situation needed in this work. We start by introducing a general optimization problem.

Let a closed set $M \subseteq \mathbb{R}^n$ and an at least continuous function $f : M \to \mathbb{R}$ be given. A general *optimization problem* or *minimization problem* is given by

$$\min\{f(\boldsymbol{x}) \mid \boldsymbol{x} \in M\} \ . \tag{2.1}$$

The set M is called the *feasible set* and the function f is called *objective function* (sometimes also simply *objective*). A point $\boldsymbol{x} \in M$ is called *feasible* . A point $\boldsymbol{x} \in \mathbb{R}^n \setminus M$ is called *infeasible*. Three cases can be distinguished:

- The set M is empty. We set $\min\{f(\boldsymbol{x}) \mid \boldsymbol{x} \in M\} := \infty$.
- The objective function f is not bounded from below on M . Then no minimum exists, we call the optimization problem *unbounded* and we set $\min\{f(\boldsymbol{x}) \mid \boldsymbol{x} \in M\} := -\infty$.

- The objective function f is bounded from below on M and M is not empty. Then, as M is closed and f is continuous, also the image $f(M)$ is closed. By the boundedness the minimum exists.

The third case can for example be enforced by requiring the feasible set M to be nonempty and compact.

Not in all literature about optimization it is assumed that the feasible set M is closed. In such cases, a minimum can not be guaranteed and one formally has to consider the following problem:

$$\inf\{f(\boldsymbol{x}) \mid \boldsymbol{x} \in M\} \ .$$

Indeed, we will introduce an optimization problem in Section 2.2 for which the feasible set is not necessarily closed, so-called *generalized semi-infinite optimization problems*. However, we will only consider problems of this type with an additional property. This property will also ensure closedness of the feasible set.

For the theoretical considerations it is enough to consider minimization problems as the following is well known:

$$\max\{-f(\boldsymbol{x}) \mid \boldsymbol{x} \in M\} = -\min\{f(\boldsymbol{x}) \mid \boldsymbol{x} \in M\} \ .$$

Different types of solutions can be characterized:

Definition 2.1. (Characterization of solutions)

- A feasible point, $\boldsymbol{x}^* \in M$, is called a *global solution*, a *global optimum* or a *global minimum*, if, for every $\boldsymbol{x} \in M$, the following holds:

$$f(\boldsymbol{x}^*) \leq f(\boldsymbol{x}) \ .$$

- A feasible point, $\boldsymbol{x}^* \in M$, is called a *local solution*, a *local optimum*, or a *local minimum* if there exists a radius $r > 0$ such that, for every $\boldsymbol{x} \in M \cap B_r(\boldsymbol{x}^*)$, the following holds:

$$f(\boldsymbol{x}^*) \leq f(\boldsymbol{x}) \ . \tag{2.2}$$

  The local solution is called *strict*, if the inequality in (2.2) can be replaced by a strict inequality, for every $\boldsymbol{x} \in M \cap B_r(\boldsymbol{x}^*)$ with $\boldsymbol{x} \neq \boldsymbol{x}^*$.

- A local minimum, $\boldsymbol{x}^*$, is called of *order* $\rho \in \mathbb{N}$, if there is a constant $L > 0$ and a radius $r > 0$ such that, for every $\boldsymbol{x} \in M \cap B_r(\boldsymbol{x})$, the following holds:

$$f(\boldsymbol{x}) - f(\boldsymbol{x}^*) \geq L\|\boldsymbol{x} - \boldsymbol{x}^*\|^\rho \ .$$

Maxima can be characterized analogously. Every local minimum of order $\rho \in \mathbb{N}$ is also a strict local minimum. The order describes how "strict" this local solution is, which means that it bounds the growth from below around this optimum. The strongest growth is given by $\rho = 1$.

For the theoretical as well as the numerical treatment the structure of f and the description of M is of great importance. In the next section we first assume that M can be described by finitely many equalities and inequalities. In Section 2.2 we consider infinitely many inequalities.

2.1 Nonlinear optimization

In this section we assume that the feasible set M is given by

$$M = \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \begin{array}{l} \boldsymbol{g}(\boldsymbol{x}) \leq 0, \\ \boldsymbol{h}(\boldsymbol{x}) = 0 \end{array} \right\} , \tag{2.3}$$

for twice continuously differentiable functions $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^{|I|}$ and $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^{|J|}$, where $|I| < \infty$ and $|J| < \infty$. An optimization problem with a feasible set given as in Equation (2.3) is called *nonlinear optimization problem*. We also write the optimization problem as:

$$\begin{array}{lll} \mathsf{P}\text{:} & \min\limits_{\boldsymbol{x} \in \mathbb{R}^n} & f(\boldsymbol{x}) \\ & s.t. & g_i(\boldsymbol{x}) \leq 0 \text{ for all } i \in I \ , \\ & & h_j(\boldsymbol{x}) = 0 \text{ for all } j \in J \ . \end{array}$$

Problem P is called *convex*, if the functions $g_i, i \in I$ and f are convex and the functions $h_j, j \in J$ are linear. In a convex optimization problem every local minimum is a global minimum (see for example [BSS06]).

As we will later introduce semi-infinite problems we will call problem P also *finite nonlinear* or simply *finite optimization problem*.

2.1.1 Optimality conditions

For the theoretical investigation and also for the design of algorithms it is of crucial importance to find equivalent conditions for a point being an optimum. We assume therefore for the rest of this section that the objective function, f, is twice

continuously differentiable. The following definitions and statements are all well known and can for example be found in [GK02] or also in the references mentioned at the beginning of this chapter.

Definition 2.2. (Stationary point, KKT conditions) A feasible point, $\boldsymbol{x}^* \in M$, is called *stationary point* or *critical point* for problem P, if there exist $\boldsymbol{\lambda}^* \in \mathbb{R}^{|I|}$ and $\boldsymbol{\mu}^* \in \mathbb{R}^{|J|}$ with:

$$Df(\boldsymbol{x}^*) + \sum_{i \in I} \lambda_i^* Dg_i(\boldsymbol{x}^*) + \sum_{j \in J} \mu_j^* Dh_j(\boldsymbol{x}^*) = 0 \ , \tag{2.4}$$

$$\lambda_i^* \geq 0 \text{ for all } i \in I \ , \tag{2.5}$$

$$\lambda_i^* g_i(\boldsymbol{x}^*) = 0 \text{ for all } i \in I \ . \tag{2.6}$$

The conditions given in (2.4)-(2.6) are called *Karush-Kuhn-Tucker conditions* (short: *KKT conditions*). The triple $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ is called *Karush-Kuhn-Tucker point* (short: *KKT point*).
If, for every $i \in I$, exactly one of the values $g_i(\boldsymbol{x}^*)$ and λ_i^* equals zero, one says that *strict complementary slackness* holds.

By introducing the *Lagrange function*, $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^{|I|} \times \mathbb{R}^{|J|} \to \mathbb{R}$, with

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) := f(\boldsymbol{x}) + \sum_{i \in I} \lambda_i g_i(\boldsymbol{x}) + \sum_{j \in J} \mu_j h_j(\boldsymbol{x}) \ ,$$

one can equivalently state condition (2.4) by

$$D_1 \mathcal{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) = 0 \ .$$

The components of the vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are called *Lagrange multipliers*.

In the literature one can find both names: stationary point and critical point. If the name critical point is used, stationary points are defined slightly differently. For this work the concept of strong stability, which we introduce in the following Section 2.1.2, is of great importance. In the corresponding literature the name stationary point is used. This is why, we use the name stationary point throughout the thesis.

The reason one is interested in stationary points is that under some further regularity conditions every local solution is a stationary point. We begin by introducing some regularity conditions and showing their connection. For $\boldsymbol{x} \in M$ we denote the set of *active indices* by

$$I_0(\boldsymbol{x}) := \{i \in I \mid g_i(\boldsymbol{x}) = 0\} \ .$$

Definition 2.3. (LICQ, MFCQ)

- A feasible point, $\boldsymbol{x} \in M$, satisfies the *Linear Independence Constraint Qualification* (short: LICQ) for problem P, if the vectors:

  $$Dg_i(\boldsymbol{x}), i \in I_0(\boldsymbol{x}) \text{ and } Dh_j(\boldsymbol{x}), j \in J$$

  are linearly independent.

- A feasible point, $\boldsymbol{x} \in M$, satisfies the *Mangasarian-Fromovitz Constraint Qualification* (short: MFCQ) for problem P, if the vectors $Dh_j(\boldsymbol{x}), j \in J$ are linearly independent and there exists a vector $\boldsymbol{\xi} \in \mathbb{R}^n$ with

  $$\begin{aligned} Dg_i(\boldsymbol{x})\boldsymbol{\xi} &\leq -1 \text{ for all } i \in I_0(\boldsymbol{x}) \ , \\ Dh_j(\boldsymbol{x})\boldsymbol{\xi} &= 0 \text{ for all } j \in J \ . \end{aligned}$$

The following connection between those regularity conditions is well known:

Lemma 2.4. *(*LICQ *implies* MFCQ*) Assume $\boldsymbol{x} \in M$ satisfies* LICQ *for* P *then $\boldsymbol{x}$ also satisfies* MFCQ*.*

The following theorem summarizes the connection between local minima and stationary points (see for example [GK02]):

Theorem 2.5. *(First order optimality conditions)*

- *Assume $\boldsymbol{x}^*$ is a local solution of* P *which satisfies the regularity condition* MFCQ*. There exist $\boldsymbol{\lambda}^* \in \mathbb{R}^{|I|}$ and $\boldsymbol{\mu}^* \in \mathbb{R}^{|J|}$ such that the KKT conditions* (2.4)-(2.6) *are fulfilled, i. e. $\boldsymbol{x}^*$ is a stationary point.*
  If even LICQ *is satisfied at $\boldsymbol{x}^*$, then the corresponding Lagrange multipliers are uniquely determined.*

- *Assume problem* P *is convex and there exists a point $\hat{\boldsymbol{x}}$ with*

  $$g_i(\hat{\boldsymbol{x}}) < 0, \text{ for } i \in I \text{ and } h_j(\hat{\boldsymbol{x}}) = 0, \text{ for } j \in J \ .$$

  Then a global solution is a stationary point.

- *Assume problem* P *is convex. Every stationary point is a global solution.*

It is clear that in the non-convex case not every stationary point will be a local solution. This can already be seen by considering the trivial example $\min\{-x^2 \mid -1 \leq x \leq 1\}$ at the point $x = 0$. Here, so-called second-order conditions are needed. We will first introduce them and then give a sufficient condition for the general case:

Definition 2.6. (SOSC) A KKT point, $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$, is said to satisfy the *Second-Order Sufficient Condition* (short: SOSC), if

$$\boldsymbol{d}^\top D_1^2 \mathcal{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)\boldsymbol{d} > 0 \text{ for every } \boldsymbol{d} \in T(\boldsymbol{x}^*), \boldsymbol{d} \neq 0 \ ,$$

where

$$T(\boldsymbol{x}^*) := \left\{ \boldsymbol{d} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} Dg_i(\boldsymbol{x}^*)\boldsymbol{d} \leq 0 \text{ for } i \in I_0(\boldsymbol{x}^*) \text{ with } \lambda_i^* = 0, \\ Dg_i(\boldsymbol{x}^*)\boldsymbol{d} = 0 \text{ for } i \in I_0(\boldsymbol{x}^*) \text{ with } \lambda_i^* > 0, \\ Dh_j(\boldsymbol{x}^*)\boldsymbol{d} = 0 \text{ for } j \in J \end{array} \right\} \ .$$

We can now state second-order necessary and sufficient conditions (again for example [GK02]).

Theorem 2.7. *(Second-order optimality conditions)*

- *Let $\boldsymbol{x}^* \in M$ be a local minimum that satisfies* LICQ*, then, with the uniquely determined Lagrange multipliers $\boldsymbol{\lambda}^*$ and $\boldsymbol{\mu}^*$ satisfying the KKT conditions, the following holds:*

$$\boldsymbol{d}^\top D_1^2 \mathcal{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)\boldsymbol{d} \geq 0 \text{ for every } \boldsymbol{d} \in T(\boldsymbol{x}^*) \ . \tag{2.7}$$

- *Let $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ be a KKT point that satisfies the second-order sufficient condition* SOSC*. Then $\boldsymbol{x}^*$ is a strict local minimum.*
  In [BS00] the authors show that the minimum is of order $\rho = 2$.

2.1.2 Strong stability of stationary points

After introducing stationary points and showing their connection to local solutions we now investigate their stability. Again we need to assume that the objective function, f, is twice continuously differentiable. Before we start with the formal introduction of strong stability, we shortly motivate why this concept is of importance for this work.

Within this thesis we do not directly solve a given problem P. Instead we generate iteratively problems which approximate the original problem (at least locally). Thus we will look at a sequence of problems, $\{\mathsf{P}_k\}_{k\in\mathbb{N}}$. In some sense they will converge and we will have a notion of a distance, $\text{dist}(\mathsf{P}_k, \mathsf{P})$, between the approximate problem and the original problem. The smaller $\text{dist}(\mathsf{P}_k, \mathsf{P})$ becomes the closer is the approximate problem to the original problem. Now consider for every $k \in \mathbb{N}$ a

stationary point $\boldsymbol{x}^k$ of P_k and a stationary point $\boldsymbol{x}^*$ of P. Assuming that $\boldsymbol{x}^k$ converge towards $\boldsymbol{x}^*$, we want to be able to bound the distance of the iterates by the means of $\operatorname{dist}(\mathsf{P}_k, \mathsf{P})$, i.e. we search an L with the property:

$$\|\boldsymbol{x}^k - \boldsymbol{x}^*\| \leq L \cdot \operatorname{dist}(\mathsf{P}_k, \mathsf{P}) \ . \tag{2.8}$$

If there exists such a bound, one can continue working on bounding the distance $\operatorname{dist}(\mathsf{P}_k, \mathsf{P})$. If no such bound exists, the iterates, $\boldsymbol{x}^k$, converge slower than the approximate problems, P_k. In the worst case we could have an arbitrarily slow convergence for the iterates, though we are able to control the distance $\operatorname{dist}(\mathsf{P}_k, \mathsf{P})$.

A possibility to ensure the existence of this bound is the stability of stationary points. Therefore we investigate perturbations of the original problem. We first introduce the considered perturbations. For an open bounded set, $U \subseteq \mathbb{R}^n$, and twice continuously differentiable functions, $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$, $\tilde{\boldsymbol{g}} : \mathbb{R}^n \to \mathbb{R}^{|I|}$ and $\tilde{\boldsymbol{h}} : \mathbb{R}^n \to \mathbb{R}^{|J|}$, we let:

$$\operatorname{norm}^1(\tilde{f}, \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{h}}, U) = \max \left\{ \begin{array}{l} \sup\limits_{\boldsymbol{x} \in U} \max \left\{ |\tilde{f}(\boldsymbol{x})|, \|D\tilde{f}(\boldsymbol{x})\| \right\}, \\ \max\limits_{i \in I} \sup\limits_{\boldsymbol{x} \in U} \max \left\{ |\tilde{g}_i(\boldsymbol{x})|, \|D\tilde{g}_i(\boldsymbol{x})\| \right\}, \\ \max\limits_{j \in J} \sup\limits_{\boldsymbol{x} \in U} \max \left\{ |\tilde{h}_j(\boldsymbol{x})|, \|D\tilde{h}_j(\boldsymbol{x})\| \right\} \end{array} \right\}$$

and

$$\operatorname{norm}^2(\tilde{f}, \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{h}}, U) = \max \left\{ \begin{array}{l} \sup\limits_{\boldsymbol{x} \in U} \max \left\{ |\tilde{f}(\boldsymbol{x})|, \|D\tilde{f}(\boldsymbol{x})\|, \|D^2\tilde{f}(\boldsymbol{x})\| \right\}, \\ \max\limits_{i \in I} \sup\limits_{\boldsymbol{x} \in U} \max \left\{ |\tilde{g}_i(\boldsymbol{x})|, \|D\tilde{g}_i(\boldsymbol{x})\|, \|D^2\tilde{g}_i(\boldsymbol{x})\| \right\}, \\ \max\limits_{j \in J} \sup\limits_{\boldsymbol{x} \in U} \max \left\{ |\tilde{h}_j(\boldsymbol{x})|, \|D\tilde{h}_j(\boldsymbol{x})\|, \|D^2\tilde{h}_j(\boldsymbol{x})\| \right\} \end{array} \right\} .$$

(For a matrix $A \in \mathbb{R}^{n \times n}$ let $\|A\| = \max\{\|A\boldsymbol{x}\| \mid \boldsymbol{x} \in \mathbb{R}^n, \|\boldsymbol{x}\| = 1\}$)

For $\delta > 0$ let

$$\mathcal{F}_\delta(U) := \left\{ (\tilde{f}, \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{h}}) : \mathbb{R}^n \to \mathbb{R}^{1+|I|+|J|} \;\middle|\; \begin{array}{c} \tilde{f}, \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{h}} \text{ twice continuously differentiable} \\ \text{and } \operatorname{norm}^2(\tilde{f}, \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{h}}, U) < \delta \end{array} \right\}$$

be the *set of perturbations* bounded by δ.

For a $\delta > 0$ and $(\tilde{f}, \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{h}}) \in \mathcal{F}_\delta(U)$, we consider the following perturbed optimization problem:

$$\begin{array}{rll} \mathsf{P}(\tilde{f}, \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{h}}) : & \min\limits_{\boldsymbol{x} \in \mathbb{R}^n} & f(\boldsymbol{x}) + \tilde{f}(\boldsymbol{x}) \\ & \text{s.t.} & g_i(\boldsymbol{x}) + \tilde{g}_i(\boldsymbol{x}) \leq 0 \text{ for all } i \in I \ , \\ & & h_j(\boldsymbol{x}) + \tilde{h}_j(\boldsymbol{x}) = 0 \text{ for all } j \in J \ . \end{array}$$

The notion of strong stability we introduce next is weaker compared to the assertion presented at the beginning of this section. Under LICQ this notion of stability already induces the existence of a constant as in (2.8). The concept has been first introduced by Kojima in [Koj80]. He studied the perturbed problems $\mathsf{P}(\tilde{f}, \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{h}})$. He demanded local uniqueness of a stationary point and a continuity property of the map of stationary solutions.

Definition 2.8. (Strong stability of stationary points [Koj80]) A stationary point, $\boldsymbol{x}^*$, is called *strongly stable*, if there exists an $\varepsilon^* > 0$ such that, for every $\varepsilon \in (0, \varepsilon^*]$, there is a $\delta > 0$ with the property that, for every $(\tilde{f}, \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{h}}) \in \mathcal{F}_\delta(B_{\varepsilon^*}(\boldsymbol{x}^*))$, problem $\mathsf{P}(\tilde{f}, \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{h}})$ has a stationary point $\tilde{\boldsymbol{x}}$ with

$$\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\| < \varepsilon \ ,$$

which is unique in $B_{\varepsilon^*}(\boldsymbol{x}^*)$.

In his work Kojima introduced for $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{\lambda} \in \mathbb{R}^{|I|}$, $\boldsymbol{\mu} \in \mathbb{R}^{|J|}$ the function

$$\Psi(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \begin{pmatrix} \left(Df(\boldsymbol{x}) + \sum_{i \in I} (\lambda_i)^+ Dg_i(\boldsymbol{x}) + \sum_{j \in J} \mu_j Dh_j(\boldsymbol{x}) \right)^\top \\ (\lambda_i)^- - g_i(\boldsymbol{x}), \quad i \in I \\ h_j(\boldsymbol{x}), \quad j \in J \end{pmatrix} \ , \tag{2.9}$$

where, for $\lambda \in \mathbb{R}$, we let $(\lambda)^+ := \max\{0, \lambda\}$ and $(\lambda)^- := \min\{\lambda, 0\}$.

In general this function will not be differentiable, but, as a composition of Lipschitz continuous functions, it will again be Lipschitz continuous. Kojima used this function to give an equivalent definition of a stationary point. A point $\boldsymbol{x}^* \in \mathbb{R}^n$ is a stationary point, if and only if there exist $\boldsymbol{\lambda}^* \in \mathbb{R}^{|I|}$ and $\boldsymbol{\mu}^* \in \mathbb{R}^{|J|}$ such that

$$\Psi(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) = 0 \ .$$

With the help of this description Kojima derived different equivalent conditions for a stationary point, $\boldsymbol{x}^* \in M$, to be strongly stable. We again need some notation to introduce this equivalent characterization:

For a KKT point, $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ and $\bar{I} \subseteq \{i \in I_0(\boldsymbol{x}^*) \mid \lambda_i^* = 0\}$, we set

$$R^{\bar{I}} = \left\{ \boldsymbol{d} \in \mathbb{R}^n \,\middle|\, \begin{array}{l} Dh_j(\boldsymbol{x}^*)\boldsymbol{d} = 0 \text{ for } j \in J, \\ Dg_i(\boldsymbol{x}^*)\boldsymbol{d} = 0 \text{ for } i \in I_0(\boldsymbol{x}^*) \text{ with } \lambda_i > 0, \\ Dg_i(\boldsymbol{x}^*)\boldsymbol{d} = 0 \text{ for } i \in \bar{I} \end{array} \right\} \ .$$

Let $Z \in \mathbb{R}^{n \times n}$ be a symmetric matrix. By Sylvesters law we can choose an arbitrary basis matrix Q of $R^{\bar{I}}$ and the sign of

$$\det\left(Q^\top Z Q\right)$$

does not change. We denote this sign by $\operatorname{sign}\det\left(Z \,\middle|\, R^{\bar{I}}\right)$.

Theorem 2.9. *(Characterization of strong stability [Koj80]) Let $\boldsymbol{x}^*$ be a stationary point of* P *assume that* LICQ *holds at $\boldsymbol{x}^*$. Then $\boldsymbol{x}^*$ is strongly stable, if and only if*

$$\operatorname{sign}\det\left(D_1^2\mathcal{L}(\boldsymbol{x}^*,\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*)\ \middle|R^{\bar{I}}\right)$$

is constant and nonvanishing for all $\bar{I}\subseteq\{i\in I_0(\boldsymbol{x}^*)\mid\lambda_i^*=0\}$, where $\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*$ are the uniquely determined Lagrange multipliers such that $(\boldsymbol{x}^*,\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*)$ is a KKT point.

In [JKT90] the authors introduced a further characterization in terms of Clark's concept of a generalized derivative (see for example [Cla90]). As this characterization has an important consequence, we will shortly introduce the concept and the results derived in [JKT90].

Given an open bounded subset, $U\subset\mathbb{R}^n$, a Lipschitz continuous function, $\boldsymbol{f}:U\to\mathbb{R}^n$ and a point $\boldsymbol{x}\in U$, the set

$$\partial f(\boldsymbol{x})=\operatorname{conv}\Big(\big\{\ Z\mid\exists\{\boldsymbol{x}^k\}_{k\in\mathbb{N}}\subseteq U,\boldsymbol{x}^k\to\boldsymbol{x},\ \boldsymbol{f}\text{ is differentiable in }\boldsymbol{x}^k\text{ and }D\boldsymbol{f}(\boldsymbol{x}^k)\to Z\big\}\Big)$$

is called the *generalized Jacobian*. Here, conv denotes the convex hull of a set. The generalized Jacobian is said to be *nonsingular* at $\boldsymbol{x}\in U$, if every $Z\in\partial f(\boldsymbol{x})$ is nonsingular.

We can now give an equivalent characterization of a strongly stable stationary point:

Theorem 2.10. *(Strong stability and nonsingularity [JKT90]) Let $\boldsymbol{x}^*$ be a stationary point of* P *and $\boldsymbol{\lambda}^*\in\mathbb{R}^{|I|}$ and $\boldsymbol{\mu}^*\in\mathbb{R}^{|J|}$ such that $\Psi(\boldsymbol{x}^*,\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*)=0$. Then $\boldsymbol{x}^*$ is strongly stable, if and only if $\partial\Psi(\boldsymbol{x}^*,\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*)$ is nonsingular.*

The nonsingularity of a differentiable function enables the application of the implicit function theorem. Exactly the same is possible for a Lipschitz continuous function and the nonsingularity of the generalized Jacobian. We are going to introduce an implicit function theorem and apply it to the strong stability under LICQ.

Define for a bounded open set, $U\subseteq\mathbb{R}^n$, the following norm, for every Lipschitz continuous function $\boldsymbol{f}:U\to\mathbb{R}^n$:

$$\|\boldsymbol{f}\|_{L,U}:=\max\left\{\sup_{\boldsymbol{x}\in U}\boldsymbol{f}(\boldsymbol{x}),\inf\left\{c\mid\|\boldsymbol{f}(\boldsymbol{x})-\boldsymbol{f}(\boldsymbol{y})\|\le c\|\boldsymbol{x}-\boldsymbol{y}\|\text{ for all }\boldsymbol{x},\boldsymbol{y}\in U\right\}\right\}.$$

The following implicit function theorem is introduced in [JKT90]:

Theorem 2.11. *(Lipschitz implicit function [JKT90]) Let U be a nonempty bounded open subset of $\mathbb{R}^n$, let $\boldsymbol{f} : U \to \mathbb{R}^n$ be a Lipschitz continuous function, and let $\boldsymbol{x}^* \in U$ be a point satisfying $\boldsymbol{f}(\boldsymbol{x}^*) = 0$. Suppose that $\partial \boldsymbol{f}(\boldsymbol{x}^*)$ is nonsingular. Then there exist real numbers $\varepsilon > 0$, $\delta > 0$ and $L > 0$ such that the following holds:*

i) For each Lipschitz continuous function $\boldsymbol{g} : U \to \mathbb{R}^n$ with $\|\boldsymbol{f} - \boldsymbol{g}\|_{L,U} < \delta$, there exists a unique solution $\boldsymbol{x}(\boldsymbol{g})$ in $B_\varepsilon(\boldsymbol{x}^*)$ to the equation $\boldsymbol{g}(\boldsymbol{x}) = 0$.

ii) for two Lipschitz continuous functions $\boldsymbol{g}^1, \boldsymbol{g}^2 : U \to \mathbb{R}^n$ with $\|\boldsymbol{f} - \boldsymbol{g}^1\|_{L,U} < \delta$ and $\|\boldsymbol{f} - \boldsymbol{g}^2\|_{L,U} < \delta$, the following holds:

$$\left\|\boldsymbol{x}(\boldsymbol{g}^1) - \boldsymbol{x}(\boldsymbol{g}^2)\right\| \leq L \sup_{\boldsymbol{x} \in U} \left\|\boldsymbol{g}^1(\boldsymbol{x}) - \boldsymbol{g}^2(\boldsymbol{x})\right\| \leq L \|\boldsymbol{g}^1 - \boldsymbol{g}^2\|_{L,U} \ .$$

Before this theorem can be applied to the function Ψ, its norm has to be bounded. In [JKT90] it is shown that for an arbitrary bounded open set, $U := U_1 \times U_2 \times U_3 \subseteq \mathbb{R}^n \times \mathbb{R}^{|I|} \times \mathbb{R}^{|J|}$, there exists an $L > 0$ such that, for every twice continuously differentiable function, $(\tilde{f}, \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{h}}) : \mathbb{R}^n \to \mathbb{R}^{1+|I|+|J|}$, the following is true:

$$\sup_{(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\mu}) \in U} \left\|\Psi(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) - \Psi_{\tilde{f},\tilde{\boldsymbol{g}},\tilde{\boldsymbol{h}}}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu})\right\| \leq L \operatorname{norm}^1(\tilde{f}, \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{h}}, U_1) \ , \tag{2.10}$$

$$\|\Psi - \Psi_{\tilde{f},\tilde{\boldsymbol{g}},\tilde{\boldsymbol{h}}}\|_{L,U} \leq L \operatorname{norm}^2(\tilde{f}, \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{h}}, U_1) \ , \tag{2.11}$$

where $\Psi_{\tilde{f},\tilde{\boldsymbol{g}},\tilde{\boldsymbol{h}}}$ is obtained by replacing $f, \boldsymbol{g}, \boldsymbol{h}$ in Equation (2.9) by $f + \tilde{f}, \boldsymbol{g} + \tilde{\boldsymbol{g}}, \boldsymbol{h} + \tilde{\boldsymbol{h}}$.

Applying Theorem 2.11 to $\Psi(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ induces together with Equation (2.10) and (2.11) the following theorem :

Theorem 2.12. *(Lipschitz constant for strong stability [JKT90]) Let $\boldsymbol{x}^*$ be a strongly stable stationary point of* P *and suppose that* LICQ *is satisfied. Let ε^* be chosen as in Definition 2.8. There exist real numbers $\delta^* > 0$ and $L > 0$ such that:*

- *for every $(\tilde{f}, \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{h}}) \in \mathcal{F}_{\delta^*}(B_{\varepsilon^*}(\boldsymbol{x}^*))$, the set $B_{\varepsilon^*}(\boldsymbol{x})$ contains a unique stationary point, $\boldsymbol{x}(\tilde{f}, \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{h}})$, of problem $\mathsf{P}(\tilde{f}, \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{h}})$.*
- *the following holds:*

$$\left\|\boldsymbol{x}^* - \boldsymbol{x}(\tilde{f}, \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{h}})\right\| \leq L \operatorname{norm}^1\left(\tilde{f}, \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{h}}, B_{\varepsilon^*}(\boldsymbol{x}^*)\right) \ .$$

Strong stability together with LICQ will therefore be enough to guarantee the property demanded in Equation (2.8).

One has to mention here that a similar concept, so called strong regularity of KKT points, has been developed by Robinson in [Rob80] nearly at the same time. In Section 2.2.2 we discuss how to extend the notion of strong stability to the semi-infinite case. There it will be easier to extend the concept of a stationary point, but more difficult to extend the notion of a KKT point. This is due to the infinite number of inequalities. However, for the finite case it is shown in [KT90] that both concepts are under LICQ equivalent.

In the remainder of this section we discuss under which conditions we can even have a differentiable function describing the stationary points locally. To this end, assume that strict complementary slackness holds and LICQ is satisfied for a KKT point $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$. In such a point the function Ψ given in Equation (2.9) is continuously differentiable. By Theorem 2.9 the derivative is invertible, if and only if $\operatorname{sign}\det\left(D_1^2\mathcal{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)\ \middle|T(\boldsymbol{x}^*)\right) \neq 0$, where under strict complementary slackness we have:

$$T(\boldsymbol{x}^*) = \left\{ \boldsymbol{d} \in \mathbb{R}^n \;\middle|\; \begin{matrix} Dg_i(\boldsymbol{x}^*)\boldsymbol{d} = 0 \text{ for } i \in I_0(\boldsymbol{x}^*) \\ Dh_j(\boldsymbol{x}^*)\boldsymbol{d} = 0 \text{ for } j \in J \end{matrix} \right\} = R^{\emptyset} \ .$$

Considering only stationary points which are potentially local solutions, i.e. fulfill the second-order necessary condition given in Theorem 2.7, Equation (2.7), one sees easily that $D\Psi(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ is invertible, if and only if:

$$\boldsymbol{d}^\top \mathcal{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)\boldsymbol{d} > 0 \text{ for all } \boldsymbol{d} \in T(\boldsymbol{x}^*), \boldsymbol{d} \neq 0 \ ,$$

which is exactly the second-order sufficient condition (SOSC).

For twice continuously differentiable functions $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $\boldsymbol{g} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{|I|}$ and $\boldsymbol{h} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{|J|}$ we consider the following so called *parametric optimization problem*:

$$\begin{aligned} \mathsf{P}(\boldsymbol{\gamma}) : \min_{\boldsymbol{x} \in \mathbb{R}^n} \quad & f(\boldsymbol{x}, \boldsymbol{\gamma}) \\ \text{s.t.} \quad & \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{\gamma}) \leq 0 \ , \\ & \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{\gamma}) \leq 0 \ . \end{aligned}$$

With the considerations above one receives applying the implicit function theorem for differentiable functions the following well known result (see for example [Fia83, JS04]):

Theorem 2.13. *(Differentiability of stationary points) For a given parameter vector, $\boldsymbol{\gamma}^* \in \mathbb{R}^m$, assume that $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ is a KKT point of $\mathsf{P}(\boldsymbol{\gamma}^*)$. Moreover, assume that for $\mathsf{P}(\boldsymbol{\gamma}^*)$ the following is true:*

- *at* $\boldsymbol{x}^*$ LICQ *is satisfied.*
- $\boldsymbol{x}^*, \boldsymbol{\lambda}^*$ *fulfill the strict complementary slackness condition.*
- $\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*$ *satisfy the second-order sufficient condition* SOSC.

Then there are $\delta > 0$, $\varepsilon > 0$ *and a continuous differentiable function,* $\boldsymbol{\nu} = (\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) : B_\delta(\boldsymbol{\gamma}^*) \to \mathbb{R}^n \times \mathbb{R}^{|I|} \times \mathbb{R}^{|J|}$, *such that, for every* $\boldsymbol{\gamma} \in B_\delta(\boldsymbol{\gamma}^*)$*:*

- *the point* $\boldsymbol{x}(\boldsymbol{\gamma})$ *is a local minimum of* $\mathsf{P}(\boldsymbol{\gamma})$ *which is unique in* $B_\varepsilon(\boldsymbol{x}^*)$.
- *the vectors* $\boldsymbol{\lambda}(\boldsymbol{\gamma}), \boldsymbol{\mu}(\boldsymbol{\gamma})$ *are the unique Lagrange multipliers such that* $\big(\boldsymbol{x}(\boldsymbol{\gamma}),$ $\boldsymbol{\lambda}(\boldsymbol{\gamma}),\ \boldsymbol{\mu}(\boldsymbol{\gamma})\big)$ *is a KKT point of* $\mathsf{P}(\boldsymbol{\gamma})$.

Remark 2.14. (Derivatives of implicit function $\boldsymbol{\nu}$) With the implicit function theorem also the derivative of $\boldsymbol{\nu}$ can be calculated. Therefore denote by

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\gamma}) = f(\boldsymbol{x}, \boldsymbol{\gamma}) + \sum_{i \in I} \lambda_i g_i(\boldsymbol{x}, \boldsymbol{\gamma}) + \sum_{j \in J} \mu_j h_j(\boldsymbol{x}, \boldsymbol{\gamma})$$

the Lagrangian of the parametric optimization problem $\mathsf{P}(\boldsymbol{\gamma})$. The following holds, for every parameter vector $\boldsymbol{\gamma}$ close to $\boldsymbol{\gamma}^*$:

$$\begin{pmatrix} D\boldsymbol{x}(\boldsymbol{\gamma}) \\ D\boldsymbol{\lambda}_0(\boldsymbol{\gamma}) \\ D\boldsymbol{\mu}(\boldsymbol{\gamma}) \end{pmatrix} = -\begin{pmatrix} D_1^2\mathcal{L}\big(\boldsymbol{x}(\boldsymbol{\gamma}), \boldsymbol{\lambda}(\boldsymbol{\gamma}), \boldsymbol{\mu}(\boldsymbol{\gamma}), \boldsymbol{\gamma}\big) & \Big(D_1 G\big(\boldsymbol{x}(\boldsymbol{\gamma}), \boldsymbol{\gamma}\big)\Big)^\top \\ D_1 G\big(\boldsymbol{x}(\boldsymbol{\gamma}), \boldsymbol{\gamma}\big) & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} D_4 D_1 \mathcal{L}\big(\boldsymbol{x}(\boldsymbol{\gamma}), \boldsymbol{\lambda}(\boldsymbol{\gamma}), \boldsymbol{\mu}(\boldsymbol{\gamma}), \boldsymbol{\gamma}\big) \\ D_2 G\big(\boldsymbol{x}(\boldsymbol{\gamma}), \boldsymbol{\gamma}\big) \end{pmatrix}, \tag{2.12}$$

where

$$\boldsymbol{\lambda}_0 = \big(\lambda_i, \quad i \in I_0(\boldsymbol{x}^*)\big)$$

and

$$G(\boldsymbol{x}, \boldsymbol{\gamma}) = \begin{pmatrix} g_i(\boldsymbol{x}, \boldsymbol{\gamma}), i \in I_0(\boldsymbol{x}^*) \\ h_j(\boldsymbol{x}, \boldsymbol{\gamma}), j \in J \end{pmatrix}.$$

For every $i \in I \setminus I_0(\boldsymbol{x}^*)$ it is true:

$$D\lambda_i(\boldsymbol{\gamma}) = 0 \ .$$

If the involved functions are even more than twice continuously differentiable, then the implicit function $\boldsymbol{\nu}$ is more then once continuously differentiable.

To check the existence of derivatives for a given point and given parameters one could check the assumptions of Theorem 2.13. Checking the second order sufficient condition directly might be hard. It is often easier to check the existence of the inverse matrix in Equation 2.12. By Theorem 2.9 and Theorem 2.10 this is equivalent for a local minimum.

2.2 Semi-infinite optimization

We turn now to the case of infinitely many inequality constraints. Let I, J be finite index sets and

$$\begin{aligned} \boldsymbol{g} &: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{|I|} \ , \\ \boldsymbol{v} &: \mathbb{R}^m \to \mathbb{R}^{|J|} \end{aligned}$$

be twice continuously differentiable functions. We consider the following so-called *semi-infinite optimization problem*.

$$\mathsf{SIP} : \min\{f(\boldsymbol{x}) \mid \boldsymbol{x} \in M\} \ ,$$

where

$$M = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) \leq 0 \text{ for all } \boldsymbol{y} \in Y\}$$

and

$$Y = \{\boldsymbol{y} \in \mathbb{R}^m \mid \boldsymbol{v}(\boldsymbol{y}) \leq 0\} \ .$$

As before we also write the optimization problem SIP as:

$$\begin{aligned} \mathsf{SIP} : \min_{\boldsymbol{x} \in \mathbb{R}^n} \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & g_i(\boldsymbol{x}, \boldsymbol{y}) \leq 0 \text{ for all } i \in I, \boldsymbol{y} \in Y \ . \end{aligned} \tag{2.13}$$

The inequalities in (2.13) are called *semi-infinite constraints*, the set Y is called *semi-infinite index-set* and its elements *index variables* or simply *indices*. In general the cardinality of Y will be infinite, which means that the feasible set M is described by infinitely many constraints. Throughout this work we assume that Y is nonempty and compact.

The presented definition can be extended in different ways. It is possible to add finitely many equalities in the description of the feasible set as well as to the description of the semi-infinite index-set. Even though many of the results presented in this work can be extended to this case, a detailed discussion of those extensions would exceed the scope of this thesis.

Finite inequality constraints do not have to be considered separately. Assume we are given a function $h : \mathbb{R}^n \to \mathbb{R}$. For an arbitrary but fixed $\hat{\boldsymbol{y}} \in Y$ and for every $\boldsymbol{x} \in \mathbb{R}^n$, the following holds:

$$h(\boldsymbol{x}) \leq 0 \Leftrightarrow h(\boldsymbol{x}) - \|\hat{\boldsymbol{y}} - \boldsymbol{y}\|^2 \leq 0 \text{ for all } \boldsymbol{y} \in Y \ .$$

(It is beneficial for the following considerations to add the additional norm.) However, this reformulation is only done to simplify the exposition of the following

theory and the derived results. For the numerical implementation it is often beneficial to consider finite inequality constraints separately.

Because there are infinitely many constraints determining whether a point is feasible becomes an optimization problem. For every $i \in I$ and $\boldsymbol{x} \in \mathbb{R}^n$, we introduce the so-called *lower-level problem*:

$$\begin{aligned} Q_i(\boldsymbol{x}) : \max_{\boldsymbol{y} \in \mathbb{R}^m} \quad & g_i(\boldsymbol{x}, \boldsymbol{y}) \\ s.t. \quad & v_j(\boldsymbol{y}) \leq 0 \text{ for all } j \in J \ . \end{aligned} \tag{2.14}$$

For every fixed $\boldsymbol{x} \in \mathbb{R}^n$, this is an "ordinary" nonlinear optimization problem considered in Section 2.1. The optimization variables, $\boldsymbol{x}$, become parameters in this problem and the former index variables, $\boldsymbol{y}$, become optimization variables. For every $i \in I$, $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{y} \in \mathbb{R}^m$ and $\boldsymbol{\mu}^i \in \mathbb{R}^{|J|}$, we denote the Lagrange function of $Q_i(\boldsymbol{x})$ by

$$\mathcal{L}_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\mu}^i) := g_i(\boldsymbol{x}, \boldsymbol{y}) - \sum_{j \in J} \mu_j^i \cdot v_j(\boldsymbol{y}) \ .$$

For every $i \in I$ the function $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ with

$$\varphi_i(\boldsymbol{x}) := \max\{g_i(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{y} \in Y\}$$

is called *optimal value function* of the i-th lower level problem.

The feasible set can be equivalently described by

$$M = \{\boldsymbol{x} \in \mathbb{R}^n \mid \varphi_i(\boldsymbol{x}) \leq 0 \text{ for all } i \in I\} \ .$$

Although this is again a nonlinear optimization problem, it is often not beneficial to treat this problem directly by this reformulation. The functions $\varphi_i, i \in I$ are only implicitly given and for each evaluation an optimization problem has to be solved. This can turn out to be a time-consuming task itself. Nevertheless, this reformulation gives rise to a for theory and solution techniques important local reduction. We discuss this further in the next paragraph.

Reduction Ansatz

We now introduce a well known technique to locally reduce the problem to a nonlinear problem. The so called *Reduction Ansatz* (see for example [HZ82, JTW92, Kla92]).

For every $i \in I$ and $\boldsymbol{x} \in M$ we denote the set of *active indices* by

$$Y_0^i(\boldsymbol{x}) := \{\boldsymbol{y} \in Y \mid g_i(\boldsymbol{x}, \boldsymbol{y}) = 0\} \ .$$

Every $\boldsymbol{y}^* \in Y_0^i(\boldsymbol{x})$ is a global solution to the lower level problem $Q_i(\boldsymbol{x})$. If we assume that LICQ holds in every $\boldsymbol{y} \in Y$, then every global solution is a stationary point. If we moreover assume that every global solution is even a strongly stable stationary point, then by compactness of Y and local uniqueness there can only be finitely many global solutions. This means we can write, for every $i \in I$:

$$Y_0^i(\boldsymbol{x}) = \{\boldsymbol{y}^{i,1}, \ldots, \boldsymbol{y}^{i,q_i}\} \ .$$

Further by strong stability (see Theorem 2.12), there exists a $\delta_1 > 0$ and an $\varepsilon > 0$ such that, for every $i \in I$ and $1 \leq k \leq q_i$, there is a Lipschitz continuous function

$$\boldsymbol{y}^{i,k} : B_{\delta_1}(\boldsymbol{x}^*) \to Y \tag{2.15}$$

such that, for every $\boldsymbol{x} \in B_{\delta_1}(\boldsymbol{x}^*)$, the point $\boldsymbol{y}^{i,k}(\boldsymbol{x})$ is the unique stationary point within $B_\varepsilon(\boldsymbol{y}^{i,k})$ for the i-th lower level problem $Q_i(\boldsymbol{x})$.

By continuity and the compactness of the sets $Y \setminus \bigcup_{k=1}^{q_i} B_\varepsilon(\boldsymbol{y}^{i,k})$ there is $\delta_2 > 0$ such that, for every $\boldsymbol{x} \in B_{\delta_2}(\boldsymbol{x}^*)$, there is no global solution within $Y \setminus \bigcup_{k=1}^{q_i} B_\varepsilon(\boldsymbol{y}^{i,k})$ to the i-th lower level problem $Q_i(\boldsymbol{x})$. Thus, all global solutions can be described by the Lipschitz continuous functions in Equation (2.15). For $\delta = \min\{\delta_1, \delta_2\}$ we have:

$$M \cap B_\delta(\boldsymbol{x}^*) = \{\boldsymbol{x} \in B_\delta(\boldsymbol{x}^*) \mid \forall\ i \in I, 1 \leq k \leq q_i : \ g_i\big(\boldsymbol{x}, \boldsymbol{y}^{i,k}(\boldsymbol{x})\big) \leq 0\} \ . \tag{2.16}$$

If we assume that strict complementary slackness holds for every $i \in I$ and $\boldsymbol{y} \in Y_0^i(\boldsymbol{x})$, then the functions $\boldsymbol{y}^{i,k}$ are even differentiable by Theorem 2.13. The calculation of the derivatives of $g_i\big(\boldsymbol{x}, \boldsymbol{y}^{i,k}(\boldsymbol{x})\big)$ is well known (see the above references). Usually a trick is used here. By Theorem 2.13 not only the local solution, $\boldsymbol{y}^{i,k}$, can be described by a differentiable function but also the unique Lagrange-multipliers, $\boldsymbol{\mu}^{i,k} \geq 0$, satisfying the KKT conditions. By complementarity the following holds, for every $i \in I$:

$$g_i\big(\boldsymbol{x}, \boldsymbol{y}^{i,k}(\boldsymbol{x})\big) = \mathcal{L}_i\big(\boldsymbol{x}, \boldsymbol{y}^{i,k}(\boldsymbol{x}), \boldsymbol{\mu}^{i,k}(\boldsymbol{x})\big) \ .$$

About the derivatives of the Lagrangian several things are known:
(1) as $\boldsymbol{y}^{i,k}(\boldsymbol{x})$ is a local solution and LICQ is satisfied, we have :

$$D_2\mathcal{L}\big(\boldsymbol{x}, \boldsymbol{y}^{i,k}(\boldsymbol{x}), \boldsymbol{\mu}^{i,k}(\boldsymbol{x})\big) = 0 \ .$$

(2) we have seen in Remark 2.14 that, for every $j \in J$, we have either $v_j(\boldsymbol{y}^{i,k}(\boldsymbol{x})) = 0$ or $D\mu_j^{i,k}(\boldsymbol{x}) = 0$, which means:

$$D_3\mathcal{L}_i\big(\boldsymbol{x}, \boldsymbol{y}^{i,k}(\boldsymbol{x}), \boldsymbol{\mu}^{i,k}(\boldsymbol{x})\big) D\boldsymbol{\mu}^{i,k}(\boldsymbol{x}) = 0 \ .$$

Combining both Equations we receive the following derivative:

$$\begin{aligned} D_{\tilde{\boldsymbol{x}}}\Big[g_i\big(\tilde{\boldsymbol{x}}, \boldsymbol{y}^{i,k}(\tilde{\boldsymbol{x}})\big)\Big]_{\tilde{\boldsymbol{x}}=\boldsymbol{x}} &= D_{\tilde{\boldsymbol{x}}}\Big[\mathcal{L}_i\big(\tilde{\boldsymbol{x}}, \boldsymbol{y}^{i,k}(\tilde{\boldsymbol{x}}), \boldsymbol{\mu}^{i,k}(\tilde{\boldsymbol{x}})\big)\Big]_{\tilde{\boldsymbol{x}}=\boldsymbol{x}} \\ &= D_1 g_i(\boldsymbol{x}, \boldsymbol{y}^{i,k}(\boldsymbol{x})) \ . \end{aligned} \tag{2.17}$$

This is again a differentiable function. Again by Remark 2.14, the following holds:

$$\begin{aligned} & D^2_{\tilde{\boldsymbol{x}}}\Big[g_i\big(\tilde{\boldsymbol{x}}, \boldsymbol{y}^{i,k}(\tilde{\boldsymbol{x}})\big)\Big]_{\tilde{\boldsymbol{x}}=\boldsymbol{x}} \\ &= D^2_1 g_i\big(\boldsymbol{x}, \boldsymbol{y}^{i,k}(\boldsymbol{x})\big) + D_2 D_1 g_i\big(\boldsymbol{x}, \boldsymbol{y}^{i,k}(\boldsymbol{x})\big) D\boldsymbol{y}^{i,k}(\boldsymbol{x}) \\ &= D^2_1 g_i\big(\boldsymbol{x}, \boldsymbol{y}^{i,k}(\boldsymbol{x})\big) - \Big(D\boldsymbol{y}^{i,k}(\boldsymbol{x})\Big)^\top \cdot D^2_2 \mathcal{L}_i\big(\boldsymbol{x}, \boldsymbol{y}^{i,k}(\boldsymbol{x}), \boldsymbol{\mu}^{i,k}(\boldsymbol{x})\big) \cdot D\boldsymbol{y}^{i,k}(\boldsymbol{x}) \ . \end{aligned} \tag{2.18}$$

This means that we can describe M locally by finitely many twice continuously differentiable functions.

We collect the needed Assumption in the following definition.

Definition 2.15. (Reduction Ansatz) Suppose LICQ holds at every point in Y and let $\boldsymbol{x}^* \in M$.
We say that the *Reduction Ansatz* is satisfied for Problem SIP at $\boldsymbol{x}^*$, if, for every $i \in I$ and every active index $\boldsymbol{y}^* \in Y_0^i(\boldsymbol{x}^*)$, strict complementary slackness and the second-order sufficient condition are satisfied for $Q_i(\boldsymbol{x}^*)$.

If the Reduction Ansatz holds at a given point, we can consider the problem locally as a finite nonlinear optimization problem discussed in Section 2.1.

Generalized semi-infinite Optimization

Before we show how the concepts, introduced in Section 2.1.1 and Section 2.1.2, can be transferred to the case of SIP, we want to introduce one generalization of these problems, so-called *generalized semi-infinite optimization problems*. Therefore we replace the function $\boldsymbol{v}$ in the definition of SIP by a twice continuously differentiable function $\boldsymbol{u} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{|\hat{J}|}$. We consider the following optimization problem.

$$\begin{aligned} \mathsf{GSIP} : \min_{\boldsymbol{x} \in \mathbb{R}^n} \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & g_i(\boldsymbol{x}, \boldsymbol{y}) \leq 0 \text{ for all } i \in I, \boldsymbol{y} \in Y(\boldsymbol{x}) \ , \end{aligned}$$

where

$$Y(\boldsymbol{x}) = \big\{\boldsymbol{y} \in \mathbb{R}^m \mid u_j(\boldsymbol{x}, \boldsymbol{y}) \leq 0 \text{ for all } j \in \hat{J}\big\} \ .$$

The difference to problem SIP is, that the infinite index-set depends on the choice of $\boldsymbol{x}$. This dependency can change the structure of the feasible set severely. Even

closedness is not necessarily given anymore. An example and an investigation of the structure and theory of GSIPs can be found in [Ste03]. Within this work we only consider GSIPs which can be transformed into a problem of type SIP:

Definition 2.16. (Transformable GSIP) Suppose for problem GSIP there exists a twice continuously differentiable function $\boldsymbol{v} : \mathbb{R}^{\tilde{m}} \to \mathbb{R}^{|J|}$ and a twice continuously differentiable function $\boldsymbol{t} : \mathbb{R}^n \times Z \to \mathbb{R}^m$, where

$$Z := \left\{ \boldsymbol{z} \in \mathbb{R}^{\tilde{m}} \mid \boldsymbol{v}(\boldsymbol{z}) \leq 0 \right\} \ .$$

Further suppose that Z is nonempty, compact and, for every $\boldsymbol{x} \in \mathbb{R}^n$, the following holds:

$$\boldsymbol{t}(\boldsymbol{x}, Z) = Y(\boldsymbol{x}) \ .$$

If such a transformation $\boldsymbol{t}$ and a set Z exist, GSIP is called *transformable*.

For a transformable GSIP and an $i \in I$ the constraint

$$g_i(\boldsymbol{x}, \boldsymbol{y}) \leq 0 \text{ for all } \boldsymbol{y} \in Y(\boldsymbol{x})$$

can be equivalently replaced by the constraint

$$\tilde{g}_i(\boldsymbol{x}, \boldsymbol{z}) := g_i\big(\boldsymbol{x}, \boldsymbol{t}(\boldsymbol{x}, \boldsymbol{z})\big) \leq 0 \text{ for all } \boldsymbol{z} \in Z \ ,$$

which is again of the same type as in (2.13). This means that we can equivalently consider a problem of type SIP. We will therefore introduce no further theory for GSIPs and assume that a given GSIP is transformable. However, for the development of algorithms it can beneficial to use the original describing functions. As it is shown for example in [Sch13] the transformation can destroy helpful properties especially convexity.

In [Sti99] it is shown that under suitable assumptions, any general semi-infinite optimization problem, can be at least locally converted into a standard one. Of practical value is such a transformation only in cases where it is defined globally.

2.2.1 Optimality conditions

In this section we show how the results for nonlinear optimization, introduced in Section 2.1.1, can be transferred to the semi-infinite case. Again we assume that the objective function f is twice continuously differentiable. All results within this section are well known and can for example be found in [LS07]. The difference to the previous definition of a stationary point (Definition 2.2) is that infinitely many constraints can be active. To transfer the results only a finite subset is chosen:

Definition 2.17. (Stationary point for SIP) A feasible point, $\boldsymbol{x}^* \in M$, is called *stationary point* of SIP, if, for every $i \in I$, there are finitely many active indices, $\boldsymbol{y}^{i,k} \in Y_0^i(\boldsymbol{x}^*), 1 \leq k \leq q_i$, and real numbers, $\lambda^{i,k} \geq 0, 1 \leq k \leq q_i$, such that:

$$Df(\boldsymbol{x}^*) + \sum_{i \in I} \sum_{k=1}^{q_i} \lambda^{i,k} Dg_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,k}) = 0 \ . \tag{2.19}$$

The real numbers $\lambda^{i,k}$ are again called *Lagrange multipliers.*

Also the regularity conditions can be transferred

Definition 2.18. (ELICQ and EMFCQ)

- A feasible point, $\boldsymbol{x} \in M$, satisfies the *Extended Linear Independence Constraint Qualification* (short: ELICQ) for problem SIP, if the vectors:

  $$D_1 g_i(\boldsymbol{x}, \boldsymbol{y}), \quad i \in I, \boldsymbol{y} \in Y_0^i(\boldsymbol{x})$$

  are linearly independent.

- A feasible point, $\boldsymbol{x} \in M$, is said to satisfies the *Extended Mangasarian-Fromovitz Constraint Qualification* (short: EMFCQ), if there is a vector $\boldsymbol{\xi} \in \mathbb{R}^n$ with

  $$D_1 g_i(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{\xi} \leq \ -1 \text{ for every } i \in I, \boldsymbol{y} \in Y_0^i(\boldsymbol{x}) \ .$$

Again, the following connection between those two conditions holds:

Lemma 2.19 (ELICQ implies EMFCQ)**.** *Assume $\boldsymbol{x} \in M$ satisfies* ELICQ *for SIP, then $\boldsymbol{x}$ also satisfies* EMFCQ*.*

A very similar connection of stationary points and local solutions as in Theorem 2.5 exists:

Theorem 2.20. *(First order optimality for* SIP*) Assume $\boldsymbol{x}^* \in M$ is a local solution of SIP and satisfies the regularity condition* EMFCQ*, then $\boldsymbol{x}^*$ is a stationary point. If also* ELICQ *is satisfied, then the multipliers in Equation* (2.19) *are uniquely determined.*

Next, we want to transfer the second-order optimality condition presented in Theorem 2.21 to the semi-infinite case. A natural way to achieve this is to use the

Reduction Ansatz. In this case, for every $i \in I$, there are only finitely many active indices, i.e.:

$$Y_0^i(\boldsymbol{x}^*) = \{\boldsymbol{y}^{i,1}, \dots, \boldsymbol{y}^{i,q_i}\} \ .$$

Moreover, there is an $\delta > 0$ and we can describe, for $i \in I$ and $1 \leq k \leq q_i$, the active indices locally by continuously differentiable functions, $\boldsymbol{y}^{i,k} : B_\delta(\boldsymbol{x}^*) \to Y$, such that $\boldsymbol{y}^{i,k}(\boldsymbol{x}^*) = \boldsymbol{y}^{i,k}$ and

$$M \cap B_\delta(\boldsymbol{x}^*) = \left\{\boldsymbol{x} \in B_\delta(\boldsymbol{x}^*) \mid \forall\ i \in I, 1 \leq k \leq q_i :\ g_i\big(\boldsymbol{x}, \boldsymbol{y}^{i,k}(\boldsymbol{x})\big) \leq 0\right\} \ .$$

We have seen that, for every $i \in I$ and $1 \leq k \leq q_i$, the function $g_i\big(\boldsymbol{x}, \boldsymbol{y}^{i,k}(\boldsymbol{x})\big)$ is twice continuously differentiable.

As we have by the Reduction Ansatz only finitely many constraints, we can introduce in contrast to Definition 2.17 a Lagrange multiplier $\lambda_{i,k} \geq 0$ for every $i \in I$ and $1 \leq k \leq q_i$. We denote the Lagrangian of the reduced problem by

$$L(\boldsymbol{x}^*, \boldsymbol{\lambda}) := f(\boldsymbol{x}^*) + \sum_{i \in I} \sum_{k=1}^{q_i} \lambda_{i,k} \cdot g_i\big(\boldsymbol{x}^*, \boldsymbol{y}^{i,k}(\boldsymbol{x}^*)\big) \ . \tag{2.20}$$

Also the cone of critical directions can be transferred. Consider a stationary point $\boldsymbol{x} \in M$ with corresponding Lagrange multipliers $\boldsymbol{\lambda}$ fulfilling Equation (2.19). Let

$$T_{\mathsf{SIP}}(\boldsymbol{x}^*) := \left\{ \boldsymbol{d} \in \mathbb{R}^n \,\middle|\, \begin{matrix} D_1 g_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,k})\boldsymbol{d} \leq 0 \text{ for } i \in I, 1 \leq k \leq q_i \text{ with } \lambda_{i,k} = 0 \\ D_1 g_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,k})\boldsymbol{d} = 0 \text{ for } i \in I, 1 \leq k \leq q_i \text{ with } \lambda_{i,k} > 0 \end{matrix} \right\} \ .$$

The following result about second-order conditions can for example be found in [LS07] and [HK93]:

Theorem 2.21. *(Second-order optimality conditions for* SIP*) For $\boldsymbol{x}^* \in M$ assume that the Reduction Ansatz is satisfied*

- *Suppose that $\boldsymbol{x}^*$ is a local minimum of* SIP *and* ELICQ *is satisfied. For the uniquely defined Lagrange-multipliers $\boldsymbol{\lambda} \geq 0$ satisfying Equation* (2.19) *the following holds:*

$$\boldsymbol{d}^\top D_1^2 L(\boldsymbol{x}^*, \boldsymbol{\lambda})\boldsymbol{d} \geq 0 \text{ for all } \boldsymbol{d} \in T_{\mathsf{SIP}}(\boldsymbol{x}^*) \ .$$

- *Suppose that $\boldsymbol{x}^*$ is a stationary point with Lagrange-multipliers $\boldsymbol{\lambda} \geq 0$ satisfying Equation* (2.19)*. If*

$$\boldsymbol{d}^\top D_1^2 L(\boldsymbol{x}^*, \boldsymbol{\lambda})\boldsymbol{d} > 0 \text{ for all } \boldsymbol{d} \in T_{\mathsf{SIP}}(\boldsymbol{x}^*), \boldsymbol{d} \neq 0 \ , \tag{2.21}$$

then $\boldsymbol{x}^*$ is a strict local minimum of order $\rho = 2$. If (2.21) *holds we say that the* extended second-order sufficient condition *(short* ESOSC*) is satisfied.*

We have calculated the first and the second derivative of the reduced problem in Equations (2.17) and (2.18). Plugging them into the definition of the Lagrangian in (2.20) shows that the first derivative of the Lagrangian only consists of derivatives with respect to $\boldsymbol{x}$. This is the reason why in the first order optimality conditions also only derivatives with respect to $\boldsymbol{x}$ appear. The situation is different for the second-order conditions. Here also a second derivative with respect to $\boldsymbol{y}$ appears, namely:

$$-\sum_{i\in I}\sum_{k=1}^{q_i}\lambda_{i,k}\left(D\boldsymbol{y}^{i,k}(\boldsymbol{x})\right)^{\top}\cdot D_2^2\mathcal{L}_i\big(\boldsymbol{x},\boldsymbol{y}^{i,k}(\boldsymbol{x}),\boldsymbol{\mu}^{i,k}(\boldsymbol{x})\big)\cdot D\boldsymbol{y}^{i,k}(\boldsymbol{x})\ ,$$

where $\boldsymbol{\mu}^{i,k}(\boldsymbol{x})$ are the to $\boldsymbol{y}^{i,k}(\boldsymbol{x})$ corresponding Lagrange-multipliers satisfying the KKT conditions for $Q_i(\boldsymbol{x})$. This term is called *shift term*. Further investigations about this term can be found in [BS98] and [Kaw88].

2.2.2 Strong stability of stationary points

We again investigate the behavior of stationary points under perturbations. The concept of strong stability was first introduced to semi-infinite optimization by Rückmann in [Rüc99]. We introduce the concept and the for this thesis needed results. Again assume for this section that the objective function f is twice continuously differentiable.

For a bounded open set, $U\subseteq\mathbb{R}^n$, and twice continuously differentiable functions, $\tilde{f}:\mathbb{R}^n\to\mathbb{R}$ and $\tilde{\boldsymbol{g}}:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^{|I|}$, we let

$$\operatorname{norm}_{\mathsf{SIP}}(\tilde{f},\tilde{\boldsymbol{g}},U):=\max\left\{\begin{array}{l}\sup\limits_{\boldsymbol{x}\in U}\max\left\{|\tilde{f}(\boldsymbol{x})|,\|D\tilde{f}(\boldsymbol{x})\|,\|D^2\tilde{f}(\boldsymbol{x})\|\right\},\\ \max\limits_{i\in I}\sup\limits_{\boldsymbol{x}\in U}\max\limits_{\boldsymbol{y}\in Y}\max\left\{|\tilde{g}_i(\boldsymbol{x},\boldsymbol{y})|,\|D\tilde{g}_i(\boldsymbol{x},\boldsymbol{y})\|,\|D^2\tilde{g}_i(\boldsymbol{x},\boldsymbol{y})\|\right\}\end{array}\right\}.$$

For $\delta>0$ let

$$\mathcal{F}_\delta(U):=\left\{(\tilde{f},\tilde{\boldsymbol{g}})\ \middle|\ \begin{array}{c}\tilde{f}:\mathbb{R}^n\to\mathbb{R}\text{ twice continuosly differentiable,}\\ \tilde{\boldsymbol{g}}:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^{|I|}\text{ twice continuously differentiable,}\\ \operatorname{norm}(\tilde{f},\tilde{\boldsymbol{g}},U)<\delta\end{array}\right\}.$$

Completely analogous to Definition 2.8 we can define strong stability in the semi-infinite case:

Definition 2.22. (Strong stability for SIP [Rüc99]) A stationary point $\boldsymbol{x}^*$ of SIP is called *strongly stable*, if there is a $\varepsilon^* > 0$ with the property that for every $\varepsilon \in (0, \varepsilon^*]$ there is a $\delta > 0$ such that, for every $(\tilde{f}, \tilde{\boldsymbol{g}}) \in \mathcal{F}_\delta(B_{\varepsilon^*}(\boldsymbol{x}^*))$, the problem

$$\begin{aligned} \mathsf{SIP}(\tilde{f}, \tilde{\boldsymbol{g}}) : \min_{\boldsymbol{x} \in \mathbb{R}^n} \quad & f(\boldsymbol{x}) + \tilde{f}(\boldsymbol{x}) \\ \text{s.t.} \quad & g_i(\boldsymbol{x}, \boldsymbol{y}) + \tilde{g}_i(\boldsymbol{x}, \boldsymbol{y}) \leq 0 \text{ for all } i \in I, \boldsymbol{y} \in Y \end{aligned}$$

has a within $B_{\varepsilon^*}(\boldsymbol{x}^*)$ unique stationary point $\boldsymbol{x}(\tilde{f}, \tilde{\boldsymbol{g}})$ and

$$\|\boldsymbol{x}^* - \boldsymbol{x}(\tilde{f}, \tilde{\boldsymbol{g}})\| < \varepsilon \ .$$

We denote by $M(\tilde{f}, \tilde{\boldsymbol{g}})$ the feasible set of problem $\mathsf{SIP}(\tilde{f}, \tilde{\boldsymbol{g}})$.

Similar to Kojima Rückmann considered an equivalent characterization of stationary points. First assume that at a given point, $\boldsymbol{x}^* \in M$, the Reduction Ansatz holds, i.e. there are $\varepsilon_r^* > 0$ and $\delta_r^* > 0$ such that, for every $(\tilde{f}, \tilde{\boldsymbol{g}}) \in \mathcal{F}_{\delta_r^*}(B_{\varepsilon_r^*}(\boldsymbol{x}^*))$ and $i \in I$, there are continuously differentiable functions, $\boldsymbol{y}^{i,k}_{g_i + \tilde{g}_i} : \overline{B}_{\varepsilon_r^*}(\boldsymbol{x}^*) \to \mathbb{R}^m, 1 \leq k \leq q_i$, such that

$$\begin{aligned} & M(\tilde{f}, \tilde{\boldsymbol{g}}) \cap B_{\varepsilon_r^*}(\boldsymbol{x}^*) \\ = \ & \{\boldsymbol{x} \in B_{\varepsilon_r^*}(\boldsymbol{x}^*) \mid \forall\, i \in I, 1 \leq k \leq q_i : \ (g_i + \tilde{g}_i)\big(\boldsymbol{x}, \boldsymbol{y}^{i,k}_{g_i + \tilde{g}_i}(\boldsymbol{x})\big) \leq 0\} \ , \end{aligned}$$

where $\overline{B}_{\varepsilon_r^*}(\boldsymbol{x}^*)$ denotes the closure of $B_{\varepsilon_r^*}(\boldsymbol{x}^*)$. We intentionally added here the semi-infinite constraining function g_i in the description to mark the dependency of the solutions, $\boldsymbol{y}^{i,k}_{g_i}$, to the semi-infinite constraints. This dependency is the reason why the previous results presented in Section 2.1.2 can not directly be applied to this situation.

In [Rüc99] Rückmann considered, for $\boldsymbol{x} \in B_{\varepsilon_r^*}(\boldsymbol{x}^*)$, the following Lipschitz continuous function

$$\Psi(\boldsymbol{x}, \boldsymbol{\lambda}) = \begin{pmatrix} \big(Df(\boldsymbol{x}) + \sum_{i \in I} \sum_{k=1}^{q_i} (\lambda^{i,k})^+ D_1 g_i\big(\boldsymbol{x}, \boldsymbol{y}^{i,k}_{g_i}(\boldsymbol{x})\big)\big)^\top \\ (\lambda^{i,k})^- - g_i\big(\boldsymbol{x}, \boldsymbol{y}^{i,k}_{g_i}(\boldsymbol{x})\big), \qquad i \in I, 1 \leq k \leq q_i \end{pmatrix} \ . \tag{2.22}$$

It is easy to see that $\boldsymbol{x}^*$ is a stationary point if and only if there exist multipliers $\boldsymbol{\lambda}^*$ with $\Psi(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) = 0$.

In [Rüc99] a similar statement to Theorem 2.9 is shown. Stating this theorem would again require a lot of notation. But in the proof of this theorem Rückmann also showed the following:

Theorem 2.23. *(Strong stability and nonsingularity [Rüc99]) Let $\boldsymbol{x}^*$ be a stationary point of SIP with multipliers $\boldsymbol{\lambda}^* \geq 0$ fulfilling Equation (2.19). Assume ELICQ is satisfied at $\boldsymbol{x}^*$ and the Reduction Ansatz holds. Then $\boldsymbol{x}^*$ is strongly stable, if an only if the generalized Jacobian $\partial\Psi(\boldsymbol{x}, \boldsymbol{\lambda}^*)$ is nonsingular.*

The following second result is shown regarding only EMFCQ and not ELICQ

Theorem 2.24. *(Strong stability under* EMFCQ *and not* ELICQ *[Rüc99]) Let* $\boldsymbol{x}^*$ *be a stationary point of* SIP*. Suppose that the Reduction Ansatz and* EMFCQ *are satisfied at* $\boldsymbol{x}^*$ *and* ELICQ *is not satisfied. Then* $\boldsymbol{x}^*$ *is strongly stable if and only if for every choice of Lagrange multipliers* $\boldsymbol{\lambda} \geq 0$ *with*

$$D_1 L(\boldsymbol{x}^*, \boldsymbol{\lambda}) = 0 \ ,$$

the following holds:

$$\boldsymbol{d}^\top D_1^2 L(\boldsymbol{x}, \boldsymbol{\lambda}) \boldsymbol{d} > 0 \text{ for all } \boldsymbol{d} \in R(\boldsymbol{x}^*), \boldsymbol{d} \neq 0 \ ,$$

where $L(\boldsymbol{x}^*, \boldsymbol{\lambda})$ *is defined as for Theorem 2.21 and*

$$R(\boldsymbol{x}^*) := \left\{ \boldsymbol{d} \in \mathbb{R}^n \mid D_1 g_i\left(\boldsymbol{x}^*, \boldsymbol{y}^{i,k}\right)\boldsymbol{d} = 0 \text{ for } i \in I, 1 \leq k \leq q_i \text{ with } \lambda_{i,k} > 0 \right\} \ .$$

The condition is even stronger then the second-order sufficient condition given in Theorem 2.21. This means that in the second case only local minima can be strongly stable.

Remark 2.25 (Multiple semi-infinite constraints)**.** In [Rüc99] the results are only shown regarding a single semi-infinite constraint. In theory multiple semi-infinite constraints can be reformulated by a single semi infinite constraint. Therefore let

$$\hat{g}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) := \sum_{i \in I} z_i g_i(\boldsymbol{x}, \boldsymbol{y}) - \sum_{i \in I} \sum_{i_2 \in I, i \neq i_2} z_i z_{i_2} \ .$$

For $Z := \{\boldsymbol{z} \in \mathbb{R}^{|I|} \mid \forall i \in I : z_i \geq 0, \sum_{i \in I} z_i = 1\}$ and for every $\boldsymbol{x} \in \mathbb{R}^n$, the following holds:

$$\begin{aligned} &\hat{g}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \leq 0 \text{ for all } (\boldsymbol{y}, \boldsymbol{z}) \in Y \times Z \\ \Leftrightarrow\ &g_i(\boldsymbol{x}, \boldsymbol{y}) \leq 0 \text{ for all } i \in I, \boldsymbol{y} \in Y \ . \end{aligned}$$

Also all other properties can be transferred.

To gain a Lipschitz type relation we would like to use again Theorem 2.11 and receive a similar statement to Theorem 2.12. This is not done in [Rüc99]. That is why we add the proofs for these statements here. As a first step we need to bound similarly as in Equations (2.10) and (2.11) the Lipschitz norm.

Lemma 2.26. *(Bound on Lipschitz norm) For* $\gamma > 0$ *and a bounded open set* $U_2 \subseteq \mathbb{R}^{\left(\sum_{i \in I} q_i\right)}$*, there exists a* $L > 0$ *and a* δ *with* $0 < \delta < \delta_r^*$ *such that, for every twice continuously differentiable function* $(\tilde{f}, \tilde{\boldsymbol{g}}) \in \mathcal{F}_\delta(B_{\varepsilon_r}(\boldsymbol{x}^*))$*, the following holds:*

$$\sup_{(\boldsymbol{x}, \boldsymbol{\lambda}) \in B_{\varepsilon_r^*}(\boldsymbol{x}^*) \times U_2} \|\Psi(\boldsymbol{x}, \boldsymbol{\lambda}) - \Psi_{\tilde{f}, \tilde{\boldsymbol{g}}}(\boldsymbol{x}, \boldsymbol{\lambda})\| \leq L \operatorname{norm}_{\mathsf{SIP}}\left(\tilde{f}, \tilde{\boldsymbol{g}}, B_\varepsilon(\boldsymbol{x}_r^*)\right) \ ,$$

$$\|\Psi - \Psi_{\tilde{f}, \tilde{\boldsymbol{g}}}\|_{L, B_\varepsilon(\boldsymbol{x}_r^*) \times U_2} \leq \gamma \ ,$$

where $\Psi_{\tilde{f}, \tilde{\boldsymbol{g}}}$ *is obtained by replacing* $f, \boldsymbol{g}$ *in* (2.22) *by* $f + \tilde{f}, \boldsymbol{g} + \tilde{\boldsymbol{g}}$*.*

Proof. We first introduce, for every $(\tilde{f}, \tilde{\boldsymbol{g}}) \in \mathcal{F}_{\delta_r^*}(B_{\varepsilon_r^*}(\boldsymbol{x}^*))$, $\boldsymbol{x} \in B_{\varepsilon_r^*}(\boldsymbol{x}^*)$, $i \in I$ and $1 \leq k \leq q_i$, an abbreviation:

$$h_{i,k}(\boldsymbol{x}) := g_i\big(\boldsymbol{x}, \boldsymbol{y}_{g_i}^{i,k}(\boldsymbol{x})\big) - (g_i + \tilde{g}_i)\big(\boldsymbol{x}, \boldsymbol{y}_{g_i+\tilde{g}_i}^{i,k}(\boldsymbol{x})\big) \ .$$

In the first part of the proof we introduce some bounds on $h_{i,k}$. The claimed inequalities then follow from these bounds.

Bounds on auxiliary function: First, there is by strong stability (see Theorem 2.12) a K_1 such that, for every $\boldsymbol{x} \in B_{\varepsilon_r^*}(\boldsymbol{x}^*)$, $i \in I$ and $1 \leq k \leq q_i$:

$$\big\|\boldsymbol{y}_{g_i}^{i,k}(\boldsymbol{x}) - \boldsymbol{y}_{g_i+\tilde{g}_i}^{i,k}(\boldsymbol{x})\big\| \leq K_1 \operatorname{norm}^1(\tilde{g}_i(\boldsymbol{x}, \cdot), 0, 0, Y) \leq K_1 \operatorname{norm}_{\mathsf{SIP}}\left(\tilde{f}, \tilde{\boldsymbol{g}}, B_{\varepsilon_r^*}(\boldsymbol{x}^*)\right) \ .$$

This means:

$$\begin{aligned}
\|h_{i,k}(\boldsymbol{x})\| &= \big\|g_i\big(\boldsymbol{x}, \boldsymbol{y}_{g_i}^{i,k}(\boldsymbol{x})\big) - (g_i + \tilde{g}_i)\big(\boldsymbol{x}, \boldsymbol{y}_{g_i+\tilde{g}_i}^{i,k}(\boldsymbol{x})\big)\big\| \\
&\leq \big\|g_i\big(\boldsymbol{x}, \boldsymbol{y}_{g_i}^{i,k}(\boldsymbol{x})\big) - g_i\big(\boldsymbol{x}, \boldsymbol{y}_{g_i+\tilde{g}_i}^{i,k}(\boldsymbol{x})\big)\big\| + \big\|\tilde{g}_i\big(\boldsymbol{x}, \boldsymbol{y}_{g_i+\tilde{g}_i}^{i,k}(\boldsymbol{x})\big)\big\|
\end{aligned}$$

by differentiability there is a K_2 such that:

$$\begin{aligned}
&\leq K_2\big\|\boldsymbol{y}_{g_i}^{i,k}(\boldsymbol{x}) - \boldsymbol{y}_{g_i+\tilde{g}_i}^{i,k}(\boldsymbol{x})\big\| + \operatorname{norm}_{\mathsf{SIP}}\left(\tilde{f}, \tilde{\boldsymbol{g}}, B_{\varepsilon_r^*}(\boldsymbol{x}^*)\right) \\
&\leq (K_2 \cdot K_1 + 1) \operatorname{norm}_{\mathsf{SIP}}\left(\tilde{f}, \tilde{\boldsymbol{g}}, B_{\varepsilon_r^*}(\boldsymbol{x}^*)\right)
\end{aligned}$$

with $L_1 = K_2 \cdot K_1 + 1$:

$$= L_1 \operatorname{norm}_{\mathsf{SIP}}\left(\tilde{f}, \tilde{\boldsymbol{g}}, B_{\varepsilon_r^*}(\boldsymbol{x}^*)\right) \ . \tag{2.23}$$

Completely analogous there is also a L_2 such that, for every $\boldsymbol{x} \in B_{\varepsilon_r^*}(\boldsymbol{x}^*)$, $i \in I$ and $1 \leq k \leq q_i$:

$$\begin{aligned}
\|Dh_{i,k}(\boldsymbol{x})\| &= \big\|D_1 g_i\big(\boldsymbol{x}, \boldsymbol{y}_{g_i}^{i,k}(\boldsymbol{x})\big) - D_1(g_i - \tilde{g}_i)\big(\boldsymbol{x}, \boldsymbol{y}_{g_i+\tilde{g}_i}^{i,k}(\boldsymbol{x})\big)\big\| \\
&\leq L_2 \operatorname{norm}_{\mathsf{SIP}}\left(\tilde{f}, \tilde{g}, B_{\varepsilon_r^*}(\boldsymbol{x}^*)\right) \ .
\end{aligned} \tag{2.24}$$

We have calculated the derivatives of $\boldsymbol{y}^{i,k}$ in Remark 2.14. By this description there is for every $\gamma_1 > 0$ a $\delta > 0$ such that, for every $(\tilde{f}, \tilde{\boldsymbol{g}}) \in \mathcal{F}_\delta(B_{\varepsilon_r^*}(\boldsymbol{x}^*))$, $\boldsymbol{x} \in B_{\varepsilon_r^*}(\boldsymbol{x}^*)$, $i \in I$ and $1 \leq k \leq q_i$ the following holds:

$$\big\|D\boldsymbol{y}_{g_i}^{i,k}(\boldsymbol{x}) - D\boldsymbol{y}_{g_i+\tilde{g}_i}^{i,k}(\boldsymbol{x})\big\| \leq \gamma_1 \ . \tag{2.25}$$

We moreover have

$$
\begin{aligned}
& D^2 h_{i,k}(\boldsymbol{x}) \\
= \quad & D_1^2 g_i\big(\boldsymbol{x}, \boldsymbol{y}_{g_i}^{i,k}(\boldsymbol{x})\big) - D_1^2(g_i+\tilde{g}_i)\big(\boldsymbol{x}, \boldsymbol{y}_{g_i+\tilde{g}_i}^{i,k}(\boldsymbol{x})\big) \\
& + D_2 D_1 g_i\big(\boldsymbol{x}, \boldsymbol{y}_{g_i}^{i,k}(\boldsymbol{x})\big) \cdot D\boldsymbol{y}_{g_i}^{i,k}(\boldsymbol{x}) \\
& - D_2 D_1 (g_i+\tilde{g}_i)\big(\boldsymbol{x}, \boldsymbol{y}_{g_i+\tilde{g}_i}^{i,k}(\boldsymbol{x})\big) \cdot D\boldsymbol{y}_{g_i+\tilde{g}_i}^{i,k}(\boldsymbol{x}) \\
= \quad & D_1^2 g_i\big(\boldsymbol{x}, \boldsymbol{y}_{g_i}^{i,k}(\boldsymbol{x})\big) - D_1^2 g_i\big(\boldsymbol{x}, \boldsymbol{y}_{g_i+\tilde{g}_i}^{i,k}(\boldsymbol{x})\big) && (2.26) \\
& - D_1^2 \tilde{g}_i\big(\boldsymbol{x}, \boldsymbol{y}_{g_i+\tilde{g}_i}^{i,k}(\boldsymbol{x})\big) && (2.27) \\
& + D_2 D_1 g_i\big(\boldsymbol{x}, \boldsymbol{y}_{g_i}^{i,k}(\boldsymbol{x})\big) \cdot \big(D\boldsymbol{y}_{g_i}^{i,k}(\boldsymbol{x}) - D\boldsymbol{y}_{g_i+\tilde{g}_i}^{i,k}(\boldsymbol{x})\big) && (2.28) \\
& - \Big(D_2 D_1 g_i\big(\boldsymbol{x}, \boldsymbol{y}_{g_i+\tilde{g}_i}^{i,k}(\boldsymbol{x})\big) - D_2 D_1 g_i\big(\boldsymbol{x}, \boldsymbol{y}_{g_i}^{i,k}(\boldsymbol{x})\big)\Big) \cdot D\boldsymbol{y}_{g_i+\tilde{g}_i}^{i,k}(\boldsymbol{x}) && (2.29) \\
& - D_2 D_1 \tilde{g}_i\big(\boldsymbol{x}, \boldsymbol{y}_{g_i+\tilde{g}_i}^{i,k}(\boldsymbol{x})\big) \cdot D\boldsymbol{y}_{g_i+\tilde{g}_i}^{i,k}(\boldsymbol{x}) \ . && (2.30)
\end{aligned}
$$

As $D\boldsymbol{y}_{g_i}^{i,k}$ is continuous on $\overline{B}_{\varepsilon_r^*}(\boldsymbol{x}^*)$ and by (2.25) the derivative $D\boldsymbol{y}_{g_i+\tilde{g}_i}^{i,k}(\boldsymbol{x})$ can be bounded. The terms (2.26) and (2.29) can than be bounded by continuity of the second derivative. The terms (2.27) and (2.30) can be directly bounded by δ. Finally the term in (2.28) can be bounded by the continuity of the second derivative and by the inequality given in (2.25). All together this means that we can choose, for every $\gamma > 0$, a $\delta > 0$ such that, for every $(\tilde{f}, \tilde{\boldsymbol{g}}) \in \mathcal{F}_\delta(B_{\varepsilon_r^*}(\boldsymbol{x}^*)), i \in I, 1 \le k \le q_i$, the following holds:

$$
\|D^2 h_{i,k}(\boldsymbol{x})\| < \gamma \ . \tag{2.31}
$$

First inequality: The first inequality now follows easily, for every $\boldsymbol{x} \in B_{\varepsilon_r^*}(\boldsymbol{x}^*)$ and $\boldsymbol{\lambda} \in U_2$

$$
\begin{aligned}
& \big\|\Psi(\boldsymbol{x}, \boldsymbol{\lambda}) - \Psi_{\tilde{f}, \tilde{\boldsymbol{g}}}(\boldsymbol{x}, \boldsymbol{\lambda})\big\| \\
\le \ & \|D\tilde{f}(\boldsymbol{x})\| + \Big\| \sum_{i \in I} \sum_{k=1}^{q_i} (\lambda^{i,k})^+ \cdot Dh_{i,k}(\boldsymbol{x}) \Big\| + \sum_{i \in I} \sum_{k=1}^{q_i} \|h_{i,k}(\boldsymbol{x})\| \\
\le \ & \left(1 + (L_2 + \lambda_{max} L_1) \sum_{i \in I} q_i\right) \cdot \mathrm{norm}_{\mathsf{SIP}}\left(\tilde{f}, \tilde{\boldsymbol{g}}, B_{\varepsilon_r^*}(\boldsymbol{x}^*)\right) \ ,
\end{aligned}
$$

where $\lambda_{max} = \max_{i \in I} \max_{1 \le k \le q_i} |\lambda^{i,k}|$ which is bounded by assumption.

Second inequality: For the second inequality it only remains to show, that we can bound the Lipschitz constant, as the first inequality already bounds the first element

of the norm. Therefore we choose a $\delta > 0$ such that, for every $(\tilde{f}, \tilde{\boldsymbol{g}}) \in \mathcal{F}_\delta(B_\varepsilon(\boldsymbol{x}^*))$, $\boldsymbol{x}^1, \boldsymbol{x}^2 \in B_{\varepsilon_r^*}(\boldsymbol{x}^*)$, $i \in I$ and $1 \leq k \leq q_i$, the following holds:

$$\|D\tilde{f}(\boldsymbol{x}^1) - D\tilde{f}(\boldsymbol{x}^2)\| < \frac{\gamma}{4}\|\boldsymbol{x}^1 - \boldsymbol{x}^2\| \ , \tag{2.32}$$

$$\|Dh_{i,k}(\boldsymbol{x}^1)\| \leq \frac{\gamma}{4 \cdot \sum_{i \in I} q_i} \ , \tag{2.33}$$

$$\|h_{i,k}(\boldsymbol{x}^1) - h_{i,k}(\boldsymbol{x}^2)\| \leq \frac{\gamma}{4 \cdot \sum_{i \in I} q_i}\|\boldsymbol{x}^1 - \boldsymbol{x}^2\| \ , \tag{2.34}$$

$$\|Dh_{i,k}(\boldsymbol{x}^1) - Dh_{i,k}(\boldsymbol{x}^2)\| \leq \frac{\gamma}{4 \cdot \sum_{i \in I} q_i \cdot \lambda_{max}}\|\boldsymbol{x}^1 - \boldsymbol{x}^2\| \ . \tag{2.35}$$

The choice of inequality (2.32) is possible by the definition of $\mathcal{F}_\delta(B_{\varepsilon_r^*}(\boldsymbol{x}^*))$, as it bounds the second derivative. The inequalities (2.33) and (2.34) follow from the inequality in Equation (2.24). Finally the choice for the last inequality is possible due to Equation (2.31). The following is then true:

$$\begin{aligned}
&\left\| \left(\Psi(\boldsymbol{x}^1, \boldsymbol{\lambda}_1) - \Psi_{\tilde{f},\tilde{\boldsymbol{g}}}(\boldsymbol{x}^1, \boldsymbol{\lambda}_1)\right) - \left(\Psi(\boldsymbol{x}^2, \boldsymbol{\lambda}_2) - \Psi_{\tilde{f},\tilde{\boldsymbol{g}}}(\boldsymbol{x}^2, \boldsymbol{\lambda}_2)\right) \right\| \\
\leq\ & \|D\tilde{f}(\boldsymbol{x}^1) - D\tilde{f}(\boldsymbol{x}^2)\| + \sum_{i \in I}\sum_{k=1}^{q_i} |\lambda_1^{i,k} - \lambda_2^{i,k}| \cdot \|Dh_{i,k}(\boldsymbol{x}^1)\| \\
&+ \sum_{i \in I}\sum_{k=1}^{q_i} |\lambda_2^{i,k}| \cdot \|Dh_{i,k}(\boldsymbol{x}^1) - Dh_{i,k}(\boldsymbol{x}^2)\| + \sum_{i \in I} \|h_{i,k}(\boldsymbol{x}^1) - h_{i,k}(\boldsymbol{x}^2)\| \\
\leq\ & \gamma \cdot \left\| \begin{pmatrix} \boldsymbol{x}^1 \\ \boldsymbol{\lambda}^1 \end{pmatrix} - \begin{pmatrix} \boldsymbol{x}^2 \\ \boldsymbol{\lambda}^2 \end{pmatrix} \right\| \ ,
\end{aligned}$$

which bounds the Lipschitz constant by γ □

Again we can give a statement about a Lipschitz constant:

Theorem 2.27. *(Lipschitz constant for strong stability) Let $\boldsymbol{x}^*$ be a strongly stable stationary point of* SIP *with Lagrange multipliers $\boldsymbol{\lambda}$. Assume* ELICQ *is satisfied at $\boldsymbol{x}^*$ and the Reduction Ansatz holds. Then there are $\delta > 0, \varepsilon > 0$ and an $L > 0$ such that, for every $(\tilde{f}, \tilde{\boldsymbol{g}}) \in \mathcal{F}_\delta(B_\varepsilon(\boldsymbol{x}))$, the perturbed problem, $\mathsf{SIP}(\tilde{f}, \tilde{\boldsymbol{g}})$, has a stationary point, $\boldsymbol{x}(\tilde{f}, \tilde{g})$, with*

$$\|\boldsymbol{x}^* - \boldsymbol{x}(\tilde{f}, \tilde{\boldsymbol{g}})\| \leq L \operatorname{norm}_{\mathsf{SIP}}\left(\tilde{f}, \tilde{\boldsymbol{g}}, B_\varepsilon(\boldsymbol{x}^*)\right) \ ,$$

which is unique within $B_\varepsilon(\boldsymbol{x}^*)$.

Proof. Let $\varepsilon = \min\{\varepsilon^*, \varepsilon_r^*\}$, where ε^* is chosen according to Definition 2.22 and ε_r^* according to the Reduction Ansatz.

Choosing a δ to guarantee existence and uniqueness is already possible by the definition of strong stability.

The equivalent characterization of strong stability in Theorem 2.23 together with the bounds introduced in Lemma 2.26 make sure that we can apply the implicit function theorem (Theorem 2.11). □

3 Convergence speed for adaptive discretization by Blankenship and Falk

In this chapter we investigate the adaptive discretization method introduced by Blankenship and Falk ([BF76]). We first motivate and introduce the algorithm. Our main interest is to quantify the convergence rate. As a first step we give in Section 3.1 a bound that connects the distance of a current iterate to a limit in terms of the maximal violation of the current iterate. In Section 3.2 we then further bound the maximal violation and obtain a quadratic rate of convergence for limit points which are a minimum of order $\rho = 1$. We end this chapter with an easy example that shows that for a minimum of higher order no quadratic rate can be expected.

The main idea of the algorithm is to reduce problem SIP to a sequence of finite nonlinear optimization problems. Therefore we consider a finite subset of the index set $\dot{Y} \subseteq Y$. Problem

$$\begin{aligned} \mathsf{SIP}(\dot{Y}): \quad & \min_{\boldsymbol{x} \in \mathbb{R}^n} f(\boldsymbol{x}) \\ s.t. \quad & g_i(\boldsymbol{x}, \boldsymbol{y}) \leq 0 \text{ for all } i \in I, \boldsymbol{y} \in \dot{Y} \end{aligned}$$

is called the *discretized problem*. We denote its feasible set by

$$M(\dot{Y}) := \left\{ \boldsymbol{x} \in \mathbb{R}^n \mid g_i(\boldsymbol{x}, \boldsymbol{y}) \leq 0 \text{ for all } i \in I, \boldsymbol{y} \in \dot{Y} \right\} .$$

We will assume that we can solve this problem by algorithms developed for finite nonlinear optimization, such as SQP-methods ([NW06]) or interior-point methods ([WB06]). The question that needs to be answered is how the discretized set $\dot{Y}$ is chosen, as the quality of the found solution will depend on this choice.

A classical way is to choose a fine grid. The benefit is that the violation of the solution and a distance of a local solution of the discretized solution can be bounded in terms of the grid size (see [Sti01]). However, a major disadvantages is that for higher dimensions the number of discretization points needed to reach a specified

grid size grows exponentially. This is cumbersome as many of the points in a discretization defined by a grid will not be active at a solution.

This motivates to not work with a fixed discretization but to update the discretization based on the solution $\boldsymbol{x}$ found so far. More precisely, for every $i \in I$, the solutions of the lower-level problems

$$\boldsymbol{y}^i \in \arg\max\{g_i(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{y} \in Y\}$$

are added to the discretization. This way the discretization can be kept small and only the points actually needed are added to the discretization. These considerations lead to the following algorithm first introduced by Blankenship and Falk in [BF76]:

Algorithm 1 Adaptive discretization by Blankenship and Falk [BF76]

1: Input: initial point $\boldsymbol{x}^1 \in \mathbb{R}^n$, initial discretization $Y^1 \subseteq Y$, $k = 1$.
2: **while** termination criterion is not met **do**
3: **for** $i \in I$ **do**
4: calculate a global maximum $\boldsymbol{y}^{i,k}$ of problem $Q_i(\boldsymbol{x}^k)$.
5: **end for**
6: $Y^{k+1} := Y^k \cup \bigcup_{i \in I}\{\boldsymbol{y}^{i,k}\}$.
7: Calculate a solution $\boldsymbol{x}^{k+1}$ of the discretized problem $\mathsf{SIP}(Y^{k+1})$.
8: $k = k + 1$.
9: **end while**

While we require to calculate a global solution in Step 4 we do not specify the type of solution found in Step 7, but every solution should at least fulfill $\boldsymbol{x}^k \in M(Y^k)$. The type of solution calculated in Step 7 will of course influence the properties of a limit point. But by the feasibility of the iterates the following is already true (see for example [BF76]):

Lemma 3.1. *(Feasibility of accumulation points) Let $\{\boldsymbol{x}^k\}_{k\in\mathbb{N}}$ be constructed according to Algorithm 1. Then, for every convergent subsequence $\{\boldsymbol{x}^{k_l}\}_{l\in\mathbb{N}}$, the accumulation point*

$$\boldsymbol{x}^* = \lim_{l\to\infty} \boldsymbol{x}^{k_l}$$

is feasible for SIP.

As a consequence if $M(Y^0)$ is compact, then for every $\alpha > 0$ there exists a finite $\tilde{k}$ such that, for every $k \geq \tilde{k}$:

$$\max_{i\in I} \max_{\boldsymbol{y}\in Y} g_i(\boldsymbol{x}^k, \boldsymbol{y}) \leq \alpha \ .$$

This is a global convergence property towards the feasible set. The next theorem shows that the properties of the limit point depend on what we can guarantee in Step 7 of Algorithm 1 (see [BF76] and [Ree94]).

Theorem 3.2. *(Basic convergence properties) Let* $\{\boldsymbol{x}^k\}_{k\in\mathbb{N}}$ *be constructed according to Algorithm* 1*. Assume there is an* $\boldsymbol{x}^*$ *such that:*

$$\lim_{k\to\infty} \boldsymbol{x}^k = \boldsymbol{x}^* \ .$$

- *Assume that, for every* $k \in \mathbb{N}$*, the iterate* $\boldsymbol{x}^k$ *is a global solution of* $\mathsf{SIP}(Y^k)$*, then* $\boldsymbol{x}^*$ *is a global solution of* SIP*.*

- *Assume that, for every* $k \in \mathbb{N}$*, the iterate* $\boldsymbol{x}^k$ *is a local solution of* $\mathsf{SIP}(Y^k)$ *with radius* r_k *and* $\inf_{k\in\mathbb{N}} r_k > 0$*, then* $\boldsymbol{x}^*$ *is a local solution of* SIP*.*

We only assume here for simplicity of notation that the sequence $\{\boldsymbol{x}^k\}_{k\in\mathbb{N}}$ is convergent. Alternatively one can assume that $M(Y^0)$ is compact and then investigate every convergent subsequence.

A question that is not addressed in the literature for adaptive discretization algorithms is, whether $\boldsymbol{x}^*$ is again a stationary point, if, for every $k \in \mathbb{N}$, the solution $\boldsymbol{x}^k$ is a stationary point of $\mathsf{SIP}(Y^k)$. In Chapter 4 we investigate this question for a new adaptive discretization method. We show that the limit point of stationary points is again a stationary point. We point out that the same proof can also be used to show this property for the original algorithm by Blankenship and Falk.

In the next section we investigate the speed in which the algorithm converges. Therefore, we first bound the distance between a current iterate $\boldsymbol{x}^k$ and the limit point $\boldsymbol{x}^*$.

3.1 Bounds based on the maximal violation

We have already seen above, that every tolerance on the semi-infinite constraints can be reached after finitely many steps. The questions addressed now is how the distance to a solution of the original problem SIP can be bounded in terms of the reached tolerance.

Still investigated in [Sti01] exactly this question in the case of a fine discretization with a maximal gird size. The main assumption made in [Sti01] can be transferred to the situation of Algorithm 1 as follows:

Assumption 3.3. *Let $\{\boldsymbol{x}^k\}_{k\in\mathbb{N}}$ be a sequence of local solutions with radius $r_k > 0$ constructed according to Algorithm 1 with:*

$$\lim_{k\to\infty} \boldsymbol{x}^k = \boldsymbol{x}^* \ .$$

Assume $\boldsymbol{x}^*$ is a local solution of order ρ and the regularity condition EMFCQ *is satisfied at $\boldsymbol{x}^*$ for problem* SIP*.*

We denote the maximal violation of the feasibility by:

$$\alpha_k := \max_{i\in I} \max_{\boldsymbol{y}\in Y} \max\left\{0, g_i(\boldsymbol{x}^k, \boldsymbol{y})\right\} = \max_{i\in I} \max\left\{0, \boldsymbol{g}(\boldsymbol{x}^k, \boldsymbol{y}^{i,k})\right\} \ .$$

Under this assumptions the following bounds are stated in [Sti01]:

Theorem 3.4. *Let Assumption 3.3 hold and the objective function f of* SIP *be Lipschitz-continuous near $\boldsymbol{x}^*$.*

(i) There is a constant $L_1 > 0$ and a $k' \in \mathbb{N}$ such that

$$0 \le f(\boldsymbol{x}^*) - f(\boldsymbol{x}^k) \le L_1\alpha_k \text{ for all } k \ge k' \ . \tag{3.1}$$

(ii) There is a constant $L_2 > 0$ and a $k'' \in \mathbb{N}$ such that

$$\|\boldsymbol{x}^* - \boldsymbol{x}^k\| \le L_2{\alpha_k}^{\frac{1}{\rho}} \text{ for all } k \ge k'' \ . \tag{3.2}$$

Unfortunately, the assumptions made in [Sti01] are not strong enough and Equation (3.1) as well as Equation (3.2) do not hold in general under Assumption 3.3. We prove this by providing a counterexample in the next section. As the bounds in [Sti01] are derived in the situation of a discretization by a fine grid, we show that our counterexample also applies to this situation. However, in Section 3.1.2 we strengthen Assumption 3.3 by requiring that the radii r_k do not vanish. We show that under this new assumption the bounds hold.

3.1.1 An example with arbitrarily slow convergence

We construct the counterexample within two steps. We first present a Lipschitz continuous function on $X = [-1, 1]^2$ with infinitely many local solutions.
Therefore let $\{\boldsymbol{c}^l\}_{l\in\mathbb{N}} \subseteq X$ with:

$$\boldsymbol{c}^l = \left(\frac{1}{3^l}, \frac{1}{l}\right)^\top \ .$$

Around every $\boldsymbol{c}^l$ we choose a circle U_l with radius $r_l := \frac{1}{3^l}$:

$$U_l := \{\boldsymbol{x} \in X \mid \|\boldsymbol{x} - \boldsymbol{c}^l\| < r_l\} \ .$$

Note that the circles U_l are disjoint. We introduce the objective function $f : X \to \mathbb{R}$ by:

$$f(\boldsymbol{x}) = \begin{cases} \|\boldsymbol{x}\| + 2\|\boldsymbol{c}^l - \boldsymbol{x}\| - 2r_l & \text{for } \boldsymbol{x} \in U_l, l \in \mathbb{N}, \\ \|\boldsymbol{x}\| & \text{otherwise .} \end{cases}$$

The function is shown in Figure 3.1. We summarize the properties of the objective

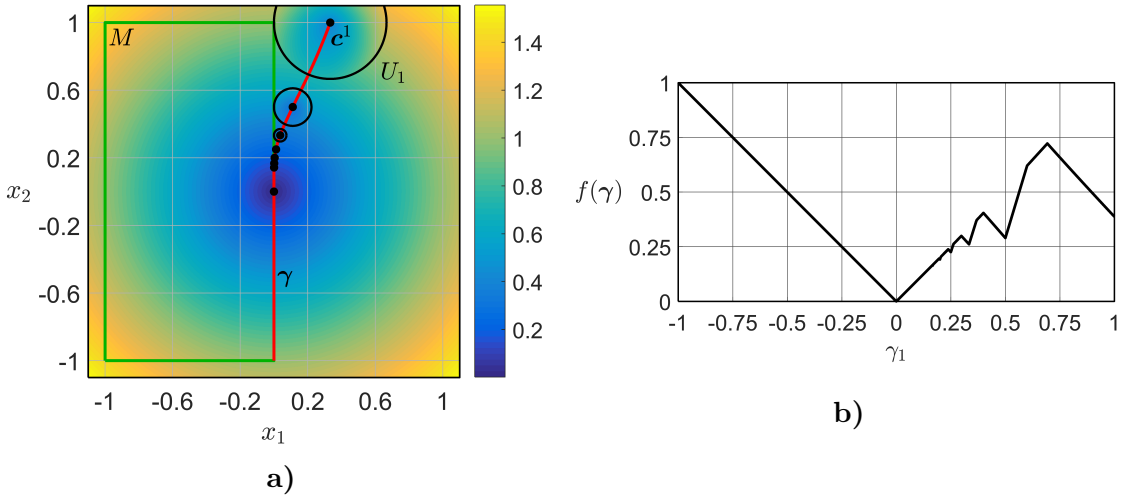


Figure 3.1: Lipschitz continuous function f with infinitely many local solutions:
a) function values over X [*colormap* - function values, *green* - border of feasible set M, *red line* - curve $\boldsymbol{\gamma}$, *black line* - border of circles U_l, *black points* - local minima]
b) function values along curve $\boldsymbol{\gamma}$

function in the following lemma:

Lemma 3.5. *For the above construction the following holds:*

i) Function f is Lipschitz continuous.

ii) For every $l \in \mathbb{N}$ the center point $\boldsymbol{c}^l$ is a strict local minimum of f with radius r_l.

Proof.

i) It is easy to see that we can write f, for every $\boldsymbol{x} \in X$, as:

$$f(\boldsymbol{x}) = \min\left\{\|\boldsymbol{x}\|, \min_{l\in\mathbb{N}}\left\{\|\boldsymbol{x}\| + 2\|\boldsymbol{c}^l - \boldsymbol{x}\| - 2r_l\right\}\right\} .$$

As every single function is Lipschitz continuous (with Lipschitz constant at most 3) the minimum is again Lipschitz continuous.

ii) Consider, for $l \in \mathbb{N}$, a point $\boldsymbol{x} \in U_l$ with $\boldsymbol{x} \neq \boldsymbol{c}^l$. The triangle inequality yields:

$$\begin{aligned} f(\boldsymbol{x}) &= \|\boldsymbol{x}\| + 2\|\boldsymbol{c}^l - \boldsymbol{x}\| - 2r_l \\ &\geq \|\boldsymbol{c}^l\| + \|\boldsymbol{c}^l - \boldsymbol{x}\| - 2r_l \\ &> \|\boldsymbol{c}^l\| - 2r_l \\ &= f(\boldsymbol{c}^l) . \end{aligned}$$

□

We consider the following semi-infinite problem:

$$\begin{aligned} \mathsf{SIP}_{\text{ex}} : \min \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & g(\boldsymbol{x}, y) := -(x_2 - y)^2 + \frac{1}{2}x_1 \leq 0 \text{ for all } y \in Y , \\ & \boldsymbol{x} \in X , \end{aligned}$$

where

$$Y := [-1, 1] .$$

The feasible set is shown in Figure 3.1. We investigate the structure of the problem in the following lemma.

Lemma 3.6. *Problem* SIP_{ex} *has the following properties:*

i) The feasible set is given by:

$$M = \{\boldsymbol{x} \in X \mid x_1 \leq 0\} .$$

ii) The origin, $\boldsymbol{x}^* = (0, 0)^\top$*, is a local minimum of order* $\rho = 1$*.*

iii) There exists a vector $\boldsymbol{\xi} \in \mathbb{R}^2$ *such that*

$$D_1 g(\boldsymbol{x}^*, y_0) \cdot \boldsymbol{\xi} \leq -1 ,$$

for every $y_0 \in Y$ *with* $g(\boldsymbol{x}^*, y_0) = 0$*. This means* EMFCQ *is satisfied at* $\boldsymbol{x}^*$ *for* SIP_{ex}*.*

Proof.

i) Let $\boldsymbol{x} \in X$. The solution of the lower-level problem is given by $y = x_2$. Thus:

$$\max_{y \in Y} g(\boldsymbol{x}, y) = \frac{1}{2} x_1 \ .$$

ii) On the feasible set M the objective function coincides with the norm, which clearly has a local minimum of order $\rho = 1$ in $\boldsymbol{x}^* = (0,0)^\top$.

iii) As by part i) the only solution of the lower-level problem for $\boldsymbol{x}^*$ is given by $y_0 = 0$, we have with $\boldsymbol{\xi} := (-2,0)^\top$:

$$D_1 g(\boldsymbol{x}^*, y_0) \cdot \boldsymbol{\xi} = \left(\frac{1}{2}, -2x_2\right) \cdot \begin{pmatrix} -2 \\ 0 \end{pmatrix} = \frac{1}{2} \cdot (-2) = -1 \ .$$

□

For an $l \in \mathbb{N}$, the violation of the strict local solution $\boldsymbol{c}^l$ is given by:

$$\max_{y \in Y} g(\boldsymbol{c}^l, y) = \frac{1}{2} \cdot c_1^l = \frac{1}{2} \cdot \frac{1}{3^l} \ .$$

However, we have for the function value as well as for the distance to the optimal solution:

$$\begin{aligned} \|\boldsymbol{c}^l - \boldsymbol{x}^*\| &\geq \frac{1}{l} \ , \\ f(\boldsymbol{c}^l) - f(\boldsymbol{x}^*) = f(\boldsymbol{c}^l) &\geq \frac{1}{l} - \frac{2}{3^l} \ . \end{aligned}$$

Thus, the local solutions are a counterexample for the bounds given in Equations (3.1) and (3.2) of Theorem 3.4. It only remains to show that $\boldsymbol{c}^l$ is actually a sequence that can be constructed by Algorithm 1.

If the current discretization consists of the points

$$Y^k = \left\{ \frac{1}{l} \mid 0 \leq l \leq k \right\} \ ,$$

the next local solution $\boldsymbol{c}^{k+1}$ is still feasible, as, for $1 \leq l \leq k$:

$$\begin{aligned} g\left(\boldsymbol{c}^{k+1}, \frac{1}{l}\right) &= -\left(\frac{1}{k+1} - \frac{1}{l}\right)^2 + \frac{1}{2} \cdot \frac{1}{3^{k+1}} \\ &\leq -\left(\frac{1}{k+1} - \frac{1}{k}\right)^2 + \frac{1}{2} \cdot \frac{1}{3^{k+1}} \\ &= -\left(\frac{1}{k \cdot (k+1)}\right)^2 + \frac{1}{2} \cdot \frac{1}{3^{k+1}} \\ &\leq 0 \ . \end{aligned}$$

The next discretization point is then $\frac{1}{k+1}$. Thus, if an empty initial discretization and $\boldsymbol{c}^1$ is chosen as first point we can generate the sequence $\{\boldsymbol{c}^l\}_{l \in \mathbb{N}}$ as iterates of Algorithm 1.

Remark 3.7. The first component of the local solutions $\frac{1}{l}$ can be replaced by an arbitrarily slow converging sequence (To keep the circles disjoint the radius can be reduced to $\frac{1}{3^{l+1}}$). This is why there can be an arbitrarily slow convergence.

As we have mentioned before, the bounds in Equations (3.1) and (3.2) are presented in [Sti01] for fine discretizations. We now show that the constructed example is also a counterexample in this case. Consider the following sequence of discretizations:

$$\begin{aligned} \dot{Y}^k = \quad & \left\{ \frac{1}{2k} + \frac{1+2l}{3^k} \,\middle|\, 0 \leq l \leq 3^k \right\} \cap Y \\ & \cup \left\{ \frac{1}{2k} - \frac{1+2l}{3^k} \,\middle|\, 0 \leq l \leq 3^k \right\} \cap Y \\ & \cup \{0, 1\} \ . \end{aligned}$$

The Hausdorff distance of the discretization to Y is given by:

$$\operatorname{dist}(\dot{Y}^k, Y) = \max_{y \in Y} \min_{\dot{y} \in \dot{Y}_k} |y - \dot{y}| = \frac{1}{3^k} \ .$$

But as, for every $\dot{y} \in \dot{Y}^k$, the following holds:

$$\begin{aligned} g\left(\boldsymbol{c}^{2k}, \dot{y}\right) &= -\left(\frac{1}{2k} - \dot{y}\right)^2 + \frac{1}{2} \cdot \frac{1}{3^{2k}} \\ &\leq -\left(\frac{1}{3^k}\right)^2 + \frac{1}{3^{2k}} \\ &= 0 \ , \end{aligned}$$

the local solution $\boldsymbol{c}^{2k}$ is feasible for the discretized problem. This means that there is no chance to achieve a bound based on the mesh size.

3.1.2 A statement using the order of a minimum

The idea of the counterexample in the previous section is to construct a sequence of local solutions, in such a way that their accumulation point is a local minimum. But as the radii r_l are vanishing, the property is not induced by the convergent sequence. Actually $\boldsymbol{x}^*$ is a global solution of f on X and not only on the feasible set M.

To exclude the counterexample we add an assumption, making sure that the property of $\boldsymbol{x}^*$ being a local minimum, is induced by the convergent sequence.

Assumption 3.8. *Let $\{\boldsymbol{x}^k\}_{k\in\mathbb{N}}$ be a sequence of local solutions with radius $r_k > 0$ generated by Algorithm 1 with:*

$$\lim_{k\to\infty} \boldsymbol{x}^k = \boldsymbol{x}^* \ .$$

Assume $\boldsymbol{x}^*$ is a local solution of order ρ and EMFCQ *holds at $\boldsymbol{x}^*$ for problem* SIP*. Moreover assume:*

$$\inf_{k\in\mathbb{N}} r_k > 0 \ . \tag{3.3}$$

The assumption coincides with Assumption 3.3 except for the inequality in Equation (3.3). This inequality is motivated by Theorem 3.2. There it is needed to prove that every accumulation point of local minima is again a local minimum. We can now again formulate the bounds in Equations (3.1) and (3.2) and give a proof with the strengthened assumption. Note that this proof follows the basic construction also introduced in [Sti01], but as we have shown with the counterexample in Section 3.1.1, the proof in [Sti01] is not fully correct. That is why, we give a corrected and complete proof here.

Theorem 3.9. *Let Assumption 3.8 hold and the objective function f of* SIP *be Lipschitz-continuous near $\boldsymbol{x}^*$.*

(i) There is a constant $L_1 > 0$ and a $k' \in \mathbb{N}$ such that

$$0 \leq f(\boldsymbol{x}^*) - f(\boldsymbol{x}^k) \leq L_1\alpha_k \text{ for all } k \geq k' \ .$$

(ii) There is a constant $L_2 > 0$ and a $k'' \in \mathbb{N}$ such that

$$\|\boldsymbol{x}^* - \boldsymbol{x}^k\| \leq L_2{\alpha_k}^{\frac{1}{\rho}} \text{ for all } k \geq k'' \ .$$

The situation and the idea of the proof are depicted in Figure 3.2. In a first step, the current infeasible iterate is shifted into the feasible set M. The size of this shift

depends on the current violation. In a second step, the function values of the three points are compared. Finally, the order of $\boldsymbol{x}^*$ is used to bound the difference of the shifted point $\bar{\boldsymbol{x}}^k$ and the optimal point $\boldsymbol{x}^*$.

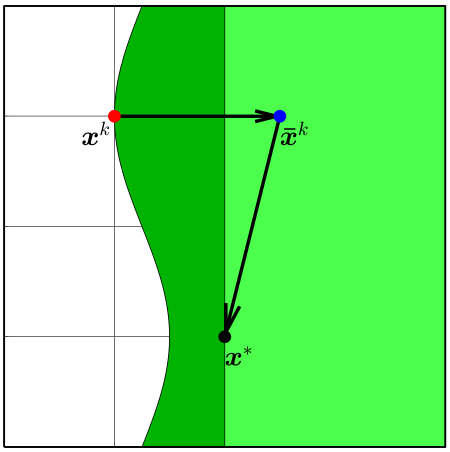


Figure 3.2: Construction for proof of Theorem 3.9 [*light green* - feasible set of SIP, *dark green* - feasible set of approximate problem SIP(Y^k), *red point* - current infeasible iterate, *blue point* - shifted feasible point, *black point* - limit point]

Proof.
Construction of a feasible point:
In a first step we construct a feasible point, $\bar{\boldsymbol{x}}^k$. According to EMFCQ there exists a $\boldsymbol{\xi}$ such that, for every $i \in I$ and for all $\boldsymbol{y} \in Y_0^i(\boldsymbol{x}^*)$:

$$D_1 g_i(\boldsymbol{x}^*, \boldsymbol{y}) \cdot \boldsymbol{\xi} \leq -1 \ .$$

For $k \in \mathbb{N}$ and $t \in [0,1]$, let:

$$\bar{\boldsymbol{x}}^k(t) := \boldsymbol{x}^k + t \cdot 2\alpha_k \boldsymbol{\xi} \ .$$

For every $i \in I$, the function $D_1 g_i$ is continuous and Y is compact. Thus, there is some $\varepsilon > 0$ such that, for every $i \in I$, $\boldsymbol{y} \in Y_0^{i,\varepsilon}(\boldsymbol{x}^*)$ and $\boldsymbol{x} \in B_\varepsilon(\boldsymbol{x}^*)$, the following holds:

$$D_1 g_i(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{\xi} \leq -\frac{1}{2} \ ,$$

where

$$Y_0^{i,\varepsilon}(\boldsymbol{x}^*) := \{\boldsymbol{y} \in Y \mid \exists\ \boldsymbol{y}^0 \in Y_0^i(\boldsymbol{x}^*) \text{ with } \|\boldsymbol{y} - \boldsymbol{y}^0\| < \varepsilon\} \ .$$

Choose a sufficiently large k_1, such that

$$\|\boldsymbol{x}^k - \boldsymbol{x}^*\| + 2\alpha_k \|\boldsymbol{\xi}\| < \varepsilon$$

for $k \geq k_1$. By the mean value theorem there is, for every $i \in I$, $\boldsymbol{y} \in Y_0^{i,\varepsilon}(\boldsymbol{x}^*)$ and $k \geq k_1$, an $s \in [0,1]$ such that

$$g_i\big(\bar{\boldsymbol{x}}^k(1), \boldsymbol{y}\big) = g_i(\boldsymbol{x}^k, \boldsymbol{y}) + D_1 g_i\big(\bar{\boldsymbol{x}}^k(s), \boldsymbol{y}\big) \cdot 2\alpha_k \boldsymbol{\xi} \leq \alpha_k - \frac{1}{2} \cdot 2\alpha_k = 0 \ .$$

As, for every $i \in I$, the set $Y \backslash Y_0^{i,\varepsilon}(\boldsymbol{x}^*)$ is compact and the functions g_i are continuous, the maximum

$$\max_{\boldsymbol{y} \in Y \backslash Y_0^{i,\varepsilon}(\boldsymbol{x}^*)} g_i(\boldsymbol{x}^*, \boldsymbol{y})$$

is attained and is strictly less than 0. By continuity there is a ε_2 such that, for $\boldsymbol{x} \in B_{\varepsilon_2}(\boldsymbol{x}^*)$ and $\boldsymbol{y} \in Y \setminus Y_0^{i,\varepsilon}(\boldsymbol{x}^*)$:

$$g_i(\boldsymbol{x}, \boldsymbol{y}) < 0 \ ,$$

Choose a sufficiently large k_2, such that

$$\|\boldsymbol{x}^k - \boldsymbol{x}^*\| + 2\alpha_k \|\boldsymbol{\xi}\| < \varepsilon_2$$

for $k \geq k_2$. Thus, for $k \geq \max(k_1, k_2)$, the point $\bar{\boldsymbol{x}}^k := \bar{\boldsymbol{x}}^k(1)$ is feasible.
Now the two claims follow easily:
Proof of i): By Assumption 3.8 the radii of the local solutions $\boldsymbol{x}^k$ do not converge to 0. This means that, for sufficiently large k, the distance of the local solution $\boldsymbol{x}^*$ to $\boldsymbol{x}^k$ is smaller than r_k and, as $M \subseteq M(Y^k)$, we have

$$f(\boldsymbol{x}^*) \geq f(\boldsymbol{x}^k) \ . \tag{3.4}$$

Again, for sufficiently large k, the feasible point $\bar{\boldsymbol{x}}^k$ is close enough to $\boldsymbol{x}^*$ such that

$$f(\bar{\boldsymbol{x}}^k) \geq f(\boldsymbol{x}^*) \ . \tag{3.5}$$

By the local Lipschitz-continuity of f, there is a constant L such that, for sufficiently large k:

$$\begin{aligned} 0 \leq f(\boldsymbol{x}^*) - f(\boldsymbol{x}^k) &\leq f(\bar{\boldsymbol{x}}^k) - f(\boldsymbol{x}^k) \\ &\leq L \cdot \|\bar{\boldsymbol{x}}^k - \boldsymbol{x}^k\| \\ &= L \cdot 2\alpha_k \|\boldsymbol{\xi}\| \ . \end{aligned} \tag{3.6}$$

Which shows the first assertion.

Proof of ii): By Assumption 3.8 $\boldsymbol{x}^*$ is a local solution of order ρ. This means that, for sufficiently large k, there is a $K > 0$ such that

$$\begin{aligned}
\|\boldsymbol{x}^* - \boldsymbol{x}^k\| &\leq \|\boldsymbol{x}^* - \bar{\boldsymbol{x}}^k\| + \|\bar{\boldsymbol{x}}^k - \boldsymbol{x}^k\| \\
&\leq \left(\frac{f(\bar{\boldsymbol{x}}^k) - f(\boldsymbol{x}^*)}{K}\right)^{\frac{1}{\rho}} + 2 \cdot \alpha_k \cdot \|\boldsymbol{\xi}\| \\
&\leq \left(\frac{f(\bar{\boldsymbol{x}}^k) - f(\boldsymbol{x}^k)}{K}\right)^{\frac{1}{\rho}} + 2 \cdot \alpha_k \cdot \|\boldsymbol{\xi}\| \\
&\leq \left(\frac{L \cdot 2\alpha_k \|\boldsymbol{\xi}\|}{K}\right)^{\frac{1}{\rho}} + 2 \cdot \alpha_k \cdot \|\boldsymbol{\xi}\| \\
&\leq \left(\left(\frac{L2\|\boldsymbol{\xi}\|}{K}\right)^{\frac{1}{\rho}} + 2\|\boldsymbol{\xi}\|\right) (\alpha_k)^{\frac{1}{\rho}} ,
\end{aligned}$$

where we've used the inequalities in Equations (3.4), (3.5) and (3.6) for the third and fourth inequality. □

The continuous differentiability of $g_i, i \in I$ and the regularity condition EMFCQ are only needed to construct a feasible point such that distance of this point to the original iterate can be bounded. Alternatively, one can assume this property in a more direct way. One could demand the existence of a neighborhood U of $\boldsymbol{x}^*$ and a constant $L \in \mathbb{R}$ such that, for every $\boldsymbol{x} \in U$, there exists a feasible $\bar{\boldsymbol{x}} \in M$ with

$$\|\boldsymbol{x} - \bar{\boldsymbol{x}}\| \leq L\alpha ,$$

where $\alpha = \max_{i \in I} \max_{\boldsymbol{y} \in Y} \max\{0, g_i(\boldsymbol{x}, \boldsymbol{y})\}$. As this assumption does not need any differentiability properties, Theorem 3.8 holds also for problems where the semi-infinite constraints are not differentiable.

The considerations show the great importance of the maximal violation α_k. The next question is of course: How does this maximal violation behave? In the next section we try to control the maximal violation of the current iterate and then derive a statement about the rate of convergence.

3.2 Quadratic rate of convergence for optima of order one

In this section we bound the maximal violation and investigate the question of a quadratic rate of convergence for the Blankenship and Falk algorithm. We put

special interest into limits $\boldsymbol{x}^*$ which are local minima of order $\rho = 1$, i.e. there is a radius $r > 0$ and an $L > 0$ such that, for every $\boldsymbol{x} \in M \cap B_r(\boldsymbol{x}^*)$:

$$L\|\boldsymbol{x}^* - \boldsymbol{x}\| \leq f(\boldsymbol{x}) - f(\boldsymbol{x}^*) \ .$$

We show in Theorem 3.11 that in this case the iterates converge with a quadratic rate. At the end of this section we present an example which shows that for higher orders no quadratic rate holds. We begin with an example which turns out to have a quadratic rate of convergence. It also shows, that quadratic convergence is the best one can show in the general case as it is not convergent in a higher order.

Example 3.10. *Consider the following semi-infinite problem:*

$$\begin{aligned} \mathsf{SIP}_{ex} : \min \quad & x \\ \text{s.t.} \quad & g(x,y) := -(x-y)^2 - x \leq 0 \text{ for all } y \in Y \ , \\ & x \in X := [-1,1] \ . \end{aligned}$$

where

$$Y = [-1,1] \ .$$

The solution of the lower-level problem, $\max_{y \in Y} g(x,y)$*, is given by* $y = x$*. This means that the feasible set is given by* $M = [0,1]$*. If all current discretization points are negative*

$$\dot{Y} \subseteq [-1,0] \ ,$$

then the point

$$x = \dot{y} - \frac{1}{2} + \sqrt{\frac{1}{4} - \dot{y}}$$

is a local solution of $M(\dot{Y})$*, where* $\dot{y} = \max \dot{Y}$*. As, for every* $x \in [-1,0)$*, the following holds:*

$$x < x - \frac{1}{2} + \sqrt{\frac{1}{4} - x} < 0 \ , \tag{3.7}$$

a possible sequence generated by Algorithm 1 is given by the recursive rule:

$$\begin{aligned} x_1 &= -1 \ , \\ x_{k+1} &= x_k - \frac{1}{2} + \sqrt{\frac{1}{4} - x_k} \ . \end{aligned}$$

By Equation (3.7) *this is a strictly increasing sequence in* $[-1,0)$ *which converges towards* 0*. By applying l'Hôpital's rule twice, one sees easily:*

$$-1 = \lim_{x \to 0} \frac{-\frac{1}{4}\left(\frac{1}{4} - x\right)^{-\frac{3}{2}}}{2} = \lim_{x \to 0} \frac{1 - \frac{1}{2}\left(\frac{1}{4} - x\right)^{-\frac{1}{2}}}{2x} = \lim_{x \to 0} \frac{x - \frac{1}{2} + \sqrt{\frac{1}{4} - x}}{x^2} \ ,$$

which shows quadratic convergence. Again, by applying l'Hôpital's rule twice, we also have, for $x < 0$:

$$\infty = \lim_{x\to 0} \frac{-\frac{1}{4}\left(\frac{1}{4}-x\right)^{-\frac{3}{2}}}{6x} = \lim_{x\to 0} \frac{1-\frac{1}{2}\left(\frac{1}{4}-x\right)^{-\frac{1}{2}}}{3x^2} = \lim_{x\to 0} \frac{x-\frac{1}{2}+\sqrt{\frac{1}{4}-x}}{x^3} \ ,$$

which shows that there is no $c \in \mathbb{R}$ such that for large k:

$$|x_{k+1}| \leq c|x_k|^3 \ .$$

After this motivating example, we now prove that under the assumption of a local solution of order $\rho = 1$ and some further assumptions concerning the lower-level problems, a quadratic rate of convergence holds.

Theorem 3.11. *Let Assumption 3.8 be satisfied. Assume that for every $i \in I$ there is exactly one active index:*

$$Y_0^i(\boldsymbol{x}^*) = \{\boldsymbol{y}^{i,*}\} \ . \tag{3.8}$$

Moreover, assume that the Reduction Ansatz holds in $\boldsymbol{x}^*$ and that $\boldsymbol{x}^*$ is a local minimum of order $\rho = 1$. Then there is a constant L such that, for sufficiently large k:

$$\|\boldsymbol{x}^{k+1} - \boldsymbol{x}^*\| \leq L\|\boldsymbol{x}^k - \boldsymbol{x}^*\|^2 \ .$$

Before proving Theorem 3.11 we need some preparation. As we assume, by the Reduction Ansatz, LICQ to be satisfied in every $\boldsymbol{y}^{i,*} \in Y$, there are, for every $i \in I$, unique Lagrange multipliers $\boldsymbol{\mu}^{i,*}$ such that:

$$\begin{aligned} D_2\mathcal{L}_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,*}, \boldsymbol{\mu}^{i,*}) &= 0 \ , \\ \mu_j^{i,*} &\geq 0 \text{ for all } j \in J \ , \\ \mu_j^{i,*} v_j(\boldsymbol{y}^{i,*}) &= 0 \text{ for all } j \in J \ . \end{aligned}$$

In the following Lemma we show that the solutions of the lower-level problem as well as the Lagrange multipliers converge. Moreover, active indices do not change for large k.

Lemma 3.12. *Assume that the same assumptions as for Theorem 3.11 hold.*

i) The solutions of the lower-level problems converge , i.e. for every $i \in I$:

$$\lim_{k\to\infty} \boldsymbol{y}^{i,k} = \boldsymbol{y}^{i,*} \ .$$

ii) As LICQ *is satisfied for every $i \in I$ and all iterates $\boldsymbol{y}^{i,k}$, there exist unique Lagrange multipliers $\boldsymbol{\mu}^{i,k}$ satisfying the KKT conditions. For those Lagrange multipliers the following is true:*

$$\lim_{k\to\infty} \boldsymbol{\mu}^{i,k} = \boldsymbol{\mu}^{i,*} \ .$$

iii) For sufficiently large k and every $i \in I$ the active indices in the lower-level do not change:

$$J_0(\boldsymbol{y}^{i,k}) = J_0(\boldsymbol{y}^{i,*}) \ ,$$

where, for $\boldsymbol{y} \in Y$:

$$J_0(\boldsymbol{y}) := \{j \in J \mid v_j(\boldsymbol{y}) = 0\}$$

denotes the set of active indices in the lower-level problem.

Proof.

i) By the Reduction Ansatz and the uniqueness in Equation (3.8), there is an $\varepsilon > 0$ and, for every $i \in I$, a continuously differentiable function

$$\boldsymbol{y}^i : B_\varepsilon(\boldsymbol{x}^*) \to Y$$

such that, for every $\boldsymbol{x} \in B_\varepsilon(\boldsymbol{x}^*)$, the point $\boldsymbol{y}^i(\boldsymbol{x})$ is the unique global solution of $Q_i(\boldsymbol{x})$. We therefore have, for sufficiently large k:

$$\boldsymbol{y}^{i,k} = \boldsymbol{y}^i(\boldsymbol{x}^k) \ .$$

The convergence follows directly from the continuity of this map.

ii) With the Reduction Ansatz we not only have continuously differentiable functions describing the global solution of the lower-level problems, but also for their unique Lagrange multipliers. The claim then follows by exactly the same arguments as in part i).

iii) For every $j \in J$ with $v_j(\boldsymbol{y}^{i,*}) > 0$, we have, for sufficiently large k, by part i) and continuity:

$$v_j(\boldsymbol{y}^{i,k}) > 0 \ ,$$

which shows, that for sufficiently large k:

$$J_0(\boldsymbol{y}^{i,k}) \subseteq J_0(\boldsymbol{y}^{i,*}) \ .$$

For an $j \in J_0(\boldsymbol{y}^{i,*})$ we know by strict complementary slackness (Reduction Ansatz) that $\mu_j^{i,*} > 0$. By part ii) this also holds for $\mu_j^{i,k}$, for sufficiently large k. This is again only possible if:

$$v_j(\boldsymbol{y}^{i,k}) = 0 \ .$$

Thus:

$$J_0(\boldsymbol{y}^{i,*}) \subseteq J_0(\boldsymbol{y}^{i,k}) \ .$$

□

We are now ready to prove the quadratic rate of convergence stated in Theorem 3.11:

Proof. (Theorem 3.11) We first develop two more inequalities. In combination with the bound derived in Theorem 3.9, they will provide the quadratic rate of convergence.

Bound on the solution of the lower-level problems: By the strong stability of $\boldsymbol{y}^{i,*}$, for every $i \in I$, there exists a $K_1 > 0$ such that, for sufficiently large k and every $i \in I$, the following holds:

$$\|\boldsymbol{y}^{i,k+1} - \boldsymbol{y}^{i,k}\| \leq K_1 \|\boldsymbol{x}^{k+1} - \boldsymbol{x}^k\| \ . \tag{3.9}$$

Bound on the next violation: By the complementarity condition of the KKT conditions, for the unique Lagrange multipliers $\boldsymbol{\mu}^{i,k}$ the following holds:

$$\mu_j^{i,k} v_j(\boldsymbol{y}^{i,k}) = 0 \ .$$

As by Lemma 3.12 the active indices do not change anymore, for sufficiently large k, we also have:

$$\mu_j^{i,k} v_j(\boldsymbol{y}^{i,k-1}) = 0 \ .$$

For every $i \in I$, we can conclude:

$$\begin{aligned}
& g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) \\
\leq\ & g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) - g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k-1}) \\
=\ & g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) - g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k-1}) - \sum_{j \in J} \mu_j^{i,k} \left(v_j(\boldsymbol{y}^{i,k}) - v_j(\boldsymbol{y}^{i,k-1}) \right)
\end{aligned}$$

by Taylor there exists a $\hat{\boldsymbol{y}} = t\boldsymbol{y}^{i,k} + (1-t)\boldsymbol{y}^{i,k-1}$ for a $t \in [0,1]$ such that

$$\begin{aligned}
=\ & \left(-D_2 g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) + \sum_{j \in J} \mu_j^{i,k} \cdot D v_j(\boldsymbol{y}^{i,k}) \right) \cdot (\boldsymbol{y}^{i,k-1} - \boldsymbol{y}^{i,k}) \\
& + \frac{1}{2} (\boldsymbol{y}^{i,k-1} - \boldsymbol{y}^{i,k})^T \cdot \left(-D_2^2 g(\boldsymbol{x}^k, \hat{\boldsymbol{y}}) + \sum_{j \in J} \mu_j^{i,k} \cdot D^2 v_j(\hat{\boldsymbol{y}}) \right) \cdot (\boldsymbol{y}^{i,k-1} - \boldsymbol{y}^{i,k})
\end{aligned}$$

by the choice of the Lagrange multipliers

$$\begin{aligned}
=\ & 0 \\
& + \frac{1}{2} (\boldsymbol{y}^{i,k-1} - \boldsymbol{y}^{i,k})^T \cdot \left(D_2^2 \mathcal{L}_i(\boldsymbol{x}^k, \hat{\boldsymbol{y}}, \boldsymbol{\mu}^{i,k}) \right) \cdot (\boldsymbol{y}^{i,k-1} - \boldsymbol{y}^{i,k}) \ .
\end{aligned}$$

As $\boldsymbol{\mu}^{i,k}, \boldsymbol{y}^{i,k}$ and $\boldsymbol{x}^k$ all converge, there is a constant K_2 such that, for sufficiently large k:

$$\alpha_k = \max_{i \in I} \max \left\{ 0, g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) \right\} \leq K_2 \|\boldsymbol{y}^{i,k} - \boldsymbol{y}^{i,k-1}\|^2 \ . \tag{3.10}$$

Combining the inequalities
By Theorem 3.9 and the assumption that $\boldsymbol{x}^*$ is a local minimum of order $\rho = 1$, there is a K_3 such that, for sufficiently large k, the following holds:

$$\|\boldsymbol{x}^k - \boldsymbol{x}^*\| \leq K_3 \alpha_k \ .$$

Combining this inequality with the inequalities given in Equations (3.9) and (3.10) we have with $L := K_1 \cdot K_2 \cdot K_3$:

$$\begin{aligned} \|\boldsymbol{x}^k - \boldsymbol{x}^*\| &\leq K_3 \alpha_k \\ &\leq K_3 K_2 \|\boldsymbol{y}^{i,k} - \boldsymbol{y}^{i,k-1}\|^2 \\ &\leq K_3 K_2 K_1 \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^2 \\ &= L \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^2 \ . \end{aligned}$$

This looks very similar to the desired quadratic rate of convergence. Indeed, as, for sufficiently large k, one has

$$\|\boldsymbol{x}^k - \boldsymbol{x}^*\| \geq 2L \left(\|\boldsymbol{x}^k - \boldsymbol{x}^*\|^2 + 2\|\boldsymbol{x}^k - \boldsymbol{x}^*\| \|\boldsymbol{x}^{k-1} - \boldsymbol{x}^*\| \right) \ ,$$

the following is true, for sufficiently large k:

$$\begin{aligned} \|\boldsymbol{x}^k - \boldsymbol{x}^*\| &\leq 2 \cdot \|\boldsymbol{x}^k - \boldsymbol{x}^*\| - 2L \left(\|\boldsymbol{x}^k - \boldsymbol{x}^*\|^2 + 2\|\boldsymbol{x}^k - \boldsymbol{x}^*\| \|\boldsymbol{x}^{k-1} - \boldsymbol{x}^*\| \right) \\ &\leq 2L \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^2 - 2L \left(\|\boldsymbol{x}^k - \boldsymbol{x}^*\|^2 + 2\|\boldsymbol{x}^k - \boldsymbol{x}^*\| \|\boldsymbol{x}^{k-1} - \boldsymbol{x}^*\| \right) \\ &\leq 2L \|\boldsymbol{x}^{k-1} - \boldsymbol{x}^*\|^2 \ . \end{aligned}$$

□

Except for the minimum being of order one, we have made a second restrictive assumption. We assumed that for every $i \in I$ there is exactly one active index. The assumption that there is at least one active index is only done to avoid case distinctions. If there is a semi-infinite constraint with no active index it can simply be dropped for large k. In the next remark we discuss the case of more then one active index.

Remark 3.13. i) By the Reduction Ansatz, there can only be finitely many active indices. We can therefore partition the sequence into subsequences such that the solutions of the lower-level problems $\boldsymbol{y}^{i,k}$ converge. For each of this subsequences we will have a quadratic rate of convergence.

ii) A possibility to still achieve a quadratic rate of convergence for the full sequence is to alter Step 4 in Algorithm 1. Instead of calculating, for every $i \in I$, only one global solution to the lower-level problem $Q_i(\boldsymbol{x}^k)$, one can calculate every local solution and add them to the discretization.

The question that arises is whether the quadratic rate of convergence also holds for optima of higher order (especially order $\rho = 2$). We show in the next example that this is not the case.

Example 3.14. *We consider the following semi-infinite optimization problem:*

$$\begin{aligned} \mathsf{SIP}_{ex}: \min_{\boldsymbol{x}\in\mathbb{R}^n} \quad & -x_1 + \frac{3}{2}x_2 \\ s.t. \quad & -y^2 + 2y \cdot x_1 - x_2 \leq 0 \quad \text{for all} \quad y \in [-1,1] \ , \\ & x_1, x_2 \in [-1,1] \ . \end{aligned}$$

One can easily see that the solution of the lower-level problem is given by $y = x_1$. This means that the feasible set is given by $M = \{\boldsymbol{x} \in [-1,1]^2 \mid x_2 \geq x_1^2\}$. Using the KKT conditions the global solution can be easily calculated and is given by:

$$\boldsymbol{x}^* = \left(\frac{1}{3}, \frac{1}{9}\right)^\top \ .$$

The corresponding solution of the lower-level problem is:

$$y^* = \frac{1}{3} \ .$$

To see that in this example, we do not have a quadratic rate of convergence, we have to investigate the sequence generated by Algorithm 1 a bit further. Starting with an empty discretization the first iterates can be calculated analytically and are given by:

$$\begin{aligned} \boldsymbol{x}^1 &= (1,-1)^\top \ , \\ y^1 &= 1 \ , \\ \boldsymbol{x}^2 &= (0,-1)^\top \ , \\ y^2 &= 0 \ . \end{aligned}$$

Thus, after the first two iterations we have two points in the discretization with:

$$y^2 \leq y^* \leq y^1 \ .$$

The following Lemma 3.15 will show that for the next iterations the following holds for suitable $k_1, k_2 < k+1$:

$$x_1^{k+1} = \frac{x_1^{k_1} + x_1^{k_2}}{2}$$

It is well known that a bisection has only a linear rate of convergence, if it does not terminate after finitely many steps. Because of the starting values the element of the bisection can, for every $k \geq 3$, be written as:

$$x_1^k = \frac{a}{2^k}$$

for a suitable natural number $a \in \mathbb{N}$. This expression can never reach $\frac{1}{3}$ by a divisibility argument.

Some iterations are shown in Figure 3.3. The iterative solutions are shown in blue and the limit in black. One can see that the iterates make some kind of a zig-zagging movement. This is due to the fact that when the approximate problem $\mathsf{SIP}(Y^k)$ is solved, we respect all constraints induced by the discretization, but we do not account for any point not in this discretization. In the worst case, the next iteration maximizes the violation between two discretization points.

The following lemma gives a rule to calculate the next iterate for the previous example.

Lemma 3.15. *Consider problem SIP_{ex} given in Example 3.14 and a finite discretization $\dot{Y} \subseteq [-1, 1]$. Assume there are $y_1, y_2 \in \dot{Y}$ with:*

$$y_1 = \max\{y \in \dot{Y} \mid y \leq y^*\} \leq y^* \leq \min\{y \in \dot{Y} \mid y \geq y^*\} = y_2 \ .$$

Then $\hat{\boldsymbol{x}}$ with

$$\begin{aligned} \hat{x}_1 &= \frac{y_1 + y_2}{2} \ , \\ \hat{x}_2 &= y_1 y_2 \end{aligned}$$

is a global solution of the discretized problem:

$$\begin{aligned} \mathsf{SIP}_{ex}(\dot{Y}) : \min_{\boldsymbol{x} \in \mathbb{R}^n} \quad & -x_1 + \frac{3}{2} x_2 \\ s.t. \quad & -y^2 + 2y \cdot x_1 - x_2 \leq 0 \text{ for all } y \in \dot{Y} \ , \\ & x_1, x_2 \in [-1, 1] \ . \end{aligned}$$

The solution is unique as long as $y^* \notin \dot{Y}$.

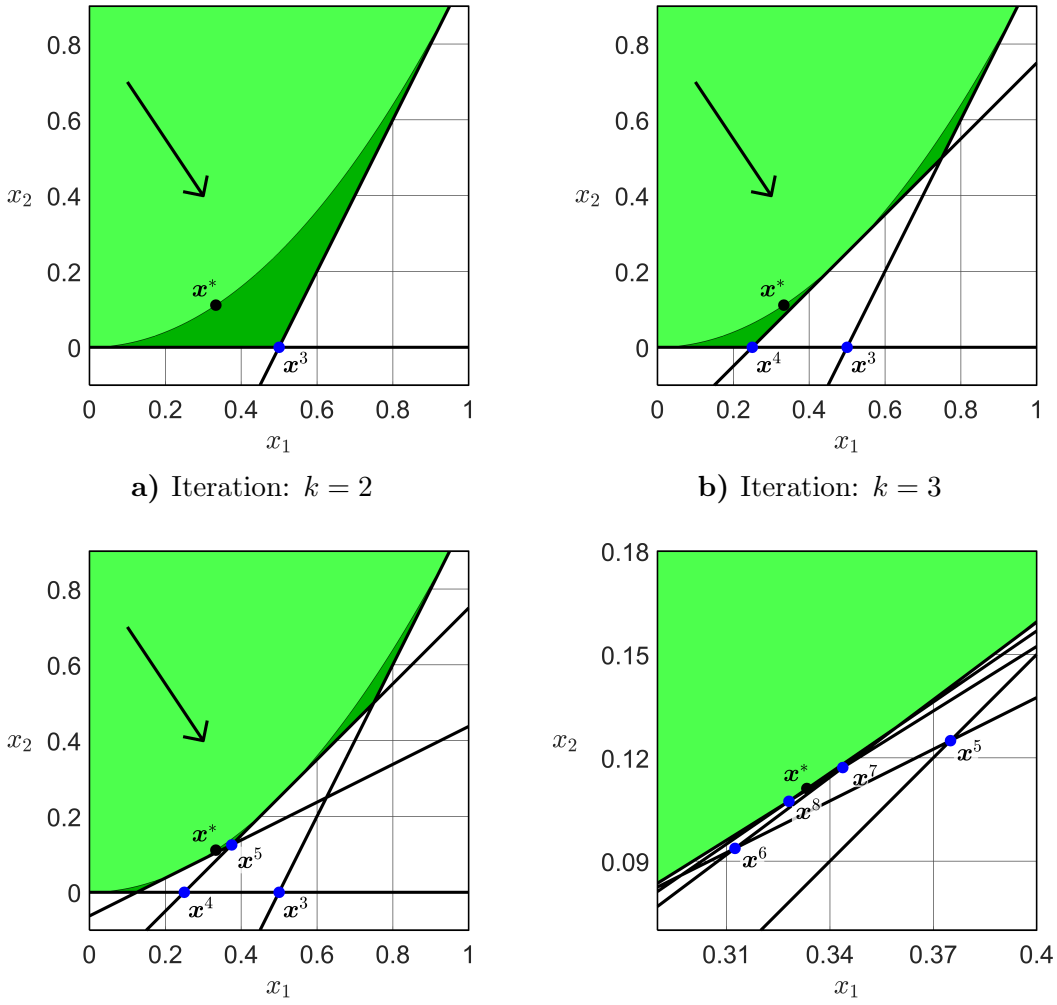
Proof. First we note that problem $\mathsf{SIP}_{\mathrm{ex}}(\dot{Y})$ is a convex problem. Thus, by Theorem 2.5 the global solutions are exactly the points satisfying the KKT conditions.

It is easy to see that for $\hat{\boldsymbol{x}}$ it holds:

$$-y_1^2 + 2 \cdot y_1 \hat{x}_1 - \hat{x}_2 = -y_2^2 + 2 \cdot y_2 \hat{x}_1 - \hat{x}_2 = 0 \ ,$$

which means that the constraints are active. By the strict concavity of the semi-infinite constraint in y, this also shows:

$$\begin{aligned} -y^2 + 2y\hat{x}_1 - \hat{x}_2 \leq & 0 \text{ for all } y \in \dot{Y} \text{ with } y \leq y_1 \ , \\ -y^2 + 2y\hat{x}_1 - \hat{x}_2 \leq & 0 \text{ for all } y \in \dot{Y} \text{ with } y \geq y_2 \ . \end{aligned}$$



a) Iteration: $k = 2$ **b)** Iteration: $k = 3$

c) Iteration: $k = 4$ **d)** Iteration: $k = 7$

Figure 3.3: Example solved by the Blankenship and Falk algorithm with a linear rate of convergence [*light green* - feasible set of SIP, *dark green* - feasible set of approximate problem $\mathsf{SIP}(Y^k)$, *black arrow* - descent direction of objective, *black lines* - constraints induced by discretization, *blue points* - iterates, *black point* - limit solution]

This means that $\hat{\boldsymbol{x}}$ is feasible for SIP_{ex}. Moreover, for $0 \leq \lambda_1 := \frac{1-3y_1}{2(y_2-y_1)}$ and $0 \leq \lambda_2 := \frac{1-3y_2}{2(y_1-y_2)}$, the following holds:

$$0 = \begin{pmatrix} -1 \\ \frac{3}{2} \end{pmatrix} + \lambda_1 \begin{pmatrix} 2y_1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2y_2 \\ -1 \end{pmatrix} .$$

This shows that $\hat{\boldsymbol{x}}$ satisfies the KKT conditions and is a global solution.

Now assume that the global solution is not unique. This means that there are two different global solutions. By convexity the complete line connecting those two solutions is feasible and by linearity of the objective also optimal. As there are infinitely many points on the line and only finitely many discretization points in $\dot{Y}$, there are two global solutions, say $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{x}} + s \cdot \boldsymbol{c}$ with $s \in \mathbb{R}$ and

$$\boldsymbol{c} = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} ,$$

with the same active index $\dot{y} \in \dot{Y}$. Thus, we have:

$$\begin{aligned} -\dot{y}^2 + 2\dot{y} \cdot \hat{x}_1 - \hat{x}_2 &= 0 \ , \\ -\dot{y}^2 + 2\dot{y} \cdot (\hat{x}_1 + s \cdot c_1) - \hat{x}_2 - s \cdot c_2 &= 0 \ . \end{aligned}$$

This is only possible if $\dot{y} = \frac{1}{3} = y^*$. □

In this chapter we have investigated the speed of convergence of the Blankenship and Falk algorithm. We showed how the distance of an iterate to an optimal solution can be bounded in terms of the maximal violation of the semi-infinite constraints. In the second part we have bounded the maximal violation. In the special case of a minimum of order $\rho = 1$ this led to a quadratic rate of convergence. With the last example we showed that for optima of higher order no quadratic rate can be expected. In the next chapter we use more information to obtain a better rate of convergence, namely the derivative of the solution of the lower-level problem.

4 An adaptive discretization method with quadratic rate of convergence

In the last example of the previous chapter we have seen that the adaptive discretization method by Blankenship and Falk (Algorithm 1) does not converge quadratically, if the order of a minimum is larger than one. In Example 3.14 we already discussed one of the reasons for this: the algorithm treats the lower-level problem and the discretized problems separately. In the k-th step for the discretized problem $\mathsf{SIP}(Y^k)$ only the discretization points Y^k are considered but no further structure of the lower-level problems and analogous in the solution of the lower-level problems, we only use the current iterate $\boldsymbol{x}^k$, but no information on how this point is affected by the lower-level. In this chapter we aim at breaking this strict separated scheme and add information of the lower-level problems to the discretized problems.

We begin by describing how information of the lower-level problem can be obtained and added by an additional constraint. We then introduce a new adaptive discretization algorithm that respects this additional constraint. The main topic of this chapter is the investigation of the new method. In Section 4.2 we prove the Quadratic Convergence Theorem, which states that the introduced algorithm has a quadratic rate of convergence. We start by considering the two main steps (Step 4 and 7) of Algorithm 1 separately and show how they are implemented in the new algorithm.

In the k-th iteration we solve, for every $i \in I$, the following lower-level problem:

$$\begin{aligned} Q_i(\boldsymbol{x}^k): \quad & \max_{\boldsymbol{y} \in \mathbb{R}^m} g_i(\boldsymbol{x}^k, \boldsymbol{y}) \\ \text{s.t.} \quad & v_j(\boldsymbol{y}) \leq 0 \text{ for all } j \in J \ . \end{aligned}$$

Assume now that the iterates converge towards a point $\boldsymbol{x}^*$. We further assume that at this point the Reduction Ansatz holds and there is, for every $i \in I$, exactly one

active index $\boldsymbol{y}^{i,*}$. We can use Theorem 2.13 to obtain an $\varepsilon > 0$ and, for every $i \in I$, differentiable functions:

$$\begin{aligned} \boldsymbol{y}^i &: B_\varepsilon(\boldsymbol{x}^*) \to Y \ , \\ \boldsymbol{\mu}^i &: B_\varepsilon(\boldsymbol{x}^*) \to \mathbb{R}^{|J|} \end{aligned}$$

such that, for every $\boldsymbol{x} \in B_\varepsilon(\boldsymbol{x}^*)$, the point $\boldsymbol{y}^i(\boldsymbol{x})$ is the unique global solution of $Q_i(\boldsymbol{x})$ and $\boldsymbol{\mu}^i(\boldsymbol{x})$ are the unique Lagrange multipliers satisfying the KKT conditions.

If the current iterate $\boldsymbol{x}^k$ is sufficiently close to $\boldsymbol{x}^*$, we know that the derivatives of the solutions of the lower-level problems exist. Around these points we can develop the functions describing the solutions of the lower-level problems linearly:

$$\begin{aligned} \overline{\boldsymbol{y}}^{i,k}(\boldsymbol{x}) &:= \boldsymbol{y}^{i,k} + D\boldsymbol{y}^i(\boldsymbol{x}^k) \cdot (\boldsymbol{x} - \boldsymbol{x}^k) \ , \\ \overline{\boldsymbol{\mu}}^{i,k}(\boldsymbol{x}) &:= \boldsymbol{\mu}^{i,k} + D\boldsymbol{\mu}^i(\boldsymbol{x}^k) \cdot (\boldsymbol{x} - \boldsymbol{x}^k) \ . \end{aligned}$$

However, for a current iterate $\boldsymbol{x}^k$ of the algorithm, we usually do not know how close we are to the limit point $\boldsymbol{x}^*$. Nevertheless, we can check for the current iterate $\boldsymbol{x}^k$ and every $i \in I$, whether the conditions for Theorem 2.13 are satisfied. (For example using Remark 2.14.) If this is the case, the solution can again be described locally and derivatives can be calculated. We collect for a given point $\boldsymbol{x} \in \mathbb{R}^n$ all indices for which the conditions of Theorem 2.13 are satisfied in

$$\bar{I}^k := \left\{ i \in I \mid \boldsymbol{y}^{i,k} \text{ satisfies all assumptions of Theorem 2.13 for } Q_i(\boldsymbol{x}^k) \right\} \ .$$

If $\lim_{k\to\infty} \boldsymbol{x}^k = \boldsymbol{x}^*$ and the Reduction Ansatz holds at $\boldsymbol{x}^*$, we have for sufficiently large k:

$$\bar{I}^k = I \ .$$

In contrast to the discretized problem introduced in Chapter 3, we now consider a problem where we add linear information about the lower-level problem. Therefore, let for $i \in \bar{I}^k$:

$$\overline{g}_i^k(\boldsymbol{x}) := \mathcal{L}_i\big(\boldsymbol{x}, \overline{\boldsymbol{y}}^{i,k}(\boldsymbol{x}), \overline{\boldsymbol{\mu}}^{i,k}(\boldsymbol{x})\big) \ ,$$

where again, for $i \in I$, the function

$$\mathcal{L}_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\mu}^i) = g_i(\boldsymbol{x}, \boldsymbol{y}) - \sum_{j \in J} \mu_j^i \cdot v_j(\boldsymbol{y})$$

is the lower-level Lagrange function. To receive the next iterate $\boldsymbol{x}^{k+1}$ we solve the following nonlinear optimization problem:

$$\begin{aligned} \overline{\mathsf{SIP}}^k(Y^{k+1}) : \quad & \min_{\boldsymbol{x} \in \mathbb{R}^n} f(\boldsymbol{x}) \\ & s.t. \quad g_i(\boldsymbol{x}, \boldsymbol{y}) \leq 0 \text{ for all } \boldsymbol{y} \in Y^{k+1}, i \in I \ , \\ & \qquad \overline{g}_i^k(\boldsymbol{x}) \leq 0 \text{ for all } i \in \bar{I}^k \ . \end{aligned}$$

One may be surprised that the Lagrange function is used here instead of the original semi-infinite constraint. The next lemma gives an answer to this question. We calculate the first and second derivatives.

Lemma 4.1. *Consider a fixed $\boldsymbol{x}^* \in \mathbb{R}^n$ and an $i \in I$. Let $\boldsymbol{y}^{i,*} \in Y$ be a global solution of the lower-level problem $Q_i(\boldsymbol{x}^*)$ satisfying all conditions for Theorem 2.13. Denote the unique Lagrange multipliers satisfying the KKT conditions by $\boldsymbol{\mu}^{i,*}$. Let*

$$\begin{aligned}\overline{\boldsymbol{y}}^i(\boldsymbol{x}) =& \boldsymbol{y}^{i,*} + D\boldsymbol{y}^i(\boldsymbol{x}^*) \cdot (\boldsymbol{x} - \boldsymbol{x}^*) \ , \\ \overline{\boldsymbol{\mu}}^i(\boldsymbol{x}) =& \boldsymbol{\mu}^{i,*} + D\boldsymbol{\mu}^i(\boldsymbol{x}^*) \cdot (\boldsymbol{x} - \boldsymbol{x}^*)\end{aligned}$$

and

$$\overline{g}_i(\boldsymbol{x}) := \mathcal{L}_i\left(\boldsymbol{x}, \overline{\boldsymbol{y}}^i(\boldsymbol{x}), \overline{\boldsymbol{\mu}}^i(\boldsymbol{x})\right) \ .$$

For the derivatives the following holds:

$$D\overline{g}_i(\boldsymbol{x}^*) = D_1 g_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,*})$$

and

$$D^2\overline{g}_i(\boldsymbol{x}^*) = D_1^2 g_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,*}) - \left(D\boldsymbol{y}^i(\boldsymbol{x}^*)\right)^\top \cdot D_2^2\mathcal{L}_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,*}, \boldsymbol{\mu}^{i,*}) \cdot D\boldsymbol{y}^i(\boldsymbol{x}^*) \ .$$

Proof. As all conditions for Theorem 2.13 are satisfied, there is an $\varepsilon > 0$ and differentiable functions $\boldsymbol{y}^i : B_\varepsilon(\boldsymbol{x}^*) \to Y$ and $\boldsymbol{\mu}^i : B_\varepsilon(\boldsymbol{x}^*) \to \mathbb{R}^{|J|}$, which locally describe the solution of the lower-level problem and the corresponding Lagrange multipliers. We show next the statements for the first and second derivatives.

First derivative: For $\boldsymbol{x} \in \mathbb{R}^n$, the following is true:

$$\begin{aligned}D\overline{g}_i(\boldsymbol{x}) =& D_{\tilde{\boldsymbol{x}}}\Big[\mathcal{L}_i\left(\tilde{\boldsymbol{x}}, \overline{\boldsymbol{y}}^i(\tilde{\boldsymbol{x}}), \overline{\boldsymbol{\mu}}^i(\tilde{\boldsymbol{x}})\right)\Big]_{\tilde{\boldsymbol{x}}=\boldsymbol{x}} \\ =& \quad D_1\mathcal{L}_i\left(\boldsymbol{x}, \overline{\boldsymbol{y}}^i(\boldsymbol{x}), \overline{\boldsymbol{\mu}}^i(\boldsymbol{x})\right) && (4.1) \\ &+ D_2\mathcal{L}_i\left(\boldsymbol{x}, \overline{\boldsymbol{y}}^i(\boldsymbol{x}), \overline{\boldsymbol{\mu}}^i(\boldsymbol{x})\right) \cdot D\boldsymbol{y}^i(\boldsymbol{x}^*) && (4.2) \\ &+ D_3\mathcal{L}_i\left(\boldsymbol{x}, \overline{\boldsymbol{y}}^i(\boldsymbol{x}), \overline{\boldsymbol{\mu}}^i(\boldsymbol{x})\right) \cdot D\boldsymbol{\mu}^i(\boldsymbol{x}^*) \ . && (4.3)\end{aligned}$$

Now for $\boldsymbol{x} = \boldsymbol{x}^*$, as the index set Y is a fixed set, the following holds:

$$D_1\mathcal{L}_i\left(\boldsymbol{x}^*, \overline{\boldsymbol{y}}^i(\boldsymbol{x}^*), \overline{\boldsymbol{\mu}}^i(\boldsymbol{x}^*)\right) = D_1\mathcal{L}_i\left(\boldsymbol{x}^*, \boldsymbol{y}^{i,*}, \boldsymbol{\mu}^{i,*}\right) = D_1 g_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,*}) \ .$$

Further, by the KKT conditions the following holds:

$$D_2\mathcal{L}_i\left(\boldsymbol{x}^*, \overline{\boldsymbol{y}}^i(\boldsymbol{x}^*), \overline{\boldsymbol{\mu}}^i(\boldsymbol{x}^*)\right) = D_2\mathcal{L}_i\left(\boldsymbol{x}^*, \boldsymbol{y}^{i,*}, \boldsymbol{\mu}^{i,*}\right) = 0$$

By Remark 2.14 we know that, for every $j \in J$, either $v_j(\boldsymbol{y}^{i,*}) = 0$ or $D\mu_j^i(\boldsymbol{x}^*) = 0$. This means:

$$D_3\mathcal{L}_i\left(\boldsymbol{x}^*, \overline{\boldsymbol{y}}^i(\boldsymbol{x}^*), \overline{\boldsymbol{\mu}}^i(\boldsymbol{x}^*)\right) \cdot D\boldsymbol{\mu}^i(\boldsymbol{x}^*) = -\sum_{j \in J} v_j(\boldsymbol{y}^{i,*}) \cdot D\mu_j^i(\boldsymbol{x}^*) = 0 \ .$$

Second derivative: We consider the three terms of the first derivative given in (4.1)-(4.3) separately. For the first term (4.1) we have, as we developed the lower-level solution and the corresponding Lagrange-multipliers linearly:

$$D_{\boldsymbol{x}}\Big[D_1\mathcal{L}_i\left(\boldsymbol{x},\overline{\boldsymbol{y}}^i(\boldsymbol{x}),\overline{\boldsymbol{\mu}}^i(\boldsymbol{x})\right)\Big]_{\boldsymbol{x}=\boldsymbol{x}^*} = D_{\boldsymbol{x}}\Big[D_1\mathcal{L}_i\left(\boldsymbol{x},\boldsymbol{y}^i(\boldsymbol{x}),\boldsymbol{\mu}^i(\boldsymbol{x})\right)\Big]_{\boldsymbol{x}=\boldsymbol{x}^*} .$$

The same calculations as for the Reduction Ansatz in Equation (2.18) show:

$$\begin{aligned} &D_{\boldsymbol{x}}\Big[D_1\mathcal{L}_i\left(\boldsymbol{x},\overline{\boldsymbol{y}}^i(\boldsymbol{x}),\overline{\boldsymbol{\mu}}^i(\boldsymbol{x})\right)\Big]_{\boldsymbol{x}=\boldsymbol{x}^*} \\ &= D_1^2 g_i(\boldsymbol{x}^*,\boldsymbol{y}^{i,*}) - \left(D\boldsymbol{y}^i(\boldsymbol{x}^*)\right)^\top \cdot D_2^2\mathcal{L}_i(\boldsymbol{x}^*,\boldsymbol{y}^{i,*},\boldsymbol{\mu}^{i,*}) \cdot D\boldsymbol{y}^i(\boldsymbol{x}^*) . \end{aligned}$$

By the KKT conditions we have, for every $\boldsymbol{x} \in B_\varepsilon(\boldsymbol{x}^*)$:

$$D_2\mathcal{L}\big(\boldsymbol{x},\boldsymbol{y}^i(\boldsymbol{x}),\boldsymbol{\mu}^i(\boldsymbol{x})\big) = 0 .$$

This means:

$$D_{\boldsymbol{x}}\Big[D_1\mathcal{L}_i\left(\boldsymbol{x},\overline{\boldsymbol{y}}^i(\boldsymbol{x}),\overline{\boldsymbol{\mu}}^i(\boldsymbol{x})\right)\Big]_{\boldsymbol{x}=\boldsymbol{x}^*} = D_{\boldsymbol{x}}\Big[D_1\mathcal{L}_i\left(\boldsymbol{x},\boldsymbol{y}^i(\boldsymbol{x}),\boldsymbol{\mu}^i(\boldsymbol{x})\right)\Big]_{\boldsymbol{x}=\boldsymbol{x}^*} = 0 ,$$

which shows that the derivative of the second term (4.2) vanishes.

For the third term (4.3) we again have that, for every $j \in J$, either $v_j(\boldsymbol{y}^i(\boldsymbol{x})) = 0$ for every $\boldsymbol{x} \in B_\varepsilon(\boldsymbol{x}^*)$ or $D\mu_j^i(\boldsymbol{x}^*) = 0$. This shows that also the third term (4.3) vanishes at $\boldsymbol{x}^*$ in the second derivative. □

Comparing the derivatives calculated in Lemma 4.1 with the derivatives calculated for the Reduction Ansatz in Equation (2.17) and (2.18), one can see that they coincide. This means that the first two derivatives of the additional constraints coincide with the derivatives of the local description by finitely many constraints. If we would chose to use the semi-infinite constraint (instead of the Lagrange function), only the first derivative would coincide.

There are two reasons why it is important to also match the second derivative: (1) we have seen in Theorem 2.21 that the second derivative is important for a point being a local minimum, (2) the approximate problem is locally closer to the true problem, which makes the quadratic rate of convergence possible.

With the above consideration we obtain the following algorithm:

Algorithm 2 Adaptive discretization method with linear information

1: Input: initial point $\boldsymbol{x}^1$, initial discretization $Y^1 \subseteq Y, k = 1$.
2: **while** termination criterion is not met **do**
3: **for** $i \in I$ **do**
4: Compute a global solution $\boldsymbol{y}^{i,k}$ and Lagrange multipliers $\boldsymbol{\mu}^{i,k}$ of $Q_i(\boldsymbol{x}^k)$.
5: Determine $D\boldsymbol{y}^i(\boldsymbol{x}^k)$ and $D\boldsymbol{\mu}^i(\boldsymbol{x}^k)$ if they exist.
6: **end for**
7: $Y^{k+1} = Y^k \cup \bigcup_{i \in I} \{\boldsymbol{y}^{i,k}\}$.
8: Determine a solution $\boldsymbol{x}^{k+1}$ of problem $\overline{\mathsf{SIP}}^k(Y^{k+1})$.
9: $k = k + 1$.
10: **end while**

Again, we do not specify which kind of solution we compute in Step 8 but we demand that at least a feasible point is found. Analogous to the statements about Algorithm 1, the same questions can be asked here:

- Is every accumulation point feasible?
- If one is able to calculate global/ local minima in Step 8, will the limit again be a global/local solution?
- Is every accumulation point of stationary points again a stationary point?

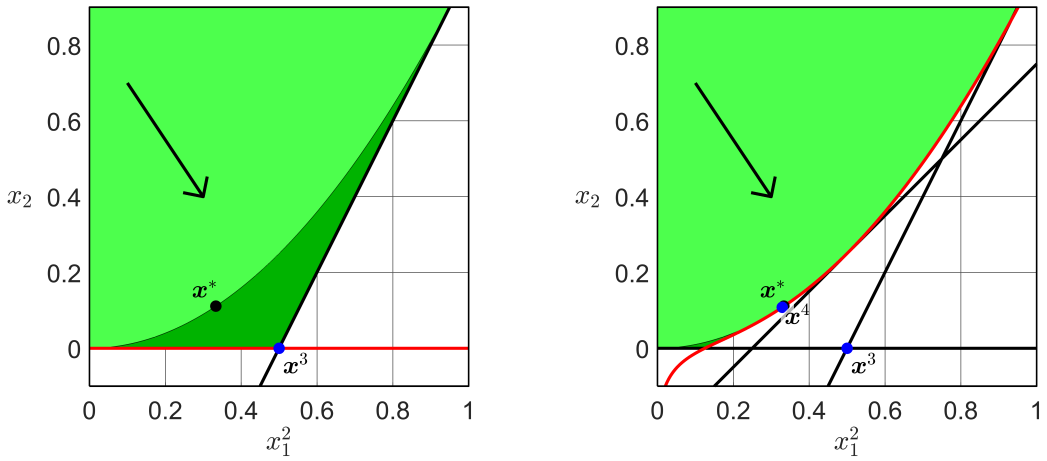
We discuss these questions in Section 4.1. The reason why we developed the new algorithm is the hope that the linear information can speed up the convergence. We show in Section 4.2 that this is indeed the case and the algorithm has a quadratic rate of convergence. Before stating all theoretical statements and proofs, we first consider a slightly modified version of Example 3.14 introduced in Section 3.2:

Example 4.2. *In Example 3.14 replace every x_1 by x_1^2. We obtain the following problem:*

$$\begin{aligned} \mathsf{SIP}_{ex}: \min \quad & -x_1^2 + \frac{3}{2}x_2 \\ s.t. \quad & -y^2 + 2y \cdot x_1^2 - x_2 \leq 0 \quad \text{for all} \quad y \in [-1,1] \ , \\ & x_1 \in [0,1], x_2 \in [-1,1] \ . \end{aligned}$$

The solution of the lower-level problem is given by $y = x_1^2$, which is differentiable. The first iterates of the modified algorithm can be calculated numerically. They are shown in Figure 4.1. Moreover, they and their distance to the optimal solution are listed in Table 4.1. As one can see, an accuracy of 10^{-6} is reached in 4 iterations. With the bisection type iterates of the Blankenship and Falk algorithm, calculated in

Example 3.14, approximately 20 *iterations are needed to reach the same accuracy. One can clearly see that the convergence is much faster and the iterates seem to converge quadratically.*



a) Iteration: $k = 2$ **b)** Iteration: $k = 3$

Figure 4.1: Adaptive discretization method with a quadratic rate of convergence [*light green* - feasible set of SIP, *dark green* - feasible set of approximate problem $\overline{\mathsf{SIP}}^{k-1}(Y^k)$, *black arrow* - descent direction of objective, *black lines* - constraints induced by discretization, *red line* - additional constraint, *blue points* - iterates, *black point* - limit solution]

Iteration k	$\boldsymbol{x}^k$	$\|\boldsymbol{x}^k - \boldsymbol{x}^*\|$
1	$(1, -1)$	1.1888
2	$(0, -1)$	1.2522
3	$(0.707107, 0)$	0.1708
4	$(0.573761, 0.108057)$	0.0047
5	$(0.57735, 0.111111)$	$2.1979 \cdot 10^{-07}$

Table 4.1: First iterates for Example 4.2

For all the statements presented in the following we assume that the Reduction Ansatz holds at a limit point. We want to comment briefly on cases where this is not true, and the Reduction Ansatz does not hold at the limit point. In this case we can still apply Algorithm 2, but linear information will not be added. If no derivative exists, the algorithm coincides with the adaptive discretization method by Blankenship and Falk investigated in Chapter 3. In other words, this means that if the Reduction Ansatz holds in a limit point we can expect a quadratic rate

of convergence and all the properties proven in the following. But if this is not the case, the algorithm does not completely fail, but coincides with the Blankenship and Falk algorithm

4.1 Basic convergence properties

In this Section we give an answer to the questions formulated above about the convergence properties. In Subsection 4.1.1 we prove that a limit of stationary points is again a stationary point. In the next subsection we then investigate the convergence of local solutions. Analogous to Lemma 3.1 we first prove that every accumulation point of a sequence constructed by Algorithm 2 is feasible.

Lemma 4.3. *Let $\{\boldsymbol{x}^k\}_{k\in\mathbb{N}}$ be constructed according to Algorithm 2. Every accumulation point $\boldsymbol{x}^*$ of $\{\boldsymbol{x}^k\}_{k\in\mathbb{N}}$ is feasible.*

Proof. By construction of the algorithm we have, for every $k' \geq k+1$ and $i \in I$:

$$g_i(\boldsymbol{x}^{k'}, \boldsymbol{y}^{i,k}) \leq 0 \ .$$

By assumption $\boldsymbol{x}^*$ is an accumulation point and Y is compact, hence we can choose subsequences $\{\boldsymbol{x}^{k_l}\}_{l\in\mathbb{N}}$ and $\{\boldsymbol{y}^{i,k_l}\}_{l\in\mathbb{N}}$, for every $i \in I$, such that

$$\begin{aligned} \lim_{l\to\infty} \boldsymbol{x}^{k_l} &= \boldsymbol{x}^* \ , \\ \lim_{l\to\infty} \boldsymbol{y}^{i,l} &= \boldsymbol{y}^{i,*} \ . \end{aligned}$$

For an arbitrary point $\boldsymbol{y} \in Y$ and every $i \in I$, the following holds by continuity:

$$\begin{aligned} g_i(\boldsymbol{x}^*, \boldsymbol{y}) &= \lim_{l\to\infty} g_i(\boldsymbol{x}^{k_l}, \boldsymbol{y}) \\ &\leq \lim_{l\to\infty} g_i(\boldsymbol{x}^{k_l}, \boldsymbol{y}^{i,k_l}) \\ &= g_i(\lim_{l\to\infty} \boldsymbol{x}^{k_l}, \lim_{l\to\infty} \boldsymbol{y}^{i,k_l}) \\ &= g_i(\lim_{l\to\infty} \boldsymbol{x}^{k_{l+1}}, \lim_{l\to\infty} \boldsymbol{y}^{i,k_l}) \\ &= \lim_{l\to\infty} g_i(\boldsymbol{x}^{k_{l+1}}, \boldsymbol{y}^{i,k_l}) \leq 0 \ . \end{aligned}$$

□

It is noteworthy that exactly the same proof can be used to show the same property (Lemma 3.1) for the original algorithm by Blankenship and Falk. The additional

constraint is not needed. The feasibility of accumulation points is a global convergence property that is directly inherited.

After showing feasibility of any accumulation point, the next question is whether we can guarantee $\boldsymbol{x}^*$ to be a global solution, if, for every $k \in \mathbb{N}$, the iterate $\boldsymbol{x}^k$ is a global solution of the current approximate problem. Unfortunately, some important properties do not hold anymore for Algorithm 2. The additional constraint is not part of the original constraints and can cut away parts of the feasible set. It may hold:

$$M \not\subseteq M^k$$

and thus also:

$$\min_{\boldsymbol{x} \in M} f(\boldsymbol{x}) < \min_{\boldsymbol{x} \in M^k} f(\boldsymbol{x}) \ ,$$

where M^k denotes the feasible set of problem $\overline{\mathsf{SIP}}^{k-1}(Y^k)$. These are important properties in the original algorithm (Algorithm 1) for the convergence of global as well as local minima. Therefore, finding a general statement asserting that an accumulation point is again a global solution is not possible.

However, the situation is better for local solutions. As we will see in Section 4.1.2, under slightly different assumptions from those in Theorem 3.2 we can still guarantee that any accumulation point of local solutions is again a local solution. The proof is very different. We start in the next Section 4.1.1 by showing that a sequence of stationary points converges to a stationary point.

To simplify the exposition of the following statements we make a general assumption.

Assumption 4.4. *Let $\{\boldsymbol{x}^k\}_{k \in \mathbb{N}}$ be constructed according to Algorithm 2. Assume there is an $\boldsymbol{x}^* \in \mathbb{R}^n$ with*

$$\lim_{k \to \infty} \boldsymbol{x}^k = \boldsymbol{x}^* \ .$$

Assume that the Reduction Ansatz holds at $\boldsymbol{x}^*$ and the objective function f is twice continuously differentiable. Further assume:

- *For every $i \in I$, there is a $\boldsymbol{y} \in Y$ with*

$$g_i(\boldsymbol{x}^*, \boldsymbol{y}) = 0 \ .$$

- *For every $k \in \mathbb{N}$ the following holds:*

$$\bar{I}^k = I \ . \tag{4.4}$$

  This means that all derivatives of the solutions of the lower-level exist.

The last two assumptions are only made to avoid case distinctions in the proofs. If there is an $i \in I$ such that no active index exists, one can simply consider the problem locally without this semi-infinite constraint. As we assume the Reduction Ansatz to hold in the limit point $\boldsymbol{x}^*$, there is a $k' \in \mathbb{N}$ such that (4.4) holds for all $k \geq k'$. If this index is larger than 1, we can start the algorithm again with the current point and the current discretization as initialization.

4.1.1 Convergence of stationary points

In this section we investigate an accumulation point of stationary points. Thus, for a sequence $\{\boldsymbol{x}^k\}_{k\in\mathbb{N}}$ constructed by Algorithm 2, we assume that, for every $k \in \mathbb{N}$, the current iterate $\boldsymbol{x}^k$ is a stationary point of $\overline{\mathsf{SIP}}^{k-1}(Y^k)$.

We already know by Lemma 4.3 that $\boldsymbol{x}^*$ is a feasible point. As the Reduction Ansatz is assumed to be satisfied, the set of active indices $Y_0^i(\boldsymbol{x}^*)$ has, for every $i \in I$, only a finite cardinality. We need to find Lagrange multipliers $\boldsymbol{\lambda}^* \geq 0$ such that

$$0 = Df(\boldsymbol{x}^*) + \sum_{i\in I} \sum_{\boldsymbol{y}\in Y_0^i(\boldsymbol{x}^*)} \lambda^*_{i,\boldsymbol{y}} D_1 g_i(\boldsymbol{x}^*, \boldsymbol{y}) \ . \tag{4.5}$$

The idea is to construct these multipliers as limits of the multipliers corresponding to the iterates. We denote, for every $i \in I$, the set of active discretization points in the k-th iteration by

$$Y_0^{i,k} := \{\boldsymbol{y} \in Y^k \mid g_i(\boldsymbol{x}^k, \boldsymbol{y}) = 0\} \ .$$

By assumption there exist, for every $k \in \mathbb{N}$, Lagrange multipliers $\boldsymbol{\lambda}^k \geq 0$ and $\overline{\boldsymbol{\lambda}}^k \geq 0$ such that:

$$0 = Df(\boldsymbol{x}^k) + \sum_{i\in I} \sum_{\boldsymbol{y}\in Y_0^{i,k}} \lambda^k_{i,\boldsymbol{y}} D_1 g_i(\boldsymbol{x}^k, \boldsymbol{y}) + \sum_{i\in I} \overline{\lambda}^k_i D\overline{g}_i^{k-1}(\boldsymbol{x}^k) \ , \tag{4.6}$$

where, for every $i \in I$, the multiplier $\overline{\lambda}^k_i$ is 0 if $\overline{g}_i^{k-1}(\boldsymbol{x}^k) < 0$.

For the construction of the Lagrange-multipliers in (4.5) we need to match the active indices and the additional constraints in a current iteration to the active indices in the limit. We do this with the next two lemmas, first for the active indices $Y_0^{i,k}(\boldsymbol{x}^k)$ then for the additional constraints $\overline{g}_i^{k-1}(\boldsymbol{x}^k)$.

Lemma 4.5. *Let Assumption 4.4 be satisfied. For every $\delta > 0$ there is a $k' \in \mathbb{N}$ such that, for all $k \geq k'$, $i \in I$ and $\boldsymbol{y} \in Y$ with*

$$g_i(\boldsymbol{x}^k, \boldsymbol{y}) = 0 \ ,$$

there is a $\boldsymbol{y}^{i,*} \in Y_0^i(\boldsymbol{x}^*)$ *such that*

$$\|\boldsymbol{y}^{i,*} - \boldsymbol{y}\| < \delta \ .$$

Proof. Fix throughout the proof an $i \in I$. The set

$$Y^{i,\delta} := Y \setminus \bigcup_{\boldsymbol{y} \in Y_0^i(\boldsymbol{x}^*)} B_\delta(\boldsymbol{y})$$

is a compact set. Thus the maximum:

$$\max \left\{ g_i(\boldsymbol{x}^*, \boldsymbol{y}) \mid \boldsymbol{y} \in Y^{i,\delta} \right\}$$

is attained and is strictly less than 0. Therefore, by continuity there is a k' such that, for $k \geq k'$ and $\boldsymbol{y} \in Y^{i,\delta}$:

$$g_i(\boldsymbol{x}^k, \boldsymbol{y}) < 0 \ ,$$

which induces the claim. □

By continuity and the lemma above we can, for every $\varepsilon > 0$, choose a $k' \in \mathbb{N}$ such that, for all $k \geq k'$, $i \in I$ and $\boldsymbol{y} \in Y^k$ with

$$g_i(\boldsymbol{x}^k, \boldsymbol{y}) = 0 \ ,$$

there is a $\boldsymbol{y}^{i,*} \in Y_0^i(\boldsymbol{x}^*)$ such that

$$\|D_1 g_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,*}) - D_1 g_i(\boldsymbol{x}^k, \boldsymbol{y})\| < \varepsilon \ . \tag{4.7}$$

Similarly we can also achieve a bound for the additional constraints:

Lemma 4.6. *Let Assumption 4.4 be satisfied. For* $i \in I$ *consider a converging subsequence* $\{\boldsymbol{y}^{i,k_l}\}_{l \in \mathbb{N}}$ *with*

$$\boldsymbol{y}^{i,*} := \lim_{l \to \infty} \boldsymbol{y}^{i,k_l} \ .$$

Then $\boldsymbol{y}^{i,*} \in Y_0^i(\boldsymbol{x}^*)$. *Moreover, for every* $\varepsilon > 0$, *there is an* $l' \in \mathbb{N}$ *such that, for every* $l > l'$ *and* $i \in I$, *the following holds:*

$$\|D_1 g_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,*}) - D\overline{g}_i^{k_l - 1}(\boldsymbol{x}^{k_l})\| < \varepsilon \ .$$

Proof. For $i \in I$, consider an arbitrary converging subsequence $\{\boldsymbol{y}^{i,k_l}\}_{l \in \mathbb{N}}$. For $\boldsymbol{y}^{i,*}$ and $\boldsymbol{y} \in Y$ the following holds:

$$\begin{aligned} g_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,*}) &= \lim_{l \to \infty} g_i(\boldsymbol{x}^{k_l}, \boldsymbol{y}^{i,k_l}) \\ &\geq \lim_{l \to \infty} g_i(\boldsymbol{x}^{k_l}, \boldsymbol{y}) \\ &= \boldsymbol{g}(\boldsymbol{x}^*, \boldsymbol{y}) \ . \end{aligned}$$

Which means $\boldsymbol{y}^{i,*}$ is a global solution of $Q_i(\boldsymbol{x}^*)$. As we have assumed to have at least one active index, we must have $\boldsymbol{y}^{i,*} \in Y_0^i(\boldsymbol{x}^*)$.

We have seen in Lemma 4.1 that, for every $i \in I$ and $k \in \mathbb{N}$, the following holds:

$$D\overline{g}_i^k(\boldsymbol{x}^k) = D_1 g(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) \ . \tag{4.8}$$

For every $\varepsilon > 0$ we can choose by continuity an $l' \in \mathbb{N}$ such that, for every $l > l'$ and $i \in I$, we have:

$$\begin{aligned} \|D_1 g_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,*}) - D_1 g_i(\boldsymbol{x}^{k_l-1}, \boldsymbol{y}^{i,k_l-1})\| &< \frac{\varepsilon}{2} \ , \\ \|D\overline{g}_i^{k_l-1}(\boldsymbol{x}^{k_l-1}) - D\overline{g}_i^{k_l-1}(\boldsymbol{x}^{k_l})\| &< \frac{\varepsilon}{2} \ . \end{aligned}$$

Combining these inequalities with Equation (4.8) shows:

$$\begin{aligned} &\|D_1 g_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,*}) - D\overline{g}_i^{k_l-1}(\boldsymbol{x}^{k_l})\| \\ \leq\ &\|D_1 g_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,*}) - D_1 g_i(\boldsymbol{x}^{k_l-1}, \boldsymbol{y}^{i,k_l-1})\| \\ &\quad + \|D\overline{g}_i^{k_l-1}(\boldsymbol{x}^{k_l-1}) - D\overline{g}_i^{k_l-1}(\boldsymbol{x}^{k_l})\| \\ <\ &\varepsilon \ . \end{aligned}$$

□

We can now construct multipliers which converge towards the Lagrange multipliers of $\boldsymbol{x}^*$. Therefore, choose a $\delta > 0$ such that, for every $i \in I$, the balls $B_\delta(\boldsymbol{y}), \boldsymbol{y} \in Y_0^i(\boldsymbol{x}^*)$ are disjoint. As $\boldsymbol{x}^k$ is a stationary point, there are Lagrange-multipliers $\boldsymbol{\lambda}^k \geq 0$ and $\overline{\boldsymbol{\lambda}}^k \geq 0$ as in (4.6).

For every $k \in \mathbb{N}, i \in I$ and $\boldsymbol{y} \in Y_0^i(\boldsymbol{x}^*)$, let

$$\hat{\lambda}_{i,\boldsymbol{y}}^k = \sum_{\substack{\dot{\boldsymbol{y}} \in Y_0^{i,k}(\boldsymbol{x}^k), \\ \|\dot{\boldsymbol{y}} - \boldsymbol{y}\| < \delta}} \lambda_{i,\dot{\boldsymbol{y}}}^k + \begin{cases} \overline{\lambda}_i^k & \text{if } \|\boldsymbol{y}^{i,k-1} - \boldsymbol{y}\| < \delta \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

We need to bound the constructed Lagrange multipliers.

Lemma 4.7. *Let Assumption 4.4 be satisfied. Assume that, for every $k \in \mathbb{N}$, the current iterate $\boldsymbol{x}^k$ is a stationary point of $\overline{\mathsf{SIP}}^{k-1}(Y^k)$ and that* EMFCQ *is satisfied at $\boldsymbol{x}^*$. There is a constant $K > 0$ such that, for sufficiently large k, the following holds:*

$$\sum_{i \in I} \sum_{\boldsymbol{y} \in Y_0^i(\boldsymbol{x}^*)} \hat{\lambda}_{i,\boldsymbol{y}}^k \leq K \ .$$

Proof. As EMFCQ holds, there is a $\boldsymbol{\xi} \in \mathbb{R}^n$ such that, for every $i \in I, \boldsymbol{y} \in Y_0^i(\boldsymbol{x}^*)$

$$D_1 g_i(\boldsymbol{x}^*, \boldsymbol{y})\boldsymbol{\xi} \leq -1 \ .$$

By Equation (4.7) and Lemma 4.6 we can choose a $k' \in \mathbb{N}$ such that, for every $k \geq k', i \in I$ and $\boldsymbol{y} \in Y_0^{i,k}$, the following is true:

$$\begin{aligned} D_1 g_i(\boldsymbol{x}^k, \boldsymbol{y})\boldsymbol{\xi} &\leq \ -0.5 \ , \\ D\overline{g}_i^{k-1}(\boldsymbol{x}^k)\boldsymbol{\xi} &\leq \ -0.5 \ . \end{aligned}$$

Multiplying the stationarity condition given in Equation (4.6) with the vector $\boldsymbol{\xi}$ shows, for $k \geq k'$:

$$\begin{aligned} 0 = & \left(Df(\boldsymbol{x}^k) + \sum_{i \in I} \sum_{\boldsymbol{y} \in Y_0^{i,k}} \lambda_{i,\boldsymbol{y}}^k D_1 g_i(\boldsymbol{x}^k, \boldsymbol{y}) + \sum_{i \in I} \overline{\lambda}_i^k D\overline{g}_i^{k-1}(\boldsymbol{x}^k) \right) \cdot \boldsymbol{\xi} \\ & \leq Df(\boldsymbol{x}^k)\boldsymbol{\xi} - 0.5 \sum_{i \in I} \sum_{\boldsymbol{y} \in Y_0^{i,k}} \lambda_{i,\boldsymbol{y}}^k - 0.5 \sum_{i \in I} \overline{\lambda}_i^k \end{aligned}$$

by resorting and using the definition of the Lagrange multipliers:

$$= Df(\boldsymbol{x}^k)\boldsymbol{\xi} - 0.5 \sum_{i \in I} \sum_{\boldsymbol{y} \in Y_0^i(\boldsymbol{x}^*)} \hat{\lambda}_{i,\boldsymbol{y}}^k \ .$$

The boundedness now follows directly from the boundedness of $Df(\boldsymbol{x}^k)$. □

We are ready to prove that the limit point is again a stationary point.

Theorem 4.8. *Let Assumption 4.4 be satisfied. Assume that, for every $k \in \mathbb{N}$, the current iterate $\boldsymbol{x}^k$ is a stationary point of $\overline{\mathsf{SIP}}^{k-1}(Y^k)$ and that* EMFCQ *is satisfied at $\boldsymbol{x}^*$. Then $\boldsymbol{x}^*$ is a stationary point of* SIP.

Proof. Choose an arbitrary $\varepsilon > 0$ and a $\delta > 0$ such that, for every $i \in I$, the balls $B_\delta(\boldsymbol{y}), \boldsymbol{y} \in Y_0^i(\boldsymbol{x}^*)$ are disjoint.

By Lemma 4.5 we can choose a $k_1 \in \mathbb{N}$ such that, for every $k \geq k_1, i \in I$ and $\boldsymbol{y} \in Y_0^{i,k}$, there is a $\boldsymbol{y}^{i,*} \in Y_0^i(\boldsymbol{x}^*)$ with

$$\begin{aligned} \|\boldsymbol{y} - \boldsymbol{y}^{i,*}\| &< \delta \ , \\ \|D_1 g_i(\boldsymbol{x}^k, \boldsymbol{y}) - D_1 g_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,*})\| &< \frac{\varepsilon}{2K} \ , \end{aligned}$$

where K is chosen as in Lemma 4.7. By Lemma 4.6 we can choose $k_2 \in \mathbb{N}$ such that, for every $k \geq k_2$ and $i \in I$, there is a $\boldsymbol{y}^{i,*} \in Y_0^i(\boldsymbol{x}^*)$ with:

$$\begin{aligned} \|\boldsymbol{y}^{i,k-1} - \boldsymbol{y}^{i,*}\| &< \delta \ , \\ \|\overline{g}_i^{k-1}(\boldsymbol{x}^k) - D_1 g_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,*})\| &< \frac{\varepsilon}{2K} \ . \end{aligned} \tag{4.9}$$

Moreover, by the convergence of $\{\boldsymbol{x}^k\}_{k\in\mathbb{N}}$ and the continuous differentiability of the objective there is a $k_3 \in \mathbb{N}$ such that, for $k \geq k_3$, the following holds:

$$\|Df(\boldsymbol{x}^*) - Df(\boldsymbol{x}^k)\| < \frac{\varepsilon}{2} \ .$$

Let $k' := \max\{k_1, k_2, k_3\}$. Combining, for $k \geq k'$, the three inequalities with the stationarity condition for $\boldsymbol{x}^k$ (see Equation (4.6)) shows:

$$\begin{aligned} &\Big\|Df(\boldsymbol{x}^*) + \sum_{i\in I} \sum_{\boldsymbol{y}\in Y_0^i(\boldsymbol{x}^*)} \hat{\lambda}_{i,\boldsymbol{y}}^k Dg_i(\boldsymbol{x}^*, \boldsymbol{y})\Big\| \\ \leq\ & \|Df(\boldsymbol{x}^*) - Df(\boldsymbol{x}^k)\| \\ &+ \sum_{i\in I} \sum_{\boldsymbol{y}\in Y_0^i(\boldsymbol{x}^*)} \sum_{\substack{\dot{\boldsymbol{y}}\in Y_0^{i,k} \\ \|\dot{\boldsymbol{y}}-\boldsymbol{y}\|<\delta}} \lambda_{i,\dot{\boldsymbol{y}}}^k \Big\|D_1 g_i(\boldsymbol{x}^*, \boldsymbol{y}) - D_1 g_i(\boldsymbol{x}^k, \dot{\boldsymbol{y}})\Big\| \\ &+ \sum_{i\in I} \overline{\lambda}_i^k \Big\|D_1 g_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,*}) - D\overline{g}_i^{k-1}(\boldsymbol{x}^k)\Big\| \\ <\ & \frac{\varepsilon}{2} + \frac{\varepsilon}{2K} \cdot \left(\sum_{i\in I} \sum_{\boldsymbol{y}\in Y_0^i(\boldsymbol{x}^*)} \hat{\lambda}_{i,\boldsymbol{y}}^k \right) \\ \leq\ & \varepsilon \ . \end{aligned} \tag{4.10}$$

where, for every $i \in I$ and $k \geq k$, the active index $\boldsymbol{y}^{i,*}$ is chosen according to (4.9). As, for every $i \in I$ and $\boldsymbol{y} \in Y_0^i(\boldsymbol{x}^*)$, by Lemma 4.7 the multiplier $\hat{\lambda}_{i,\boldsymbol{y}}^k$ are bounded, we can choose a subsequence $\{\boldsymbol{x}^{k_l}\}_{l\in\mathbb{N}}$ such that, for every $i \in I$ and $\boldsymbol{y} \in Y_0^i(\boldsymbol{x}^*)$:

$$\lambda_{i,\boldsymbol{y}}^* := \lim_{l\to\infty} \hat{\lambda}_{i,\boldsymbol{y}}^{k_l}$$

exists. From the inequality in Equation (4.10) it follows:

$$Df(\boldsymbol{x}^*) + \sum_{i\in I} \sum_{\boldsymbol{y}\in Y_0^i(\boldsymbol{x}^*)} \lambda_{i,\boldsymbol{y}}^* D_1 g_i(\boldsymbol{x}^*, \boldsymbol{y}) = 0 \ .$$

□

Remark 4.9. We presented the proof here in the context of Algorithm 2, but the same proof can be used to show a similar statement for the Blankenship and Falk algorithm (Algorithm 1). The only difference lies in the fact that no additional constraint has to be considered, which only simplifies the proof.

4.1.2 Convergence of local solutions

Now we turn our interest to the convergence of local solutions. Therefore we assume that for every iteration the calculated solution $\boldsymbol{x}^k$ is a local minimum of $\overline{\mathsf{SIP}}^{k-1}(Y^k)$. The main question is under which conditions an accumulation point is again a local solution. We have seen in Theorem 3.2 that for the algorithm by Blankenship and Falk (Algorithm 1) the only property needed are that the radii do not vanish in the limit. The next example shows that the situation is more complicated in case of Algorithm 2.

Example 4.10. *We consider the following semi-infinite optimization problem:*

$$\begin{aligned} \mathsf{SIP}_{ex}: \min_{\boldsymbol{x}\in\mathbb{R}^2} \quad & x_2 \\ s.t. \quad & -x_2 + (1+x_1^2)(4x_1 \cdot y_1 + y_2 - 3x_1^2) + x_1^2 \leq 0 \text{ for all } \boldsymbol{y} \in Y \ , \\ & x_1, x_2 \in [-1,1] \ , \end{aligned}$$

where

$$Y := \{\boldsymbol{y} \in [-2,2]^2 \mid y_2 + 2y_1^2 \leq 0\} \ .$$

It is easy to see that for every $\boldsymbol{x} \in [-1,1]^2$ the point and the multiplier

$$\begin{aligned} \boldsymbol{y}(\boldsymbol{x}) &= \begin{pmatrix} x_1 \\ -2x_1^2 \end{pmatrix} \ , \\ \mu(\boldsymbol{x}) &:= (1+x_1^2) \end{aligned}$$

make up the unique KKT point of the lower-level problem $Q(\boldsymbol{x})$. As LICQ *is satisfied at every point, $\boldsymbol{y}(\boldsymbol{x})$ is the unique global solution. Plugging in the lower-level solution into the semi-infinite constraint shows that the feasible set can be described by:*

$$M := \{\boldsymbol{x} \in [-1,1]^2 \mid -x_2 - x_1^4 \leq 0\} \ .$$

As it can be seen in Figure 4.2, the feasible set is non-convex. There are two global solutions $(-1,-1)^\top, (1,-1)^\top$ and one stationary point $(0,0)^\top$ of the semi-infinite problem SIP_{ex}.

Using $\boldsymbol{x}^0 = (0,1)^\top$ as an initial point, the first solution of the lower-level problem and the Lagrange multiplier are $\boldsymbol{y}^1 = (0,0)^\top, \mu^1 = 1$. The linearly developed functions are given by

$$\begin{aligned} \overline{\boldsymbol{y}}^{i,1}(\boldsymbol{x}) &= \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \\ \overline{\mu}^1(\boldsymbol{x}) &= 1 \ . \end{aligned}$$

The additional constraint is then given by

$$\overline{g}_i^1(\boldsymbol{x}) = -x_2 + (1 + x_1^2)(4x_1^2 - 3x_1^2) + x_1^2 - 2x_1^2 = -x_2 + x^4 \ . \tag{4.11}$$

The next iterate is a solution of the following problem:

$$\min\{x_2 \mid x_2 \geq x_1^4\} \ .$$

(the discretization can be dropped, as the feasible set is a subset of M and all feasible sets induced by a discretization are supersets.) The feasible set of the approximate problem is also shown in Figure 4.2. The problem is convex and has a unique global solution at $\boldsymbol{x}^2 = (0,0)^\top$. In the next iteration the solution of the lower-level and its linearization do not change. Which means that for all $k > 1$:

$$\boldsymbol{x}^k = (0,0)^\top$$

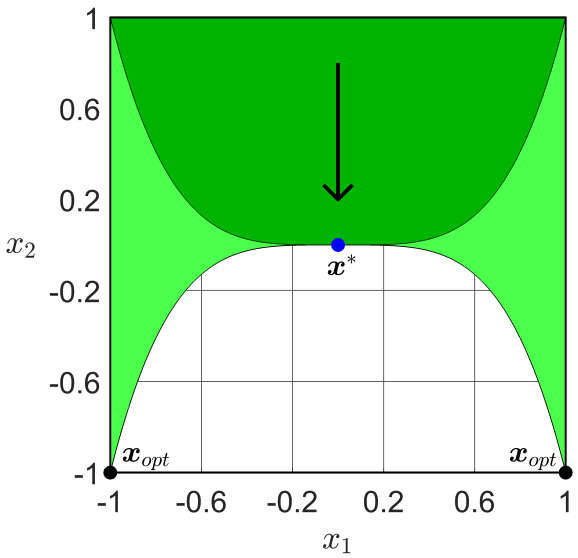


Figure 4.2: Feasible sets of approximate and original problem in Example 4.10 [*light green* - feasible set of SIP_{ex}, *dark green* - feasible set of $\overline{\mathsf{SIP}}^{k-1}(Y^k)$ for $k > 1$, *black arrow* - descent direction of objective, *blue point* - limit point of constructed sequence, *black points* - global solutions of SIP_{ex}]

The example shows that even if we are able to calculate a global solution of the approximate problem $\overline{\mathsf{SIP}}^{k-1}(Y^k)$ in every iteration, it does not mean that the iterates converge towards a local solution of the original problem SIP. In the example the iterates do not converge towards a global or local minimum. But the limit point is a local solution of a different problem.

The next lemma shows that this is always the case:

Lemma 4.11. *Let $\{\boldsymbol{x}^k\}_{k\in\mathbb{N}}$ be constructed according to Algorithm 2. Assume that, for every $k \in \mathbb{N}$, the iterate $\boldsymbol{x}^k$ is a local solution of $\overline{\mathsf{SIP}}^{k-1}(Y^k)$ with radius r_k. Assume:*

$$\begin{aligned} \lim_{k\to\infty} \boldsymbol{x}^k &= \boldsymbol{x}^* \ , \\ \lim_{k\to\infty} \boldsymbol{y}^{i,k} &= \boldsymbol{y}^{i,*} \ , \\ r := \inf_{k\in\mathbb{N}} r_k &> 0 \ . \end{aligned}$$

Further assume that the Reduction Ansatz and EMFCQ *hold at $\boldsymbol{x}^*$. Then $\boldsymbol{x}^*$ is a local solution of*

$$\begin{aligned} \overline{\mathsf{SIP}} : \min_{\boldsymbol{x}\in\mathbb{R}^n} \quad & f(\boldsymbol{x}) \\ s.t. \quad & g_i(\boldsymbol{x},\boldsymbol{y}) \leq 0 \text{ for all } i \in I, \boldsymbol{y} \in Y \ , \\ & \overline{g}_i(\boldsymbol{x}) \leq 0 \text{ for all } i \in I \ , \end{aligned}$$

where, for $i \in I$:

$$\overline{g}_i(\boldsymbol{x}) = \mathcal{L}_i\big(\boldsymbol{x}, \boldsymbol{y}^{i,*} + D\boldsymbol{y}^i(\boldsymbol{x}^*)(\boldsymbol{x}-\boldsymbol{x}^*), \boldsymbol{\mu}^{i,*} + D\boldsymbol{\mu}^i(\boldsymbol{x}^*)(\boldsymbol{x}-\boldsymbol{x}^*)\big) \ .$$

Proof. In a first step we try to construct for a point $\dot{\boldsymbol{x}} \in \mathbb{R}^n$ feasible for $\overline{\mathsf{SIP}}$, a point which is feasible for the iterate problem $\overline{\mathsf{SIP}}^{k-1}(Y^k)$.

By EMFCQ there is a vector $\boldsymbol{\xi}$ with

$$D_1 g_i(\boldsymbol{x}^*, \boldsymbol{y})\boldsymbol{\xi} \leq -1$$

for every $i \in I$ and $\boldsymbol{y}$ with $g_i(\boldsymbol{x}^*, \boldsymbol{y}) = 0$.

By Lemma 4.1 there is a $\delta_1 > 0$ such that, for sufficiently large k and every $\boldsymbol{x} \in B_{\delta_1}(\boldsymbol{x}^*)$, the following holds:

$$D\overline{g}_i^{k-1}(\boldsymbol{x})\boldsymbol{\xi} \leq -\frac{1}{2} \ .$$

By continuity there is also a $\delta_2 > 0$ such that, for every $i \in I$, the following holds:

$$D_1 g_i(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{\xi} \leq -\frac{1}{2}$$

for every $\boldsymbol{y} \in Y_0^{i,\delta_2}$ and $\boldsymbol{x} \in B_{\delta_2}(\boldsymbol{x}^*)$, where

$$Y_0^{i,\delta_2}(\boldsymbol{x}^*) := \bigcup_{\boldsymbol{y} \in Y_0^i(\boldsymbol{x}^*)} B_{\delta_2}(\boldsymbol{y})$$

Let $\delta := \min\{\delta_1, \delta_2, r\}$.

Consider an arbitrary but fixed $\dot{\boldsymbol{x}} \in B_{\delta/2}(\boldsymbol{x}^*)$ which is feasible for $\overline{\mathsf{SIP}}$.

The chosen point is not always feasible for $\overline{\mathsf{SIP}}^{k-1}(Y^k)$. It fulfills all constraints induced by the discretization points, but may be infeasible for the additional constraints. We can measure the violation by

$$\alpha^k = \max_{i \in I} \max\{0, \overline{g}_i^{k-1}(\dot{\boldsymbol{x}})\} \ .$$

By the feasibility for $\overline{\mathsf{SIP}}$ we know that α^k converges towards 0. We can now construct a point which is feasible for $\overline{\mathsf{SIP}}^{k-1}(Y^k)$. Let

$$\bar{\boldsymbol{x}}^k := \dot{\boldsymbol{x}} + 2\alpha^k \boldsymbol{\xi} \ .$$

Choose k sufficiently large such that $\|\bar{\boldsymbol{x}}^k - \dot{\boldsymbol{x}}\| \leq \delta/2$.

After we have constructed the point we now briefly prove that it is indeed feasible for sufficiently large k.

By Taylor there is, for sufficiently large k and $i \in I$, an $\tilde{\boldsymbol{x}} := t\dot{\boldsymbol{x}} + (1-t)\bar{\boldsymbol{x}}^k$, for a suitable $t \in [0,1]$, such that

$$\overline{g}_i^{k-1}(\bar{\boldsymbol{x}}^k) = \overline{g}_i^{k-1}(\dot{\boldsymbol{x}}) + D\overline{g}_i^{k-1}(\tilde{\boldsymbol{x}}) 2\alpha^k \boldsymbol{\xi} \leq \alpha^k - 2\frac{1}{2}\alpha^k = 0 \ .$$

Completely analogous, for sufficiently large k, $i \in I$ and $\boldsymbol{y} \in Y_0^{i,\delta}$, the following also holds:

$$g(\bar{\boldsymbol{x}}^k, \boldsymbol{y}) \leq -\alpha^k \ .$$

For every $i \in I$, the set $\overline{Y} := Y^k \setminus Y_0^{i,\delta}$ is a compact set. Thus, $\max_{\boldsymbol{y} \in \overline{Y}} g_i(\boldsymbol{x}^*, \boldsymbol{y})$ is attained and is strictly less than 0. By continuity we can choose k sufficiently large such that, for every $\boldsymbol{y} \in \overline{Y}$, the following holds:

$$g_i(\bar{\boldsymbol{x}}^k, \boldsymbol{y}) < 0 \ .$$

The three inequalities prove feasibility.

By definition we have $\delta \leq r$. Together with the feasibility this shows, for sufficiently large k:

$$f(\boldsymbol{x}^k) \leq f(\bar{\boldsymbol{x}}^k) \ .$$

As $\bar{\boldsymbol{x}}^k$ converges towards $\dot{\boldsymbol{x}}$, we have:

$$f(\boldsymbol{x}^*) = \lim_{k\to\infty} f(\boldsymbol{x}^k) \leq \lim_{k\to\infty} f(\bar{\boldsymbol{x}}^k) = f(\dot{\boldsymbol{x}}) \ .$$

The claim is shown as $\dot{\boldsymbol{x}}$ was chosen arbitrarily in $B_{\delta/2}(\boldsymbol{x}^*)$ and feasible for $\overline{\mathsf{SIP}}$. □

If the local solutions of $\overline{\mathsf{SIP}}$ and SIP coincide, the limit of local minima is again a local solution. But we have seen in Example 4.10 that this is not always the case. We already know by the convergence result about stationary points (Theorem 4.8) that $\boldsymbol{x}^*$ is a stationary point of the original problem SIP. The question that arises now is: under which additional conditions one can be sure that a found point is indeed a local solution and not only a stationary point.

A possibility is to check for the second-order sufficient condition. We therefore assume for the remainder of this section that the objective function f is twice continuously differentiable and that there is for every $i \in I$ a unique active index:

$$Y_0^i(\boldsymbol{x}^*) = \{\boldsymbol{y}^{i,*}\} \ .$$

By the Reduction Ansatz there is an $\varepsilon > 0$ and, for every $i \in I$, a continuously differentiable function

$$\boldsymbol{y}^i : B_\varepsilon(\boldsymbol{x}^*) \to Y \ ,$$

which locally describes the unique solution of the i-th lower-level problem. By Theorem 2.21 a sufficient condition for $\boldsymbol{x}^*$ to be a local solution of $\overline{\mathsf{SIP}}$, is given by the existence of Lagrange multipliers $\boldsymbol{\lambda} \geq 0$ and $\overline{\boldsymbol{\lambda}} \geq 0$ such that

$$Df(\boldsymbol{x}^*) + \sum_{i\in I} \lambda_i D_{\boldsymbol{x}} \Big[g_i\big(\boldsymbol{x}, \boldsymbol{y}^i(\boldsymbol{x})\big) \Big]_{\boldsymbol{x}=\boldsymbol{x}^*} + \sum_{i\in I} \overline{\lambda}_i D\overline{g}_i(\boldsymbol{x}^*) = 0$$

and for all $\boldsymbol{d} \in \overline{T}(\boldsymbol{x}^*), \boldsymbol{d} \neq 0$:

$$\boldsymbol{d}^\top \cdot \left(D^2 f(\boldsymbol{x}^*) + \sum_{i\in I} \lambda_i D^2_{\boldsymbol{x}} \Big[g_i\big(\boldsymbol{x}, \boldsymbol{y}^i(\boldsymbol{x})\big) \Big]_{\boldsymbol{x}=\boldsymbol{x}^*} + \sum_{i\in I} \overline{\lambda}_i D^2\overline{g}_i(\boldsymbol{x}^*) \right) \cdot \boldsymbol{d} > 0 \ ,$$

where

$$\overline{T}(\boldsymbol{x}^*) := \left\{ \boldsymbol{d} \in \mathbb{R}^n \;\middle|\; \begin{array}{r} D_{\boldsymbol{x}} \Big[g_i\big(\boldsymbol{x}, \boldsymbol{y}^i(\boldsymbol{x})\big) \Big]_{\boldsymbol{x}=\boldsymbol{x}^*} \boldsymbol{d} \leq 0 \text{ for all } i \in I, \\ D_{\boldsymbol{x}} \Big[g_i\big(\boldsymbol{x}, \boldsymbol{y}^i(\boldsymbol{x})\big) \Big]_{\boldsymbol{x}=\boldsymbol{x}^*} \boldsymbol{d} = 0 \text{ for all } i \in I, \text{ with } \lambda_i > 0, \\ D\overline{g}_i(\boldsymbol{x}^*)\boldsymbol{d} \leq 0 \text{ for all } i \in I, \\ D\overline{g}_i(\boldsymbol{x}^*)\boldsymbol{d} = 0 \text{ for all } i \in I, \text{ with } \overline{\lambda}_i > 0 \end{array} \right\} \ .$$

By Lemma 4.1 we know that, for every $i \in I$:

$$D_{\boldsymbol{x}}\Big[g_i\big(\boldsymbol{x}, \boldsymbol{y}^i(\boldsymbol{x})\big)\Big]_{\boldsymbol{x}=\boldsymbol{x}^*} = D\overline{g}_i(\boldsymbol{x}^*) \ ,$$
$$D^2_{\boldsymbol{x}}\Big[g_i\big(\boldsymbol{x}, \boldsymbol{y}^i(\boldsymbol{x})\big)\Big]_{\boldsymbol{x}=\boldsymbol{x}^*} = D^2\overline{g}_i(\boldsymbol{x}^*) \ .$$

This means that the sufficient conditions for the problems $\overline{\mathsf{SIP}}$ and SIP coincide. As a consequence if the limit point satisfies the second-order sufficient condition for $\overline{\mathsf{SIP}}$, we can be sure that it is also a local minimum of the original problem SIP.

In the remainder of this section we give conditions on the sequence $\{\boldsymbol{x}^k\}_{k\in\mathbb{N}}$, which ensure that the limit point satisfies the sufficient condition ESOSC. We assume that the limit point satisfies the regularity condition EMFCQ. From Lemma 4.5 it then follows that MFCQ is satisfied, for sufficiently large k, for $\boldsymbol{x}^k$. Thus, the iterate is a stationary point and there are Lagrange multipliers $\boldsymbol{\lambda}^k \geq 0$ and $\overline{\boldsymbol{\lambda}}^k \geq 0$ such that:

$$Df(\boldsymbol{x}^k) + \sum_{i\in I}\sum_{\boldsymbol{y}\in Y_0^{i,k}} \lambda^k_{i,\boldsymbol{y}} Dg_i(\boldsymbol{x}^k, \boldsymbol{y}) + \sum_{i\in I} \overline{\lambda}^k_i D\overline{g}^{k-1}_i(\boldsymbol{x}^k) = 0 \ ,$$

where $\overline{\lambda}^k_i = 0$ if $\overline{g}^{k-1}_i(\boldsymbol{x}^k) < 0$. As there can be active constraints enforced by the discretization the cone of critical directions $T(\boldsymbol{x}^k)$ for problem $\overline{\mathsf{SIP}}^{k-1}(Y^k)$ can be small and a limit condition is harder to formulate based on its sufficient conditions. This is why we consider a modified condition based only on the additional constraints. Therefore let, for sufficiently large k and $i \in I$:

$$\hat{\lambda}^k_i := \overline{\lambda}^k_i + \sum_{\boldsymbol{y}\in Y_0^i(\boldsymbol{x}^k)} \lambda^k_{i,\boldsymbol{y}}$$
$$T^k := \{\boldsymbol{d} \in \mathbb{R}^n \mid D\overline{g}^{k-1}_i(\boldsymbol{x}^k)\boldsymbol{d} = 0 \text{ for all } i \in I \text{ with } \hat{\lambda}^k > 0\}$$

and

$$\kappa^k := \min_{\boldsymbol{d}\in T^k, \|\boldsymbol{d}\|=1} \boldsymbol{d}^\top \left(D^2 f(\boldsymbol{x}) + \sum_{i\in I} \hat{\lambda}^k D^2 \overline{g}^{k-1}_i(\boldsymbol{x}^k) \right) \boldsymbol{d} \ .$$

We are now ready to prove the following theorem about the convergence of local solutions.

Theorem 4.12. *Let the regularity Assumption 4.4 be satisfied. Assume that, for every $k \in \mathbb{N}$, the solution $\boldsymbol{x}^k$ is a local minimum of $\overline{\mathsf{SIP}}^{k-1}(Y^k)$ and, for every $i \in I$, the following holds:*

$$\lambda^*_i := \liminf_{k\to\infty} \hat{\lambda}^k_i > 0 \ ,$$
$$\kappa := \liminf_{k\to\infty} \kappa^k > 0 \ .$$

Assume that the Reduction Ansatz and ELICQ *are satisfied at* $\boldsymbol{x}^*$ *for* SIP. *Further assume that at* $\boldsymbol{x}^*$ *the solution of the lower-level problem is unique, for every* $i \in I$. *Then* $\boldsymbol{x}^*$ *is local minimum of* SIP.

The first limit ensures strict complementary slackness. The second one makes sure that the curvature does not vanish.

Proof. By the proof of Theorem 4.8 we know that:

$$Df(\boldsymbol{x}^*) + \sum_{i \in I} \lambda_i^* D_1 g_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,*}) = 0 \ .$$

This means that $\boldsymbol{x}^*$ is a stationary point. It remains to show that the second-order sufficient condition is satisfied.

Choose a fixed $\boldsymbol{d} \in \overline{T}(\boldsymbol{x}^*)$ with $\|\boldsymbol{d}\| = 1$. As, for every $i \in I$, we have $\lambda_i^* > 0$, the following holds:

$$D_1 g_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,*}) \cdot \boldsymbol{d} = 0 \ .$$

By Lemma 4.6 the following is also true:

$$\lim_{k \to \infty} \boldsymbol{y}^{i,k} = \boldsymbol{y}^{i,*} \ .$$

As the derivatives of the constraints are by assumption linearly independent in the limit we can construct for sufficiently large k vectors $\boldsymbol{d}^k$ with $\|\boldsymbol{d}^k\| = 1$ such that, for every $i \in I$:

$$D_1 g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) \cdot \boldsymbol{d}^k = 0 \tag{4.12}$$

and

$$\lim_{k \to \infty} \boldsymbol{d}^k = \boldsymbol{d} \ .$$

By Equation (4.12) we have $\boldsymbol{d}^k \in T^k$ and the following holds by definition:

$$\left(\boldsymbol{d}^k\right)^\top \left(f(\boldsymbol{x}^k) + \sum_{i \in I} \hat{\boldsymbol{\lambda}}^k D^2 \overline{g}_i^{k-1}(\boldsymbol{x}^k) \right) \boldsymbol{d}^k \geq \kappa_k \ .$$

Taking the limit inferior shows:

$$\boldsymbol{d}^\top \left(D^2 f(\boldsymbol{x}^*) + \sum_{i \in I} \lambda_i^* D^2 \overline{g}_i(\boldsymbol{x}^*) \right) \boldsymbol{d} \geq \kappa \ .$$

Which shows that the second-order sufficient condition is satisfied for $\overline{\text{SIP}}$. We have seen that this is equivalent to the second-order sufficient condition for SIP. □

The benefit of this theorem is that we can check the introduced second-order condition at an iterate. The iterates are the only thing we have at hand as we need to terminate the algorithm after finitely many steps.

In this Section we have shown statements about the properties of a limit point. We have seen that under mild conditions every accumulation point of stationary points is again a stationary point and an accumulation point of local solutions is again a local solution. We now turn to the main reason why we have introduced the new algorithm, namely the quadratic rate of convergence.

4.2 Quadratic rate of convergence

We have already shown in Section 4.1.1 that the limit of stationary points is again a stationary point. We therefore assume this in the following and show that the iterates converge with a quadratic rate. The main statement of this section is the Quadratic Convergence Theorem. We begin by strengthening our previous assumptions. To prove the quadratic rate of convergence, we assume the following:

Assumption 4.13. *(General regularity assumptions for quadratic convergence) Let the describing functions $\boldsymbol{g}$ and $\boldsymbol{v}$ be three times continuously differentiable and the objective f be twice continuously differentiable. Let $\{\boldsymbol{x}^k\}_{k\in\mathbb{N}}$ be constructed according to Algorithm 2. Assume that, for every $k \in \mathbb{N}$, the point $\boldsymbol{x}^k$ is a stationary-point of $\overline{\mathsf{SIP}}^{k-1}(Y^k)$. Assume there is a strongly stable stationary point $\boldsymbol{x}^* \in M$ with*

$$\lim_{k\to\infty} \boldsymbol{x}^k = \boldsymbol{x}^* \ .$$

Moreover assume that $\boldsymbol{x}^*$ has the following properties:

- *The constraint qualification* EMFCQ *is satisfied.*
- *The Reduction Ansatz holds at $\boldsymbol{x}^*$.*
- *For every $i \in I$, the global solution $\boldsymbol{y}^{i,*}$ of the lower-level problem $Q_i(\boldsymbol{x}^*)$ is unique and*

$$g_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,*}) = 0 \ . \tag{4.13}$$

We finally assume that $\bar{I}^k = I$, for every $k \in \mathbb{N}$.

The last assumption and Equation (4.13) are, similar to Assumption 4.4, only added to avoid case distinctions in the proof. We will comment on the uniqueness

of the lower-level solutions after the Quadratic Convergence Theorem and point out a possible extension to the case of multiple lower-level solutions. All remaining assumptions are basic regularity assumptions.

Before starting to prove the quadratic rate of convergence, we summarize the further properties induced by Assumption 4.13. Note that by the differentiability assumptions, the Reduction Ansatz yields a $\varepsilon > 0$ and, for every $i \in I$, the twice continuously differentiable functions:

$$\begin{aligned} \boldsymbol{y}^i &: B_\varepsilon(\boldsymbol{x}^*) \to Y \ , \\ \boldsymbol{\mu}^i &: B_\delta(\boldsymbol{x}^*) \to \mathbb{R}^{|J|} \end{aligned}$$

such that for every $\boldsymbol{x} \in B_\varepsilon(\boldsymbol{x}^*)$ the point $\boldsymbol{y}^i(\boldsymbol{x})$ is the unique global solution of $Q_i(\boldsymbol{x})$ and $\boldsymbol{\mu}^i(\boldsymbol{x})$ are the unique corresponding Lagrange multipliers satisfying the KKT conditions. Thus, for sufficiently large k, we have:

$$\begin{aligned} \boldsymbol{y}^{i,k} &= \boldsymbol{y}^i(\boldsymbol{x}^k) \ , \\ \boldsymbol{\mu}^{i,k} &= \boldsymbol{\mu}^i(\boldsymbol{x}^k) \ . \end{aligned}$$

The following lemma follows easily by continuity and the proof is completely analogous to the proof of Lemma 3.12.

Lemma 4.14. *Let Assumption 4.13 hold. Then, for every $i \in I$, also the following is true:*

$$\begin{aligned} \lim_{k\to\infty} \boldsymbol{y}^{i,k} &= \boldsymbol{y}^{i,*} \ , \\ \lim_{k\to\infty} \boldsymbol{\mu}^{i,k} &= \boldsymbol{\mu}^{i,*} \ . \end{aligned}$$

Moreover, for every $i \in I$, the set of active indices in the lower-level does not change for sufficiently large k:

$$J_0(\boldsymbol{y}^{i,k}) = J_0(\boldsymbol{y}^{i,*}) \ .$$

To prove the quadratic rate of convergence, we have to show that there exists a constant $L > 0$ such that, for sufficiently large k, we have

$$\|\boldsymbol{x}^{k+1} - \boldsymbol{x}^*\| \leq L\|\boldsymbol{x}^k - \boldsymbol{x}^*\|^2 \ .$$

The basic line of the proof is to first find a perturbed semi-infinite problem. We do this in such a way that the current iterate $\boldsymbol{x}^k$ is a stationary point of this problem. As a next step we have to bound the perturbation. After we have achieved bounds, we can use the strong stability of a limit point to obtain the quadratic rate of convergence. We begin by introducing the considered perturbation.

For $i \in I$, we denote again the set of active discretization points in iteration k by

$$Y_0^{i,k} := \{\boldsymbol{y} \in Y^k \mid g_i(\boldsymbol{x}^k, \boldsymbol{y}) = 0\}$$

As $\boldsymbol{x}^k$ is a stationary point, we know that there are multipliers $\boldsymbol{\lambda}^k \geq 0$ and $\overline{\boldsymbol{\lambda}}^k \geq 0$ such that

$$0 = Df(\boldsymbol{x}^k) + \sum_{i \in I} \sum_{\boldsymbol{y} \in Y_0^{i,k}} \lambda_{i,\boldsymbol{y}}^k D_1 g_i(\boldsymbol{x}^k, \boldsymbol{y}) + \sum_{i \in I} \overline{\lambda}_i^k D\overline{g}_i^{k-1}(\boldsymbol{x}^k) \ . \tag{4.14}$$

The main idea of the proof is to replace, for $i \in I$, every derivative of the constraints by $D_1 g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k})$. Equation (4.14) will not hold with an equality anymore, but we will introduce a $\boldsymbol{\beta}^k \in \mathbb{R}^n$ such that

$$Df(\boldsymbol{x}^k) + \sum_{i \in I} \hat{\lambda}_i^k D_1 g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) = \boldsymbol{\beta}^k \ ,$$

where, for $i \in I$, we let $\hat{\lambda}_i^k := \overline{\lambda}_i^k + \sum_{\boldsymbol{y} \in Y_0^{i,k}} \lambda_{i,\boldsymbol{y}}^k$. We will also introduce for every $i \in I$ an $\alpha_i \in \mathbb{R}$ such that:

$$\begin{aligned} g_i(\boldsymbol{x}^k, \boldsymbol{y}) &\leq \alpha_i \text{ for all } \boldsymbol{y} \in Y \ , \\ g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) &= \alpha_i \ . \end{aligned}$$

This means that $\boldsymbol{x}^k$ is a stationary point of a perturbed semi-infinite optimization problem. The perturbation can be controlled by $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. To obtain a quadratic rate of convergence, we need to bound these parameters. Therefore, we first investigate in the next two lemmas (Lemmas 4.15 and 4.16) the difference of the additional constraint $\overline{g}_i^{k-1}(\boldsymbol{x}^k)$ to the constraint $g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k})$. The following two lemmas (Lemmas 4.17 and 4.18) will then investigate the active discretization points. We start by bounding the distance of the lower-level solution added in the next iteration to our linear approximation.

Lemma 4.15. *Let Assumptions 4.13 hold. There exists a $K_1 \in \mathbb{R}$ such that, for $i \in I$ and sufficiently large k, the following holds:*

$$\begin{aligned} \|\boldsymbol{y}^{i,k} - \overline{\boldsymbol{y}}^{i,k-1}(\boldsymbol{x}^k)\| &\leq K_1 \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^2 \ , \\ \|\boldsymbol{\mu}^{i,k} - \overline{\boldsymbol{\mu}}^{i,k-1}(\boldsymbol{x}^k)\| &\leq K_1 \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^2 \ . \end{aligned}$$

Proof. Consider throughout the proof a fixed index $i \in I$. The function $\boldsymbol{y}^i(\boldsymbol{x})$ describing the lower-level solution is twice continuously differentiable. Thus, by a Taylor expansion of $\boldsymbol{y}^i(\boldsymbol{x}^k)$ around $\boldsymbol{x}^{k-1}$, there is an $\hat{\boldsymbol{x}} = t \cdot \boldsymbol{x}^k + (1-t) \cdot \boldsymbol{x}^{k-1}$, for an appropriate $t \in [0,1]$, such that

$$\begin{aligned} \|\boldsymbol{y}^{i,k} - \overline{\boldsymbol{y}}^{i,k-1}(\boldsymbol{x}^k)\| &= \|\boldsymbol{y}^i(\boldsymbol{x}^k) - \boldsymbol{y}^i(\boldsymbol{x}^{k-1}) - D\boldsymbol{y}^i(\boldsymbol{x}^{k-1})(\boldsymbol{x}^k - \boldsymbol{x}^{k-1})\| \\ &= \frac{1}{2} \|(\boldsymbol{x}^k - \boldsymbol{x}^{k-1})^\top \cdot D^2 \boldsymbol{y}(\hat{\boldsymbol{x}}) \cdot (\boldsymbol{x}^k - \boldsymbol{x}^{k-1})\| \ . \end{aligned}$$

The statement now follows easily by the continuity of $D^2\boldsymbol{y}$. The proof for the Lagrange multipliers $\boldsymbol{\mu}^{i,k}$ is completely analogous. □

We can now bound the next values and derivatives of the semi-infinite constraints.

Lemma 4.16. *Let Assumption 4.13 hold.*

i) There is a $K_2 \in \mathbb{R}$ such that, for sufficiently large k and $i \in I$, the following holds:

$$|g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) - \overline{g}_i^{k-1}(\boldsymbol{x}^k)| \leq K_2 \cdot \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^4 \ .$$

ii) There is a $K_3 \in \mathbb{R}$ such that, for sufficiently large k and $i \in I$, the following holds:

$$\|D_1 g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) - D\overline{g}_i^{k-1}(\boldsymbol{x}^k)\| \leq K_3 \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^2 \ .$$

Proof. Consider throughout the proof a fixed $i \in I$.

i) Note that, for every $k \in \mathbb{N}$, we have

$$g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) = \mathcal{L}_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}, \boldsymbol{\mu}^{i,k}) \ ,$$

by the complementarity conditions. By definition the following is also true:

$$\overline{g}_i^{k-1}(\boldsymbol{x}^k) = \mathcal{L}_i\big(\boldsymbol{x}^k, \overline{\boldsymbol{y}}^{i,k-1}(\boldsymbol{x}^k), \overline{\boldsymbol{\mu}}^{i,k-1}(\boldsymbol{x}^k)\big) \ .$$

By the KKT conditions for the lower-level problem we have

$$D_2\mathcal{L}_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}, \boldsymbol{\mu}^{i,k}) \cdot (\overline{\boldsymbol{y}}^{i,k-1}(\boldsymbol{x}^k) - \boldsymbol{y}^{i,k}) = 0 \ .$$

As by Lemma 4.14 the active indices do not change, for sufficiently large k, we also have:

$$D_3\mathcal{L}_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}, \boldsymbol{\mu}^{i,k}) \cdot (\overline{\boldsymbol{\mu}}^{i,k-1}(\boldsymbol{x}^k) - \boldsymbol{\mu}^{i,k}) = 0 \ .$$

This means that, for every $i \in I$ and sufficiently large k, by a Taylor expansion in $(\boldsymbol{y}^{i,k}, \boldsymbol{\mu}^{i,k})$, there are

$$\begin{aligned} \hat{\boldsymbol{y}} &= t \cdot \overline{\boldsymbol{y}}^{i,k-1}(\boldsymbol{x}^k) + (1-t)\boldsymbol{y}^{i,k} \ , \\ \hat{\boldsymbol{\mu}} &= t \cdot \overline{\boldsymbol{\mu}}^{i,k-1}(\boldsymbol{x}^k) + (1-t)\boldsymbol{\mu}^{i,k} \ , \end{aligned}$$

for an appropriate $t \in [0,1]$, such that:

$$\begin{aligned}
&\|g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) - \overline{g}_i^{k-1}(\boldsymbol{x}^k)\| \\
&= \|\mathcal{L}_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}, \boldsymbol{\mu}^{i,k}) - \mathcal{L}_i(\boldsymbol{x}^k, \overline{\boldsymbol{y}}^{i,k-1}(\boldsymbol{x}^k), \overline{\boldsymbol{\mu}}^{i,k-1}(\boldsymbol{x}^k))\| \\
&= \frac{1}{2}\| \begin{pmatrix} \overline{\boldsymbol{y}}^{i,k-1}(\boldsymbol{x}^k) - \boldsymbol{y}^{i,k} \\ \overline{\boldsymbol{\mu}}^{i,k-1}(\boldsymbol{x}^k) - \boldsymbol{\mu}^{i,k} \end{pmatrix}^\top \cdot D^2_{2,3}\mathcal{L}_i(\boldsymbol{x}^k, \hat{\boldsymbol{y}}, \hat{\boldsymbol{\mu}}) \cdot \begin{pmatrix} \overline{\boldsymbol{y}}^{i,k-1}(\boldsymbol{x}^k) - \boldsymbol{y}^{i,k} \\ \overline{\boldsymbol{\mu}}^{i,k-1}(\boldsymbol{x}^k) - \boldsymbol{\mu}^{i,k} \end{pmatrix} \|
\end{aligned}$$

By continuity there is a $K \in \mathbb{R}$ which bounds the second derivative and we have together with Lemma 4.15, for sufficiently large k:

$$\begin{aligned}
&\|g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) - \overline{g}_i^{k-1}(\boldsymbol{x}^k)\| \\
&\leq \frac{1}{2}K \cdot \left(\|\overline{\boldsymbol{y}}^{i,k-1}(\boldsymbol{x}^k) - \boldsymbol{y}^{i,k}\|^2 + \|\overline{\boldsymbol{\mu}}^{i,k-1}(\boldsymbol{x}^k) - \boldsymbol{\mu}^{i,k}\|^2\right) \\
&\leq K \cdot K_1 \cdot \|\boldsymbol{x}^{k+1} - \boldsymbol{x}^k\|^4 \ .
\end{aligned}$$

ii) We first bound each term and then combine these terms.

Analogous to part i) we have, for sufficiently large k:

$$\begin{aligned}
D_2\mathcal{L}_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}, \boldsymbol{\mu}^{i,k}) D\boldsymbol{y}^i(\boldsymbol{x}^{k-1}) = 0 \ , \\
D_3\mathcal{L}_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}, \boldsymbol{\mu}^{i,k}) D\boldsymbol{\mu}^i(\boldsymbol{x}^{k-1}) = 0 \ .
\end{aligned}$$

As all involved functions are at least twice continuously differentiable, there is an L_1 with

$$\begin{aligned}
&\|D_{2,3}\mathcal{L}_i(\boldsymbol{x}^k, \overline{\boldsymbol{y}}^{i,k-1}(\boldsymbol{x}^k), \overline{\boldsymbol{\mu}}^{i,k-1}(\boldsymbol{x}^k)) \cdot \begin{pmatrix} D\boldsymbol{y}^i(\boldsymbol{x}^{k-1}) \\ D\boldsymbol{\mu}^i(\boldsymbol{x}^{k-1}) \end{pmatrix} \| \\
&\leq L_1(\|\overline{\boldsymbol{y}}^{i,k-1}(\boldsymbol{x}^k) - \boldsymbol{y}^{i,k}\| + \|\overline{\boldsymbol{\mu}}^{i,k-1}(\boldsymbol{x}^k) - \boldsymbol{\mu}^{i,k}\|) \\
&\leq L_1 \cdot 2 \cdot K_1 \cdot \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^2 \ ,
\end{aligned}$$

where we used again Lemma 4.15 in the last inequality. Analogously there is also an $L_2 > 0$ such that, for sufficiently large k:

$$\begin{aligned}
&\|D_1 g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) - D_1\mathcal{L}_i(\boldsymbol{x}^k, \overline{\boldsymbol{y}}^{i,k-1}(\boldsymbol{x}^k), \overline{\boldsymbol{\mu}}^{i,k-1}(\boldsymbol{x}^k))\| \\
=&\|D_1 g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) - D_1 g_i(\boldsymbol{x}^k, \overline{\boldsymbol{y}}^{i,k-1}(\boldsymbol{x}^k))\| \\
\leq& L_2 \cdot K_1 \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^2
\end{aligned}$$

We can now bound the derivative with the help of the chain-rule and the above inequalities, for sufficiently large k:

$$\begin{aligned}
&\|D_1 g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) - D\overline{g}_i^{k-1}(\boldsymbol{x}^k)\| \\
=&\|D_1 g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) - D_1\mathcal{L}_i(\boldsymbol{x}^k, \overline{\boldsymbol{y}}^{i,k-1}(\boldsymbol{x}^k), \overline{\boldsymbol{\mu}}^{i,k-1}(\boldsymbol{x}^k)) \\
&- D_{2,3}\mathcal{L}_i(\boldsymbol{x}^k, \overline{\boldsymbol{y}}^{i,k-1}(\boldsymbol{x}^k), \overline{\boldsymbol{\mu}}^{i,k-1}(\boldsymbol{x}^k)) \cdot \begin{pmatrix} D\boldsymbol{y}^i(\boldsymbol{x}^{k-1}) \\ D\boldsymbol{\mu}^i(\boldsymbol{x}^{k-1}) \end{pmatrix} \| \\
\leq& L_1 \cdot 2 \cdot K_1 \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^2 + L_2 \cdot K_1 \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^2 \ .
\end{aligned}$$

Letting $K_3 = (2L_1 + L_2)K_1$ the claim follows.

□

If the only active constraints in the stationarity conditions (4.14) are the additional constraints $\overline{g}_i^{k-1}(\boldsymbol{x}^k)$, then the inequalities derived in the above lemma would already suffice to bound the parameter $\boldsymbol{\beta}^k$. But the points in Y^k are needed to guarantee convergence to a feasible point and we cannot exclude the possibility that some of the constraints

$$g_i(\boldsymbol{x}^k, \boldsymbol{y}) \leq 0 \text{ for all } i \in I, \boldsymbol{y} \in Y^k$$

are active. We need to investigate these constraints further in the next two lemmas. The goal is it to show in Lemma 4.18 that, for every $i \in I$, the distance of any active index in Y^k to the solution $\boldsymbol{y}^{i,k}$ can be bounded quadratically.

The next lemma gives a lower growth condition. It is important that the constant given in this lemma is independent of the current iterate.

Lemma 4.17. *Let Assumption 4.13 hold. There is a constant $K_4 > 0$, a $k' \in \mathbb{N}$ and an $\varepsilon > 0$ such that, for every $i \in I$, $k \geq k'$ and $\boldsymbol{y} \in Y$ with $\|\boldsymbol{y} - \boldsymbol{y}^{i,*}\| < \varepsilon$, the following holds:*

$$g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) - g_i(\boldsymbol{x}^k, \boldsymbol{y}) \geq K_4 \|\boldsymbol{y}^{i,k} - \boldsymbol{y}\|^2 \ .$$

Proof. As we assumed the Reduction Ansatz to hold in $\boldsymbol{x}^*$, the global solution $\boldsymbol{y}^{i,*}$ is strongly stable. By exactly this property there is an $\varepsilon > 0$, an $L' > 0$ and a $k' \in \mathbb{N}$ such that, for every $0 < L \leq L'$ and $k \geq k'$, there exists a unique stationary point of

$$\max_{\boldsymbol{y} \in Y} g_i(\boldsymbol{x}^k, \boldsymbol{y}) + L\|\boldsymbol{y}^{i,k} - \boldsymbol{y}\|^2 \tag{4.15}$$

in $B_\varepsilon(\boldsymbol{y}^{i,*})$. For every $k \in \mathbb{N}$, the KKT conditions are not affected at the point $\boldsymbol{y}^{i,k}$ by the added term $L\|\boldsymbol{y} - \boldsymbol{y}^{i,k}\|^2$. This means that this point is still a stationary point. We need to show that it is still a local maximum. By compactness the maximum

$$\max_{\substack{\boldsymbol{y} \in Y \\ \|\boldsymbol{y} - \boldsymbol{y}^{i,*}\| = \varepsilon}} \quad g_i(\boldsymbol{x}^*, \boldsymbol{y}) - g_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,*})$$

is attained and is strictly less than 0. We can therefore choose a k'' and a L'' such that, for every $k \geq k''$, $L \leq L''$ and $\boldsymbol{y} \in Y$ with $\|\boldsymbol{y} - \boldsymbol{y}^{i,*}\| = \varepsilon$, the following holds:

$$g_i(\boldsymbol{x}^k, \boldsymbol{y}) + L\|\boldsymbol{y}^{i,k} - \boldsymbol{y}\|^2 < g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) \ .$$

As, for sufficiently large k, the iterate of the lower-level, $\boldsymbol{y}^{i,k}$, lies within $B_\varepsilon(\boldsymbol{y}^{i,*})$ there must be a local minimum within $B_\varepsilon(\boldsymbol{y}^{i,*}) \cap Y$ for the problem given in Equation

(4.15). As LICQ holds everywhere and by uniqueness of the stationary point, this point must be, for sufficiently large k, the stationary point $\boldsymbol{y}^{i,k}$.

For every $\boldsymbol{y} \in Y \cap B_\varepsilon(\boldsymbol{y}^{i,*})$ we then have for $K_4 := \min\{L', L''\}$:

$$g_i(\boldsymbol{x}^k, \boldsymbol{y}) + K_4 \|\boldsymbol{y}^{i,k} - \boldsymbol{y}\|^2 \leq g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k})$$

□

We can now bound the difference of all active constraints to the current solution of the lower-level problem:

Lemma 4.18. *Let Assumption 4.13 hold. For sufficiently large k, $i \in I$ and a $\boldsymbol{y} \in Y$ with $g_i(\boldsymbol{x}^k, \boldsymbol{y}) = 0$, the following holds:*

$$\|\boldsymbol{y}^{i,k} - \boldsymbol{y}\| \leq \sqrt{\frac{K_2}{K_4}} \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^2 \ ,$$

where K_2 and K_4 are chosen according to Lemma 4.16 and 4.17.

Proof. Consider throughout this proof a fixed $i \in I$. If $g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) < 0$, there cannot be any active $\boldsymbol{y} \in Y$, because $\boldsymbol{y}^{i,k}$ is the global maximum. For the remainder of the proof we therefore assume $g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) \geq 0$. As $\boldsymbol{x}^k$ is feasible for $\overline{\mathsf{SIP}}^{k-1}(Y^k)$, we know by Lemma 4.15, for sufficiently large k:

$$g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) \leq K_2 \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^4 \ .$$

By Lemma 4.17 there is an $\varepsilon \geq 0$ and a $k' \in \mathbb{N}$ such that, for $k > k'$ and $\boldsymbol{y} \in Y$ with $\|\boldsymbol{y} - \boldsymbol{y}^{i,*}\| < \varepsilon$:

$$g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) - g_i(\boldsymbol{x}^k, \boldsymbol{y}) \geq K_4 \|\boldsymbol{y}^{i,k} - \boldsymbol{y}\|^2 \ .$$

If we now further have $g_i(\boldsymbol{x}^k, \boldsymbol{y}) = 0$, we obtain:

$$K_2 \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^4 \geq g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) - g_i(\boldsymbol{x}^k, \boldsymbol{y}) \geq K_4 \|\boldsymbol{y}^{i,k} - \boldsymbol{y}\|^2 \ ,$$

which proves the claim for all $\boldsymbol{y} \in Y$ with $\|\boldsymbol{y} - \boldsymbol{y}^{i,*}\| < \varepsilon$.

Note that the set $Y_\varepsilon^i := \{\boldsymbol{y} \in Y \mid \|\boldsymbol{y} - \boldsymbol{y}^{i,*}\| \geq \varepsilon\}$ is compact. Thus the maximum $\max_{\boldsymbol{y} \in Y_\varepsilon^i} g(\boldsymbol{x}^*, \boldsymbol{y})$ is attained and is strictly less then 0. By continuity we have, for sufficiently large k and for all $\boldsymbol{y} \in Y_\varepsilon^i$:

$$g_i(\boldsymbol{x}^k, \boldsymbol{y}) < 0 \ .$$

□

As a direct consequence of this lemma, there is a constant K_5 such that, for sufficiently large k and $\boldsymbol{y} \in Y_0^{i,k}$, the following holds:

$$\|D_1 g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) - g_i(\boldsymbol{x}^k, \boldsymbol{y})\| \leq K_5 \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^2 \ . \tag{4.16}$$

We have now obtained all necessary bounds needed to prove the quadratic rate of convergence. As already mentioned in the beginning of this section, for every sufficiently large k, there are Lagrange-multipliers $\boldsymbol{\lambda}^k \geq 0$ and $\overline{\boldsymbol{\lambda}}^k \geq 0$ such that Equation (4.14) holds. Again, we denote the sum of Lagrange multipliers, for $i \in I$, by $\hat{\lambda}_i^k = \overline{\lambda}_i^k + \sum_{\boldsymbol{y} \in Y_0^{i,k}} \lambda_{i,\boldsymbol{y}}^k$.

Let for $k \in \mathbb{N}$ and $i \in I$:

$$\boldsymbol{\beta}^k := Df(\boldsymbol{x}^k) + \sum_{i \in I} \hat{\lambda}_i^k D_1 g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) \ , \tag{4.17}$$

$$\alpha_i^k := \begin{cases} \max\{0, g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k})\} & \text{if } \hat{\lambda}_i^k = 0 \ , \\ g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) & \text{otherwise} \ . \end{cases} \tag{4.18}$$

We are now ready to prove the quadratic rate of convergence.

Quadratic Convergence Theorem. *Let the general regularity assumptions (Assumption 4.13) be satisfied. There is a constant L such that, for sufficiently large k, the following holds:*

$$\|\boldsymbol{x}^{k+1} - \boldsymbol{x}^*\| \leq L \|\boldsymbol{x}^k - \boldsymbol{x}^*\|^2 \ .$$

Proof. We first consider the case where ELICQ is satisfied. In a second proof we show the rate of convergence if ELICQ is not satisfied but EMFCQ is satisfied.

First case: ELICQ holds at $\boldsymbol{x}^*$:
By Equation (4.17) and (4.18) the next iterate, $\boldsymbol{x}^{k+1}$, is a stationary point for the modified SIP:

$$\begin{aligned} \mathsf{SIP}_{\text{mod}}^k : \min_{\boldsymbol{x} \in \mathbb{R}^n} \quad & f(\boldsymbol{x}) - \boldsymbol{\beta}^{k+1} \cdot \boldsymbol{x} \\ s.t. \quad & g_i(\boldsymbol{x}, \boldsymbol{y}) - \alpha_i^{k+1} \leq 0 \text{ for all } i \in I, \boldsymbol{y} \in Y \end{aligned}$$

By the strong stability of $\boldsymbol{x}^*$, we know by Theorem 2.27 that there is a constant L' such that, for sufficiently large k:

$$\|\boldsymbol{x}^{k+1} - \boldsymbol{x}^*\| \leq L' \cdot \left(\|\boldsymbol{\alpha}^{k+1}\| + \|\boldsymbol{\beta}^{k+1}\| \right) \ . \tag{4.19}$$

We now need to bound the two parameters $\boldsymbol{\alpha}^k$ and $\boldsymbol{\beta}^k$. By Equation (4.17) and the definition of $\hat{\lambda}_i^k$:

$$\begin{aligned}\|\boldsymbol{\beta}^k\| &= \|Df(\boldsymbol{x}^k) + \sum_{i\in I} \hat{\lambda}_i^k D_1 g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k})\| \\ &= \|\sum_{i\in I} \overline{\lambda}_i^k \big(D_1 g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) - D\overline{g}_i^{k-1}(\boldsymbol{x}^k)\big) \\ &\quad + \sum_{i\in I} \sum_{\boldsymbol{y}\in Y_0^{i,k}} \lambda_{\boldsymbol{y}}^{i,k} \big(D_1 g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) - D_1 g_i(\boldsymbol{x}^k, \boldsymbol{y})\big)\| \\ &=: (A) \ .\end{aligned}$$

By Lemma 4.16 and the inequality given in Equation (4.16), the following holds :

$$\begin{aligned}(A) &\leq \left(\sum_{i\in I} \overline{\lambda}_i^k + \sum_{i\in I} \sum_{\boldsymbol{y}\in Y_0^{i,k}} \lambda_{\boldsymbol{y}}^{i,k}\right) \max\{K_3, K_5\} \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^2 \\ &=: (B) \ .\end{aligned}$$

By Lemma 4.7 the Lagrange multipliers are bounded and there is a $L_1 > 0$ such that

$$(B) \leq L_1 \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^2 \ . \tag{4.20}$$

To bound the absolute value of α_i^k, for $i \in I$, we have to make a case distinction.

First case: assume that $g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) \geq 0$. As $\overline{g}_i^k(\boldsymbol{x}^{k+1}) \leq 0$, we have by Lemma 4.16:

$$\alpha_i^k \leq K_2 \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^4 \ . \tag{4.21}$$

Second case: assume that $g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) < 0$. If $\hat{\lambda}_i^k = 0$, then it holds $\alpha_i^k = 0$, by definition. If $\hat{\lambda}_i^k > 0$, then we must have $\overline{\lambda}_i^k > 0$, as no $\boldsymbol{y} \in Y^k$ can be active. We have by the complementarity conditions:

$$\overline{g}_i^{k-1}(\boldsymbol{x}^k) = 0 \ .$$

Again, Lemma 4.16 shows the same inequality as in Equation (4.21). In summary, there is an $L_2 > 0$ with:

$$\alpha_i^k \leq L_2 \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^2 \ . \tag{4.22}$$

Using the inequality given in Equation (4.19) in combination with the inequalities given in Equations (4.20) and (4.22), we receive further, for sufficiently large k:

$$\begin{aligned}\|\boldsymbol{x}^{k+1} - \boldsymbol{x}^*\| \leq & L'(L_1 + L_2)\|\boldsymbol{x}^{k+1} - \boldsymbol{x}^k\|^2 \\ \leq & L'(L_1 + L_2)\Big(\|\boldsymbol{x}^{k+1} - \boldsymbol{x}^*\|^2 + \|\boldsymbol{x}^{k+1} - \boldsymbol{x}^*\| \cdot \|\boldsymbol{x}^k - \boldsymbol{x}^*\| + \|\boldsymbol{x}^k - \boldsymbol{x}^*\|^2\Big) \\ \leq & \frac{1}{2}\|\boldsymbol{x}^{k+1} - \boldsymbol{x}^*\| + L'(L_1 + L_2)\|\boldsymbol{x}^k - \boldsymbol{x}^*\|^2 \ , \end{aligned} \tag{4.23}$$

where the last inequality holds for sufficiently large k, as the first two terms converge faster to 0 than $\|\boldsymbol{x}^{k+1} - \boldsymbol{x}^*\|$. The claim is shown for $L = 2L'(L_1 + L_2)$.

Second case: ELICQ does not hold at $\boldsymbol{x}^*$:
Completely analogous to the first case, there is a K_2 such that Equation (4.21) holds. We now first introduce some neighborhoods of $\boldsymbol{x}^*$.

(1) As ELICQ does not hold at $\boldsymbol{x}^*$, but EMFCQ does, we know from Theorem 2.24 that the strongly stable stationary point $\boldsymbol{x}^*$ satisfies the second-order sufficient condition ESOSC. By Theorem 2.21, the limit point $\boldsymbol{x}^*$ is a local minimum and there is an $\varepsilon_1 > 0$ and an $L_1 > 0$ such that, for all $\boldsymbol{x} \in M \cap B_{\varepsilon_1}(\boldsymbol{x}^*)$, the following holds:

$$f(\boldsymbol{x}) - f(\boldsymbol{x}^*) \geq L_1 \cdot \|\boldsymbol{x}^* - \boldsymbol{x}\|^2 \ . \tag{4.24}$$

(2) We denote the feasible set of problem $\mathsf{SIP}^k_{\text{mod}}$ by M^k. By the strong stability there is an $\varepsilon_2 > 0$ such that, for sufficiently large k, there is an unique stationary point of $\mathsf{SIP}^k_{\text{mod}}$ within

$$M^k \cap B_{\varepsilon_2}(\boldsymbol{x}^*) \ .$$

(3) As EMFCQ holds at $\boldsymbol{x}^*$ for SIP, there is an $\boldsymbol{\xi}$ such that, for every $i \in I$, we have:

$$D_1 g_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,*}) \leq -1 \ .$$

We can choose an $\varepsilon_3 > 0$ and a $\delta > 0$ such that, for all $\boldsymbol{x} \in B_{\varepsilon_3}(\boldsymbol{x}^*)$, $i \in I$ and $\boldsymbol{y} \in B_\delta(\boldsymbol{y}^{i,*})$, the following holds:

$$D_1 g_i(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{\xi} \leq -\frac{1}{2} \ . \tag{4.25}$$

(4) For every $i \in I$, the set $Y^{i,\delta} := Y \setminus B_\delta(\boldsymbol{y}^{i,*})$ is compact. The maximum $\max_{\boldsymbol{y} \in Y^{i,\delta}} g_i(\boldsymbol{x})$ is attained for every $i \in I$. By continuity we can choose an ε_4 such that, for every $\boldsymbol{x} \in B_{\varepsilon_4}(\boldsymbol{x}^*)$, $i \in I$ and $\boldsymbol{y} \in Y^{i,\delta}$, the following holds:

$$g_i(\boldsymbol{x}, \boldsymbol{y}) \leq 0 \ . \tag{4.26}$$

Let $\varepsilon := \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$. Denote by $\hat{\boldsymbol{x}}^k \in M^k \cap \overline{B}_{\varepsilon/2}(\boldsymbol{x}^*)$ a point with:

$$f(\hat{\boldsymbol{x}}^k) = \min_{\boldsymbol{x} \in M^k \cap \overline{B}_{\varepsilon/2}(\boldsymbol{x}^*)} f(\boldsymbol{x}) \ .$$

We will see in the following that $\boldsymbol{x}^k = \hat{\boldsymbol{x}}^k$. We first move the constructed point towards feasibility. As $\max_{i \in I} \alpha_i^k$ converges to 0, we have, for sufficiently large k:

$$2\|\boldsymbol{\xi}\| \max_{i \in I} \alpha_i^k < \varepsilon/2 \ . \tag{4.27}$$

By (4.26) and using a first-order Taylor expansion together with (4.25), the point:

$$\bar{\boldsymbol{x}}^k := \hat{\boldsymbol{x}}^k + 2\boldsymbol{\xi} \max_{i \in I} \alpha_i^k$$

is, for sufficiently large k, feasible for SIP (see for details the proof of Theorem 3.9). By continuity and the inequality given in (4.21) there is an $L_2 > 0$ such that, for sufficiently large k:

$$f(\bar{\boldsymbol{x}}^k) - f(\hat{\boldsymbol{x}}^k) \leq L_2 \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^4 \ .$$

As by construction $f(\hat{\boldsymbol{x}}^k) \leq f(\boldsymbol{x}^*)$ we obtain using (4.24), for sufficiently large k:

$$\|\hat{\boldsymbol{x}}^k - \boldsymbol{x}^*\| \leq \sqrt{\frac{L_2}{L_1}} \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^2 \ .$$

This means that, for sufficiently large k, the point $\hat{\boldsymbol{x}}^k$ is contained in $B_{\varepsilon/2}(\boldsymbol{x}^*)$ and is a local minimum of $\mathsf{SIP}^k_{\text{mod}}$. From (4.25), it follows that EMFCQ holds for $\mathsf{SIP}^k_{\text{mod}}$. This means that $\hat{\boldsymbol{x}}^k$ is a stationary point. By uniqueness we have:

$$\hat{\boldsymbol{x}}^k = \boldsymbol{x}^k \ .$$

The quadratic convergence now follows completely analogous to (4.23). □

In Remark 3.13, we commented on the assumption of exactly one active index in the case of Theorem 3.11. We pointed out that there are two ways to treat the case of multiple active indices. One of them was the quadratic convergence of subsequences. This cannot be easily applied here, as the most important bounds derived in Lemma 4.15 are not true anymore, if we have multiple active indices.

The second possibility introduced in Remark 3.13 was to alter the algorithm such that in Step 4, one calculates all local minima of the lower-level problems, rather than one global solution. This approach can be transferred here directly.

We have introduced in this chapter a new adaptive discretization algorithm. The key idea for achieving a quadratic rate of convergence is to include additional

information of the lower-level problem in the approximate problems. We did this using a linearization of the solutions of the lower-level problems. We have shown in Section 4.1 that many convergence properties of the Blankenship and Falk algorithm also hold for the new adaptive discretization method. In Section 4.2 we presented and proved the Quadratic Convergence Theorem. Under the Reduction Ansatz and some further regularity assumptions, we can assume to observe a quadratic rate of convergence. In the next Chapter we transfer the ideas and the theoretical statements, to the case of an variable index set.

5 The generalized semi-infinite case

In this chapter we discuss different possibilities how to extend the developed Algorithm 2 to generalized semi-infinite optimization problems. In the next section we present two different possibilities how the ideas can be transferred. As one possibility is directly covered by the analysis of Algorithm 2, we mainly focus on the second possibility and use the ideas to develop Algorithm 3. In Section 5.2 we show that all results presented in the previous chapter, including the Quadratic Convergence Theorem, can be transferred.

5.1 Two algorithmic variants

The difference of a GSIP compared to a SIP consists in the $\boldsymbol{x}$-dependency of the index set. As we want to use a discretization method, we assume that problem GSIP is transformable (see Definition 2.16), i.e. there is a nonempty compact set

$$Z = \{\boldsymbol{z} \in \mathbb{R}^{\tilde{m}} \mid \forall j \in J : v_j(\boldsymbol{z}) \leq 0\}$$

and a twice continuously differentiable function $\boldsymbol{t} : \mathbb{R}^n \times Z \to \mathbb{R}^m$ such that, for every $\boldsymbol{x} \in \mathbb{R}^n$:

$$\boldsymbol{t}(\boldsymbol{x}, Z) = Y(\boldsymbol{x}) \ .$$

We have seen in Section 2.2 that every transformable GSIP can be equivalently described by a problem of type SIP. We denote the by the transformation induced problem by:

$$\widetilde{\mathsf{SIP}} : \min_{\boldsymbol{x} \in \mathbb{R}^n} \quad f(\boldsymbol{x})$$
$$\tilde{g}_i(\boldsymbol{x}, \boldsymbol{z}) \leq 0 \text{ for all } i \in I, \boldsymbol{z} \in Z \ ,$$

where

$$\tilde{g}_i(\boldsymbol{x}, \boldsymbol{z}) := g_i(\boldsymbol{x}, \boldsymbol{t}(\boldsymbol{x}, \boldsymbol{z})) \ .$$

In [Sch13] it is shown how this transformation can be used to extend the adaptive discretization method by Blankenship and Falk (Algorithm 1) to solve problems of type GSIP.

To construct the next discretization points in the k-th iteration one has two possibilities. One can either solve, for $i \in I$, the original lower-level problems:

$$Q_i(\boldsymbol{x}^k) : \max_{\boldsymbol{y} \in Y(\boldsymbol{x}^k)} g_i(\boldsymbol{x}^k, \boldsymbol{y}) \ ,$$

or one can solve, for $i \in I$, the lower-level problems of the standard semi-infinite problem $\widetilde{\mathsf{SIP}}$ induced by the transformation:

$$\tilde{Q}_i(\boldsymbol{x}^k) : \max_{\boldsymbol{z} \in Z} \tilde{g}_i(\boldsymbol{x}^k, \boldsymbol{z}) \ .$$

If one calculates a global solution $\boldsymbol{y}^{i,*}$ of Problem $Q_i(\boldsymbol{x}^k)$, then $\boldsymbol{z}^*$ with $\boldsymbol{t}(\boldsymbol{x}^k, \boldsymbol{z}^*) = \boldsymbol{y}^{i,*}$ is a global solution of $\tilde{Q}_i(\boldsymbol{x}^k)$. In [Sch13] an example is presented which shows that it can be beneficial to use the original problem, as properties like convexity can get lost by the transformation.

In both cases one solves in iteration k for a given discretization Z^k the following discretized problem:

$$\begin{aligned} \widetilde{\mathsf{SIP}}(Z^k) : \min_{\boldsymbol{x} \in \mathbb{R}^n} \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & \tilde{g}_i(\boldsymbol{x}, \boldsymbol{z}) \leq 0 \text{ for all } i \in I, \boldsymbol{z} \in Z^k \ . \end{aligned}$$

Also for adding linear information one has two possibilities.

- One can consider, for an $i \in I$, the transformed problem $\tilde{Q}_i(\boldsymbol{x}^k)$. Assuming that the derivative of the lower level solution $D\boldsymbol{z}^i(\boldsymbol{x}^k)$ and the corresponding Lagrange-multipliers $D\boldsymbol{\mu}^i(\boldsymbol{x}^k)$ exist and denoting the lower level Lagrangian by $\widetilde{\mathcal{L}}_i$, the additional constraint would be given by

  $$\widetilde{\mathcal{L}}_i(\boldsymbol{x}, \boldsymbol{z}^{i,k} + D\boldsymbol{z}^i(\boldsymbol{x}^k) \cdot (\boldsymbol{x} - \boldsymbol{x}^k), \boldsymbol{\mu}^{i,k} + D\boldsymbol{\mu}^i(\boldsymbol{x}^k)(\boldsymbol{x} - \boldsymbol{x}^k)) \leq 0 \ .$$

  The algorithm then completely coincides with Algorithm 2 applied to the transformed problem $\widetilde{\mathsf{SIP}}$. (With the exception that the original lower-level problem and the transformation can be used to obtain a solution $\boldsymbol{z}^{i,k}$)

- A second possibility consists in considering the original lower-level problems $Q_i(\boldsymbol{x})$. For $i \in I$, $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{y} \in \mathbb{R}^m$ and $\boldsymbol{\mu}^i \in \mathbb{R}^{|J|}$, the lower-level Lagrange function is given by:

  $$\mathcal{L}_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\mu}^i) = g_i(\boldsymbol{x}, \boldsymbol{y}) - \sum_{j \in J} \mu_j^i u_j(\boldsymbol{x}, \boldsymbol{y}) \ .$$

  The difference to the case SIP is the $\boldsymbol{x}$-dependency of the constraints.

If the Reduction Ansatz holds at a given point $\boldsymbol{x} \in M$ for SIP, we have, for every $i \in I$ and every $\boldsymbol{z} \in Z$ with $g_i(\boldsymbol{x}, \boldsymbol{z}) = 0$, an $\varepsilon > 0$ and a continuous differentiable function $\boldsymbol{z} : B_\varepsilon(\boldsymbol{x}) \to Z$, such that locally the maximum of the lower-level problem can be described by one of these functions. Now letting for $\boldsymbol{x} \in B_\varepsilon(\boldsymbol{x}^*)$

$$\boldsymbol{y}(\boldsymbol{x}) := \boldsymbol{t}(\boldsymbol{x}, \boldsymbol{z}(\boldsymbol{x})) \ ,$$

we obtain differentiable functions which describe the global solution for the original lower-level problem $Q_i(\boldsymbol{x}^*)$.

We collect at a current iterate $\boldsymbol{x}^k$ all indices in $\bar{I}^k$ for which the above properties hold. For every $i \in \bar{I}^k$ we can consider the following constraint with linear information:

$$\widehat{g}_i^k(\boldsymbol{x}) := \mathcal{L}_i\big(\boldsymbol{x}, \boldsymbol{y}^{i,k} + D\boldsymbol{y}^i(\boldsymbol{x}^k) \cdot (\boldsymbol{x} - \boldsymbol{x}^k), \boldsymbol{\mu}^{i,k} + D\boldsymbol{\mu}^i(\boldsymbol{x}^k) \cdot (\boldsymbol{x} - \boldsymbol{x}^k)\big) \ .$$

We then solve the following optimization problem to find the next iterate $\boldsymbol{x}^{k+1}$:

$$\begin{aligned} \widehat{\mathsf{SIP}}^k(Z^{k+1}) : \min_{\boldsymbol{x} \in \mathbb{R}^n} \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & \tilde{g}_i(\boldsymbol{x}, \boldsymbol{z}) \leq 0 \text{ for all } i \in I, \boldsymbol{z} \in Z^{k+1} \ , \\ & \widehat{g}_i^k(\boldsymbol{x}) \leq 0 \text{ for all } i \in \bar{I}^k \ . \end{aligned}$$

Depending on the transformation one of both approaches might lead to problems with a structure that is easier to solve. For example: one might lead to convex problems, while the other formulation leads to non-convex problems. For complicated transformations also the condition can be worse of the transformed problem. In this case it can be beneficial to solve problem $\widehat{\mathsf{SIP}}^k(Z^{k+1})$. We show their different performance with some numerical examples in Chapter 6.

The consideration above give rise to the following algorithm solving GSIP.

Algorithm 3 Adaptive discretization with quadratic rate of convergence solving GSIP

Input: initial point $\boldsymbol{x}^1 \in \mathbb{R}^n$, initial discretization $Z^1 \subseteq Z, k = 1$.
while termination criterion is not met **do**
  for $i \in I$ **do**
    Compute a global solution $\boldsymbol{y}^{i,k}$ and Lagrange multipliers $\boldsymbol{\mu}^{i,k}$ of $Q_i(\boldsymbol{x}^k)$.
    Determine $D\boldsymbol{y}^i(\boldsymbol{x}^k)$ and $D\boldsymbol{\mu}^i(\boldsymbol{x}^k)$ if they exist.
    Determine a $\boldsymbol{z}^{i,k}$ such that:

$$\boldsymbol{y}^{i,k} = \boldsymbol{t}(\boldsymbol{x}^k, \boldsymbol{z}^{i,k}) \ .$$

  end for
  $Z^{k+1} = Z^k \cup \bigcup_{i \in I} \{\boldsymbol{z}^{i,k}\}$.
  Determine a solution $\boldsymbol{x}^{k+1}$ of problem $\widehat{\mathsf{SIP}}^k(Z^{k+1})$.
  $k = k + 1$.
end while

Formally this is a new algorithm and the same questions already investigated in Chapter 4 can be asked. We discuss in the next section under which assumption the results can be transferred. We do not prove all the statements again, but point out why the proof is either completely analogous or which argument can be used to transfer the statements.

5.2 Transfer of the convergence results

One of the important properties of Algorithm 2 is that every accumulation point of the iterates $\boldsymbol{x}^k$ is feasible. As the proof of Lemma 4.3 only uses discretization points the same holds for Algorithm 3.

Lemma 5.1. *Let $\{\boldsymbol{x}^k\}_{k \in \mathbb{N}}$ be constructed according to Algorithm 3. Every accumulation point $\boldsymbol{x}^*$ of $\{\boldsymbol{x}^k\}_{k \in \mathbb{N}}$ is feasible.*

After showing that any accumulation point of the iterates is feasible we investigated a sequence of stationary points and local minima. An important auxiliary statement in the case of an fixed index set investigated the derivatives of the additional constraints. It stated that the first and the second derivatives coincide with the derivatives of the functions locally describing the feasible set. The next Lemma shows that this is also the case for the functions $\widehat{g}_i^k, i \in I$.

Lemma 5.2. *Consider a fixed $\boldsymbol{x}^* \in \mathbb{R}^n$ and $i \in I$. Let $\boldsymbol{z}^{i,*} \in Z$ be a global solution of the lower level problem $\tilde{Q}_i(\boldsymbol{x}^*)$ satisfying all conditions for Theorem 2.13. Denote the unique Lagrange multipliers satisfying the KKT conditions by $\boldsymbol{\mu}^{i,*}$. Let:*

$$\begin{aligned}\overline{\boldsymbol{y}}^i(\boldsymbol{x}) =& \boldsymbol{y}^{i,*} + D\boldsymbol{y}^i(\boldsymbol{x}^*) \cdot (\boldsymbol{x} - \boldsymbol{x}^*) \ , \\ \overline{\boldsymbol{\mu}}^i(\boldsymbol{x}) =& \boldsymbol{\mu}^{i,*} + D\boldsymbol{\mu}^i(\boldsymbol{x}^*) \cdot (\boldsymbol{x} - \boldsymbol{x}^*) \ .\end{aligned}$$

and

$$\widehat{g}_i(\boldsymbol{x}) := \mathcal{L}_i\left(\boldsymbol{x}, \overline{\boldsymbol{y}}^i(\boldsymbol{x}), \overline{\boldsymbol{\mu}}^i(\boldsymbol{x})\right) \ .$$

For the derivatives the following holds:

$$D\widehat{g}_i(\boldsymbol{x}^*) = D_{\boldsymbol{x}}\Big[g_i\big(\boldsymbol{x}, \boldsymbol{y}^i(\boldsymbol{x})\big)\Big]_{\boldsymbol{x}=\boldsymbol{x}^*} = D_{\boldsymbol{x}}\Big[\tilde{g}_i\big(\boldsymbol{x}, \boldsymbol{z}^i(\boldsymbol{x})\big)\Big]_{\boldsymbol{x}=\boldsymbol{x}^*} = D_1\tilde{g}_i(\boldsymbol{x}^*, \boldsymbol{z}^{i,*})$$

and

$$D^2\widehat{g}_i(\boldsymbol{x}^*) = D^2_{\boldsymbol{x}}\Big[g_i\big(\boldsymbol{x}, \boldsymbol{y}^i(\boldsymbol{x})\big)\Big]_{\boldsymbol{x}=\boldsymbol{x}^*} = D^2_{\boldsymbol{x}}\Big[\tilde{g}_i\big(\boldsymbol{x}, \boldsymbol{z}^i(\boldsymbol{x})\big)\Big]_{\boldsymbol{x}=\boldsymbol{x}^*} \ .$$

The proof is completely analogous to the proof of Lemma 4.1. The equality to the derivatives of $\tilde{g}_i$, for $i \in I$, follows easily, as we have by construction:

$$g_i\big(\boldsymbol{x}, \boldsymbol{y}^i(\boldsymbol{x})\big) = \tilde{g}_i\big(\boldsymbol{x}, \boldsymbol{z}^i(\boldsymbol{x})\big) \ .$$

To simplify the following statements we collect the needed regularity assumptions (similar to Assumption 4.4):

Assumption 5.3. *Let $\{\boldsymbol{x}^k\}_{k\in\mathbb{N}}$ be constructed according to Algorithm 3. Assume there is a $\boldsymbol{x}^* \in \mathbb{R}^n$ with*

$$\lim_{k\to\infty} \boldsymbol{x}^k = \boldsymbol{x}^* \ .$$

Suppose the Reduction Ansatz and EMFCQ *hold at $\boldsymbol{x}^*$ for $\widetilde{\mathsf{SIP}}$ and the objective function f is twice continuously differentiable. Further assume:*

- *For every $i \in I$ there is at least one $\boldsymbol{y} \in Y(\boldsymbol{x}^*)$ with*

$$g_i(\boldsymbol{x}^*, \boldsymbol{y}) = 0 \ .$$

- *For every $k \in \mathbb{N}$ the following holds:*

$$\bar{I}^k = I \ .$$

As by Lemma 5.2 the derivatives of the additional constraints coincide, also their limits are the same. As a consequence the construction for stationary points can be transferred directly and we have similar to Theorem 4.8 the following result:

Theorem 5.4. *Let Assumption 5.3 be satisfied. Moreover assume that, for every $k \in \mathbb{N}$, the current iterate $\boldsymbol{x}^k$ is a stationary point of $\widehat{\mathsf{SIP}}^{k-1}(Z^k)$. Then the limit $\boldsymbol{x}^*$ is a stationary point of $\widehat{\mathsf{SIP}}$.*

The next question we considered in Subsection 4.1.2 is the convergence of local solutions. We have seen that an accumulation point of local solutions is not necessarily a local solution of the original problem. Completely analogous of Lemma 4.11 the following holds for GSIP.

Lemma 5.5. *Let $\{\boldsymbol{x}^k\}_{k\in\mathbb{N}}$ be constructed according to Algorithm 2, Assume that, for every $k \in \mathbb{N}$, the solution $\boldsymbol{x}^k$ is a local solution of $\widehat{\mathsf{SIP}}(Z^k)$ with radius r_k. Assume:*

$$\begin{aligned} \lim_{k\to\infty} \boldsymbol{x}^k &= \boldsymbol{x}^* \ , \\ \lim_{k\to\infty} \boldsymbol{y}^{i,k} &= \boldsymbol{y}^{i,*} \ , \\ r := \inf_{k\in\mathbb{N}} r_k &> 0 \ . \end{aligned}$$

Further assume that the Reduction Ansatz and EMFCQ *hold at $\boldsymbol{x}^*$. Then $\boldsymbol{x}^*$ is a local solution of*

$$\begin{aligned} \widehat{\mathsf{SIP}} : \min_{\boldsymbol{x}\in\mathbb{R}^n} \quad & f(\boldsymbol{x}) \\ s.t. \quad & g_i(\boldsymbol{x},\boldsymbol{y}) \le 0 \text{ for all } i \in I, \boldsymbol{y} \in Y(\boldsymbol{x}) \ , \\ & \widehat{g}_i(\boldsymbol{x}) \le 0 \ , \end{aligned}$$

where

$$\widehat{g}_i(\boldsymbol{x}) := \mathcal{L}_i\big(\boldsymbol{x}, \boldsymbol{y}^{i,*} + D\boldsymbol{y}^i(\boldsymbol{x}^*)(\boldsymbol{x}-\boldsymbol{x}^*), \boldsymbol{\mu}^{i,*} + D\boldsymbol{\mu}^i(\boldsymbol{x}^*)(\boldsymbol{x}-\boldsymbol{x}^*)\big) \ .$$

We again assume for the following that the objective function f is twice continuously differentiable and the solution of the lower-level problem $Q_i(\boldsymbol{x}^*)$ is unique. By Lemma 5.2 the first and second derivative coincide with the first and second derivative of the functions which locally describe the feasible set. This means that the second-order sufficient condition for problem $\widehat{\mathsf{SIP}}$ and the transformed problem $\widehat{\mathsf{SIP}}$ coincide. In other words: if the second-order sufficient condition for $\widehat{\mathsf{SIP}}$ hold at a point $\boldsymbol{x}^* \in M$, the point is local minimum of GSIP.

We presented in Subsection 4.1.2 conditions that guarantee the second-order sufficient condition in the limit. As, for sufficiently large k, the point $\boldsymbol{x}^k$ is a stationary point there are Lagrange-multipliers $\overline{\boldsymbol{\lambda}}^k \ge 0$ and $\boldsymbol{\lambda}^k \ge 0$ such that:

$$0 = Df(\boldsymbol{x}^k) + \sum_{i \in I} \sum_{\boldsymbol{z} \in Z_0^k} \lambda_{i,\boldsymbol{z}}^k D_1 \tilde{g}_i(\boldsymbol{x}^k, \boldsymbol{z}) + \sum_{i \in I} \overline{\lambda}_i^k D\widehat{g}_i^{k-1}(\boldsymbol{x}^k) \ ,$$

where $Z_0^k := \{\boldsymbol{z} \in Z^k \mid \tilde{g}_i(\boldsymbol{x}^k, \boldsymbol{z}) = 0\}$. Again there can be active constraints enforced by the discretization. That is why we consider similarly to Theorem 4.12 a modified condition based only on the additional constraints. Therefore let, for $k \in \mathbb{N}$ and $i \in I$:

$$\hat{\lambda}_i^k := \overline{\lambda}_i^k + \sum_{\boldsymbol{y} \in Y_0^i(\boldsymbol{x}^k)} \lambda_{i,\boldsymbol{y}}^k \ ,$$
$$T^k := \{\boldsymbol{d} \in \mathbb{R}^n \mid D\widehat{g}_i^{k-1}(\boldsymbol{x}^k)\boldsymbol{d} = 0 \text{ for all } i \in I \text{ with } \hat{\lambda}_i^k > 0\}$$

and

$$\kappa^k := \min_{\boldsymbol{d} \in T^k, \|\boldsymbol{d}\|=1} \boldsymbol{d}^\top \left(D^2 f(\boldsymbol{x}) + \sum_{i \in I} \hat{\lambda}_i^k D^2 \widehat{g}_i^{k-1}(\boldsymbol{x}^k) \right) \boldsymbol{d} \ .$$

Theorem 5.6. *Let Assumption 5.3 be satisfied, Assume that, for every $k \in \mathbb{N}$, the solution $\boldsymbol{x}^k$ is a local solution of $\widehat{\mathsf{SIP}}^{k-1}(Z^k)$ and the following holds, for every $i \in I$:*

$$\lambda_i^* := \liminf_{k \to \infty} \hat{\lambda}_i^k > 0 \ ,$$
$$\kappa^k := \liminf_{k \to \infty} \kappa^k > 0 \ .$$

Assume that the Reduction Ansatz and ELICQ *hold at $\boldsymbol{x}^*$ for $\widetilde{\mathsf{SIP}}$. Further assume that, for every $i \in I$, the solution of the lower-level problem $Q_i(\boldsymbol{x}^*)$ is unique. Then $\boldsymbol{x}^*$ is a local solution of* GSIP.

The proof is completely analogous to the proof of Theorem 4.12

The last property we have shown in Chapter 4 is the quadratic rate of convergence. We briefly discuss how the derived bounds can be transferred to the case GSIP and then state the quadratic convergence in this situation. We therefore assume additionally to Assumption 5.3 that the describing functions $\boldsymbol{g}, \boldsymbol{u}$ of GSIP and the transformation function, $\boldsymbol{t}$, are three times continuously differentiable. We further assume that there is, for every $i \in I$, a unique $\boldsymbol{y}^{i,*}$ with $g_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,*}) = 0$.

First the bounds (Lemma 4.15 and 4.16) concerning the additional constraints can be shown with exactly the same proofs considering, for $i \in I$, the original lower-level problem $Q_i(\boldsymbol{x})$ of GSIP. We have again the following bounds:

$$\|\tilde{g}_i(\boldsymbol{x}^k, \boldsymbol{z}^{i,k}) - \widehat{g}_i^{k-1}(\boldsymbol{x}^k)\| = \|g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) - \widehat{g}_i^{k-1}(\boldsymbol{x}^k)\| \leq K_1 \|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^4$$

and

$$\|D_1\tilde{g}_i(\boldsymbol{x}^k, \boldsymbol{z}^{i,k}) - D\widehat{g}_i^{k-1}(\boldsymbol{x}^k)\| = \|D\widehat{g}_i^{k}(\boldsymbol{x}^k) - D\widehat{g}_i^{k-1}(\boldsymbol{x}^k)\| \leq K_1\|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^2 \ .$$

We now switch, for every $i \in I$, to the lower-level problems $\tilde{Q}_i(\boldsymbol{x})$ of the transformed problem. Similar to Lemma 4.17 and 4.18 we can find, with the help of the strong stability of $\boldsymbol{z}^{i,*}$, a $K_2 > 0$ such that, for sufficiently large k, $i \in I$ and a $\boldsymbol{z} \in Z$ with $\tilde{g}_i(\boldsymbol{x}^k, \boldsymbol{z}) = 0$, the following holds:

$$\|\boldsymbol{z}^{i,k} - \boldsymbol{z}\| \leq K_2\|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\|^2 \ .$$

Similar to the considerations in section 4.2 the current iterate $\boldsymbol{x}^k$ is a stationary point of the following semi-infinite problem:

$$\begin{aligned} \mathsf{SIP}^k_{mod}: \quad \min_{\boldsymbol{x}\in\mathbb{R}^n} \quad & f(\boldsymbol{x}) - \boldsymbol{\beta}^k \cdot \boldsymbol{x} \\ s.t. \quad & \tilde{g}_i(\boldsymbol{x}, \boldsymbol{z}) - \alpha_i^k \leq 0 \text{ for all } i \in I, \boldsymbol{z} \in Z \ , \end{aligned}$$

for appropriate $\boldsymbol{\alpha}^k$ and $\boldsymbol{\beta}^k$. The inequalities above can be used to bound both parameters. Analogous to the Quadratic Convergence Theorem we obtain the following statement:

Theorem 5.7. *Let Assumption 5.3 be satisfied. Further assume:*

- *the describing functions $\boldsymbol{g}, \boldsymbol{u}$ of* GSIP *and the transformation function $\boldsymbol{t}$ are three times continuously differentiable.*
- *for every $i \in I$, there is a unique $\boldsymbol{y}^{i,*} \in Y(\boldsymbol{x}^*)$ with $g_i(\boldsymbol{x}^*, \boldsymbol{y}^{i,*}) = 0$.*
- *the current iterate $\boldsymbol{x}^k$ is, for every $k \in \mathbb{N}$, a stationary point of $\widehat{\mathsf{SIP}}^{k-1}(Z^k)$.*

Finally assume that the limit $\boldsymbol{x}^*$ is a strongly stable stationary point. There is a $L > 0$ such that, for sufficiently large k, the following holds:

$$\|\boldsymbol{x}^{k+1} - \boldsymbol{x}^*\| \leq L\|\boldsymbol{x}^k - \boldsymbol{x}^*\|^2 \ .$$

6 Numerical aspects

In the previous chapters we have seen that for Algorithm 2 and Algorithm 3 a quadratic rate of convergence can be expected. In this chapter we want to show that this quadratic rate and an improvement in the number of iterations and compute time can be observed also for numerical examples. We compare the two approaches introduced in Chapter 5, which extended Algorithm 2 to generalized semi-infinite problems. One of the approaches transforms the generalized semi-infinite problem to a standard semi-infinite problem and then applies Algorithm 2. We therefore only consider problems which are transformable generalized semi-infinite optimization problems.

We begin this chapter by summarizing further details of our implementation. We then introduce so-called design-centering problems, which are the source of the considered test problems. In Section 6.1 we discuss the solution of three numerical examples. For one of them we give an analytical solution. The benefit is that we can compare the calculated iterates with the true solution of the problem. With the second example we investigate the effects of an increasing dimension. The third example is non-convex. Using this example we study the question whether discretization points are needed. In Section 6.2 we discuss the growth of the problem dimension. We compare the Blankenship and Falk algorithm to the newly developed Algorithms 2 and 3. In Section 6.3 we give a short summary of our observations.

Details of implementation
All implementations were done in MATLAB® [MAT16]. All finite nonlinear problems were solved using the SQP method provided by the MATLAB® function `fmincon` (part of MATLAB® Optimization Toolbox™). The standard settings were used except for the accuracy. We required an accuracy in the optimization variables, the objective function and the constraints of 10^{-8}. All derivatives were calculated analytically and provided to `fmincon` as well as to the calculation of the linearization. To check the existence and to calculate the derivatives of the implicit functions we use Remark 2.14.

As mentioned in the introduction we consider in the following only generalized semi-infinite optimization problems. We solve the problems of type GSIP with three

different approaches. All problems are transformable, which means that there is an equivalent standard semi-infinite optimization problem $\widetilde{\mathsf{SIP}}$. The first two solution approaches consist of applying the Blankenship and Falk algorithm (Algorithm 1) and the newly developed algorithm (Algorithm 2) to the induced standard semi-infinite problem $\widetilde{\mathsf{SIP}}$. To calculate a solution of the lower-level problems, we use the original lower-level problem and transform it back as suggested in [Sch13] and discussed in Chapter 5. The third solution approach consists of applying Algorithm 3 to the generalized semi-infinite problem. The three approaches together with their abbreviation are summarized in Table 6.1.

$\mathcal{A}_{B\&F}$	Blankenship and Falk algorithm applied to equivalent problem $\widetilde{\mathsf{SIP}}$.
$\mathcal{A}_{\mathsf{SIP}}$	Algorithm 2 applied to equivalent problem $\widetilde{\mathsf{SIP}}$.
$\mathcal{A}_{\mathsf{GSIP}}$	Algorithm 3 applied to problem GSIP.

Table 6.1: Notation for the tested algorithms

So far we did not specify a stopping criterion for any of the introduced algorithms. One possibility is to set a tolerance on the maximal violation of the new points, i.e.:

$$\alpha^k = \max_{i \in I} g_i(\boldsymbol{x}^k, \boldsymbol{y}^{i,k}) \ .$$

We have seen in Theorem 3.9 that the distance to the limit can then be bounded. Unfortunately, if the local solution is of order two, there is a constant L such that:

$$\|\boldsymbol{x}^k - \boldsymbol{x}^*\| \leq L\sqrt{\alpha^k} \ .$$

This leads to the following negative effect: if, for example $L = 1$, then even if the violation is approximately 10^{-6}, the distance to an optimal solution can still be approximately 10^{-3}. This is why, we do not use the current violation. Instead we use the size of the last step as a stopping criterion for all three algorithms. We terminate the algorithm as soon as

$$\|\boldsymbol{x}^{k+1} - \boldsymbol{x}^k\| < 10^{-5}$$

holds.

Design-centering problems

A well studied class of examples for GSIPs are so called *design-centering problems* (see for example [Ste03, HB17]). There, one considers a fixed set $C \subseteq \mathbb{R}^m$, the so-called *container*, and, for every $\boldsymbol{x} \in \mathbb{R}^n$, a second parametrized set $Y(\boldsymbol{x}) \subseteq \mathbb{R}^m$, the so-called *design*. The task is to maximize the volume of $Y(\boldsymbol{x})$ under the condition:

$$Y(\boldsymbol{x}) \subseteq C \ . \tag{6.1}$$

Assume that we can describe the sets C and $Y(\boldsymbol{x})$ for every $\boldsymbol{x} \in \mathbb{R}^n$ by:

$$C := \{\boldsymbol{y} \in \mathbb{R}^m \mid g_i(\boldsymbol{y}) \leq 0 \text{ for all } i \in I\} ,$$
$$Y(\boldsymbol{x}) := \{\boldsymbol{y} \in \mathbb{R}^m \mid u_j(\boldsymbol{x}, \boldsymbol{y}) \leq 0 \text{ for all } j \in J\}$$

with twice continuously differentiable functions $\boldsymbol{g} : \mathbb{R}^m \to \mathbb{R}^{|I|}, \boldsymbol{u} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{|J|}$ and finite index sets I, J. If such a description exists, one can reformulate a problem with a set inclusion condition given as in Equation (6.1) by a GSIP:

$$\begin{aligned} \mathsf{GSIP}_{DC} : \max_{\boldsymbol{x} \in \mathbb{R}^n} \quad & \operatorname{vol}\big(Y(\boldsymbol{x})\big) \\ \text{s.t.} \quad & g_i(\boldsymbol{y}) \leq 0 \text{ for all } i \in I, \boldsymbol{y} \in Y(\boldsymbol{x}) . \end{aligned}$$

As we have seen in Chapter 2, finitely many inequalities on the parameters $\boldsymbol{x}$ can be added easily.

The following three examples are all design-centering problems. For every example we describe the considered design and container. Furthermore, we provide a transformation function and describe the additional restrictions on the parameters.

6.1 Three numerical examples

We begin with a two dimensional example which can be solved analytically. The benefit of an analytic solution is that the algorithms are easy to compare and different effects, described already in the previous chapters theoretically, can be found again in the numerics.

Example with an analytic solution: To demonstrate the quadratic rate numerically, we consider as a first example the embedding of an ellipse into a triangle. Therefore, consider the following description of an ellipse:

$$Y(\boldsymbol{x}) := \Big\{\boldsymbol{y} \in \mathbb{R}^2 \mid \big(\boldsymbol{y} - \boldsymbol{c}(\boldsymbol{x})\big)^\top \big(A(\boldsymbol{x})A(\boldsymbol{x})^\top\big)^{-1} \big(\boldsymbol{y} - \boldsymbol{c}(\boldsymbol{x})\big) \leq 1\Big\} ,$$

where

$$\boldsymbol{c}(\boldsymbol{x}) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad A(\boldsymbol{x}) = \begin{pmatrix} x_3 & x_5 \\ 0 & x_4 \end{pmatrix} .$$

To make sure that the inverse of $A(\boldsymbol{x})$ exists, the lower bounds of x_3 and x_4 are set to 10^{-6}. The volume is given by:

$$\operatorname{vol}(Y\big(\boldsymbol{x}\big)) = \pi x_3 x_4 .$$

A transformation from the unit disk to $Y(\boldsymbol{x})$ is given by:

$$\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{z}) = A(\boldsymbol{x})\boldsymbol{z} + \boldsymbol{c}(\boldsymbol{x}) \ .$$

We choose the following triangle as a container:

$$C := \left\{ \boldsymbol{y} \in \mathbb{R}^2 \,\middle|\, \begin{array}{r} -y_1 - 1 \leq 0, \\ -y_2 - 1 \leq 0, \\ \frac{1}{4}y_1 + y_2 - \frac{3}{4} \leq 0 \end{array} \right\} \ . \tag{6.2}$$

The exact same triangle was for example considered as a test problem in [Sch13]. Solving the resulting optimization problem analytically yields the following solution

$$\boldsymbol{x}^* = \left(\frac{5}{3}, -\frac{1}{3}, \frac{4\sqrt{3}}{3}, \frac{2}{3}, -\frac{4}{3} \right)^\top \ .$$

The solution $\boldsymbol{x}^*$ of the design centering problem is depicted in Figure 6.1.

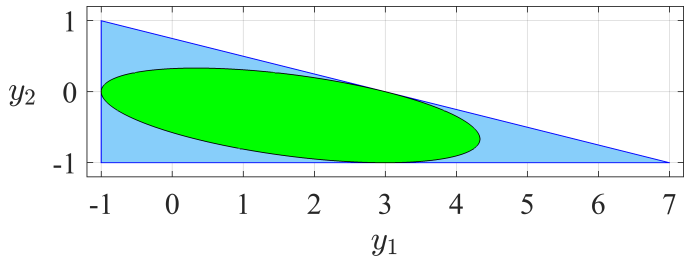


Figure 6.1: Largest ellipse in the Triangle C [*green* - design, *blue* - container]

As initial values for the algorithms we use a unit disk:

$$\boldsymbol{x}^1 = (0, 0, 1, 1, 0)^\top \ .$$

The distances of the first iterates to the optimal solution, calculated by the different algorithms, are given in Table 6.2. The two algorithms with linear information needed both 5 iterations which took approximately 0.7 seconds for Algorithm $\mathcal{A}_{\mathsf{SIP}}$ and 0.97 seconds for Algorithm $\mathcal{A}_{\mathsf{GSIP}}$. The algorithm by Blankenship and Falk $\mathcal{A}_{B\&F}$ needed 21 iterations to meet the stopping criterion (2.36 seconds).

Many properties can be explained from the theoretical analysis we have done before. Considering the first 4 iterates calculated by $\mathcal{A}_{B\&F}$ and by $\mathcal{A}_{\mathsf{GSIP}}$ one can see that the distance to the optimal solution is similar. This is due to Lemma 4.3. For both algorithms the convergence is guaranteed by the discretization points. Starting with iteration 5 we are close enough so that the quadratic rate of convergence can

Iteration	$\mathcal{A}_{B\&F}$	$\mathcal{A}_{\mathsf{SIP}}$	$\mathcal{A}_{\mathsf{GSIP}}$
1	2.5480	2.5480	2.5480
2	3.4975	2.156	3.5146
3	0.3374	$3.9415 \cdot 10^{-2}$	0.4730
4	0.7586	$1.0997 \cdot 10^{-5}$	0.1002
5	0.2543	$8.9662 \cdot 10^{-6}$	$3.5645 \cdot 10^{-4}$
6	$6.5552 \cdot 10^{-2}$	$7.1661 \cdot 10^{-7}$	$6.8389 \cdot 10^{-8}$
7	$2.3665 \cdot 10^{-2}$	-	-
8	$1.7667 \cdot 10^{-2}$	-	-
	$\vdots$		

Table 6.2: Distances of the iterates to the optimal solution, $\|\boldsymbol{x}^k - \boldsymbol{x}^*\|$, for the three algorithms under consideration

be observed for $\mathcal{A}_{\mathsf{GSIP}}$. The convergence for Algorithm $\mathcal{A}_{B\&F}$ is still only enforced by the discretization points. This is why now a much slower convergence can be observed for Algorithm $\mathcal{A}_{B\&F}$. Comparing only the 'final steps' the algorithm with linear information only needs 2 iterations from an accuracy of 0.1 to $7 \cdot 10^{-8}$. The Blankenship and Falk algorithm needs for these 'final steps' further 15 iterations.

We comment next on the difference between the iterates calculated by Algorithm $\mathcal{A}_{\mathsf{SIP}}$ and the ones calculated by Algorithm $\mathcal{A}_{\mathsf{GSIP}}$. One can first observe that for $\mathcal{A}_{\mathsf{SIP}}$, the range where a quadratic rate holds, is already reached in iteration 3. In the next example we observe this effect again. We comment there why Algorithm $\mathcal{A}_{\mathsf{SIP}}$ reaches the quadratic rate faster than Algorithm $\mathcal{A}_{\mathsf{GSIP}}$. Unfortunately, there is also a drawback: in iteration 5 the numerical error is already too large so that no further quadratic rate can be observed. A possible reason why the numerical error is larger for $\mathcal{A}_{\mathsf{SIP}}$ is the following: to calculate a solution of the transformed lower-level problem $\tilde{Q}_i(\boldsymbol{x}^k)$, for $i \in I$, we use the original lower-level problem $Q_i(\boldsymbol{x}^k)$, for $i \in I$ and then use the transformation to obtain a solution to the transformed lower-level problem. This is why all tolerances on the accuracy hold for the original lower-level problem. The error for the transformed problem might be larger. This means also the error for the derivatives can be larger.

Example with a higher problem dimension: For the next example we consider again an ellipse, but this time in an arbitrary dimension $m \geq 3$. Completely analogous to the first example we can describe the design by

$$Y(\boldsymbol{x}) := \left\{ \boldsymbol{y} \in \mathbb{R}^m \mid \left(\boldsymbol{y} - \boldsymbol{c}(\boldsymbol{x})\right)^\top \left(A(\boldsymbol{x})A(\boldsymbol{x})^\top\right)^{-1} \left(\boldsymbol{y} - \boldsymbol{c}(\boldsymbol{x})\right) \leq 1 \right\},$$

where $\boldsymbol{c}(\boldsymbol{x})$ is the vector consisting of the first m optimization variables and $A(\boldsymbol{x})$ is an m-dimensional upper triangular matrix. Again, we assume that its entries

on the diagonal are larger than 10^{-6}. The volume is given as the product of the diagonal elements and the constant volume of a unit ball in m dimensions. As a transformation we can use as before:

$$\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{z}) = A(\boldsymbol{x})\boldsymbol{z} + \boldsymbol{c}(\boldsymbol{x}) \ .$$

As a container we consider the unit simplex in m dimensions. To avoid numerical difficulties in higher dimension we scale the length of the edges by the dimension:

$$C = \left\{ \boldsymbol{y} \in \mathbb{R}^m \middle| \begin{array}{l} \sum_{i=1}^{m} y_i \leq m \\ \quad -y_i \leq 0 \text{ for all } 1 \leq i \leq m \end{array} \right\} \ .$$

The resulting GSIP consists of $m + \frac{(m+1)\cdot m}{2}$ optimization variables, $m+1$ semi-infinite constraints and m index variables. The problem and the optimal solution for $m = 3$ are shown in Figure 6.2. For $m = 3$ and edge-length 1 the example was considered as a test-problem in [Ste11].

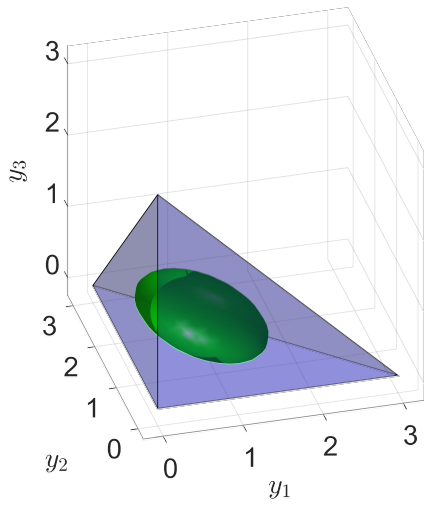


Figure 6.2: Optimal solution for design-centering problem: Embedding of three dimensional Ellipse into a simplex [*green* - design, *blue* - container]

As initial parameters we set $\boldsymbol{c}(\boldsymbol{x}) = 0$ and set $A(\boldsymbol{x})$ to be the identity matrix. The number of iterations and the time needed for increasing dimension are given in Table 6.3.

As expected, one can observe that the number of iteration for the Blankenship and Falk algorithm grows much faster than for the other two algorithms. While

m	$\mathcal{A}_{B\&F}$	$\mathcal{A}_{\mathsf{SIP}}$	$\mathcal{A}_{\mathsf{GSIP}}$
3	33	4	7
4	53	4	10
5	82	4	15
6	108	4	19

a)

m	$\mathcal{A}_{B\&F}$	$\mathcal{A}_{\mathsf{SIP}}$	$\mathcal{A}_{\mathsf{GSIP}}$
3	6.21	1.88	2.08
4	37.17	1.95	6.60
5	177.80	4.11	20.13
6	593.60	5.70	49.34

b)

Table 6.3: Number of iterations and time needed to solve embedding of higher dimensional ellipse into simplex: a) Number of iterations, b) Time in seconds

the number of iterations for algorithm $\mathcal{A}_{\mathsf{SIP}}$ remains constant, one can observe that the number of iterations needed by $\mathcal{A}_{\mathsf{GSIP}}$ also grows moderately. In Table 6.4 we have divided the iterations into the iterations that are needed to reach a step size of less than 10^{-1} and the remaining steps with smaller step size: While

m	steps $> 10^{-1}$	steps $< 10^{-1}$
3	4	3
4	7	3
5	10	5
6	16	3

Table 6.4: Number of steps needed by Algorithm $\mathcal{A}_{\mathsf{GSIP}}$ with size larger than 10^{-1} and less than 10^{-1}

the number of iteration with a step size larger than 10^{-1} grows, The number of iterations with a step size less than 10^{-1} is almost constant. Both observations can be explained with the previously developed theory: (1) We have already discussed in the previous section and in Lemma 4.3 that the convergence is guaranteed by the discretization points. This is true for the algorithms with linear information as well as for the algorithm by Blankenship and Falk. As the number of discretization points needed increases for Algorithm $\mathcal{A}_{B\&F}$, the same can be expected for the first iterations of Algorithm $\mathcal{A}_{\mathsf{GSIP}}$. (2) Once the step size becomes small enough, the quadratic convergence, guaranteed by Theorem 5.7, becomes dominant and only a few iterations, independent of the dimension, remain.

Another interesting aspect, one can observe in Table 6.3, is that the number of steps stays constant, if we use Algorithm $\mathcal{A}_{\mathsf{SIP}}$. This means that for this example it is better to use a linearization of the solution of the transformed lower-level problems $\tilde{Q}_i(\boldsymbol{x}^k)$. Independent of the dimension, the current iterate $\boldsymbol{x}^k$ is already after one step close enough to a limit point. The quadratic rate of convergence can be applied and after three additional iterations the step size is less than 10^{-5}. The behavior for the last three iterations is similar to the ones of Algorithm $\mathcal{A}_{\mathsf{GSIP}}$. The difference lies in the earlier iterations in which we need to come close enough to

a limit point. To explain this difference further, we consider the first iteration in more detail.

For both algorithms the size of the first step is similar:

$$\begin{aligned}\|\boldsymbol{x}^1_{\mathsf{GSIP}} - \boldsymbol{x}^2_{\mathsf{GSIP}}\| \approx & 2.2466 \ , \\ \|\boldsymbol{x}^1_{\mathsf{SIP}} - \boldsymbol{x}^2_{\mathsf{SIP}}\| \approx & 1.5354 \ ,\end{aligned}$$

where $\boldsymbol{x}^k_{\mathsf{GSIP}}, \boldsymbol{x}^k_{\mathsf{SIP}}$ denote the iterates generated by Algorithm $\mathcal{A}_{\mathsf{GSIP}}$ and Algorithm $\mathcal{A}_{\mathsf{SIP}}$ respectively. Both algorithms calculate an approximate to the next lower-level solution. Algorithm $\mathcal{A}_{\mathsf{GSIP}}$ uses a linearization in the original lower-level problem. We denote the approximate solution by $\overline{\boldsymbol{y}}^1_{\mathsf{GSIP}}$. In Algorithm $\mathcal{A}_{\mathsf{SIP}}$ we linearize the solutions in the transformed lower-level problem. We denote the approximate solution by $\overline{\boldsymbol{z}}^1_{\mathsf{SIP}}$. We can calculate the difference to the true solution:

$$\begin{aligned}\|\boldsymbol{y}^{i,2}_{\mathsf{GSIP}} - \overline{\boldsymbol{y}}^1_{\mathsf{GSIP}}\| \approx & \ 1.8246 \ , \\ \|\boldsymbol{y}^{i,2}_{\mathsf{SIP}} - \boldsymbol{t}(\boldsymbol{x}^2_{\mathsf{SIP}}, \overline{\boldsymbol{z}}^1_{\mathsf{SIP}})\| \approx & \ 0.2435 \ .\end{aligned}$$

One can see that the approximation found by Algorithm $\mathcal{A}_{\mathsf{SIP}}$ is closer to the true solution. As a consequence the true feasible set is better approximated and the next iterate is closer to the limit point.

This example shows that the approximation calculated in the transformed lower-level problem can be closer to the original problem on a larger range. Unfortunately this depends on the chosen transformation and no general statement can be made. However, we have seen in Chapter 4 and 5 that both approximations have locally the same quality.

A non-convex problem As a third example we consider a slightly modified version of a design called boat introduced in [Sch13]. First let

$$Z = \left\{ \boldsymbol{z} \in \mathbb{R}^2 \,\middle|\, \begin{matrix}(z_1 - \frac{1}{2})^2 + z_2^2 - 1 \leq 0, \\ (z_1 + \frac{1}{2})^2 + z_2^2 - 1 \leq 0\end{matrix} \right\} \ .$$

For $\boldsymbol{x} \in \mathbb{R}^4$ we consider a rotation matrix:

$$R(\boldsymbol{x}) = \begin{pmatrix} \cos(x_2) & -\sin(x_2) \\ \sin(x_2) & cos(x_2) \end{pmatrix} \ .$$

As design we then consider for every $\boldsymbol{x} \in \mathbb{R}^4$:

$$Y(\boldsymbol{x}) = \left\{ R(\boldsymbol{x}) \cdot x_1 \cdot \boldsymbol{z} + \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \mid \boldsymbol{z} \in Z \right\} \ . \tag{6.3}$$

To make sure that $Y(\boldsymbol{x})$ has an infinite cardinality, we demand further: $x_1 \geq 10^{-6}$. A transformation to a fixed set is already given in the description in Equation (6.3).

As a container we use a concavified version of the triangle given in (6.2):

$$C := \left\{ \boldsymbol{y} \in \mathbb{R}^2 \;\middle|\; \begin{array}{r} -y_1 - 1 \leq 0, \\ -y_2 - 1 \leq 0, \\ \frac{1}{4}y_1 + y_2 - \frac{3}{4} \leq 0, \\ -y_1 - y_2^2 \leq 0 \end{array} \right\} .$$

This container was first used in [SS03] and [Ste03] as a test object.

We use the following initial point:

$$\boldsymbol{x}^1 = (0, 1, 0, 0)^\top .$$

Running the algorithms we observe that all three terminate with the same solution:

$$\boldsymbol{x}^* \approx (1.4199, 1.5708, 1.1456, -0.2900)^\top . \tag{6.4}$$

While the Algorithms $\mathcal{A}_{\mathsf{SIP}}$ and $\mathcal{A}_{\mathsf{GSIP}}$ both needed 6 iterations, algorithm $\mathcal{A}_{B\&F}$ needed 18 iterations. The design and the container for the optimally chosen parameters are shown in Figure 6.3.

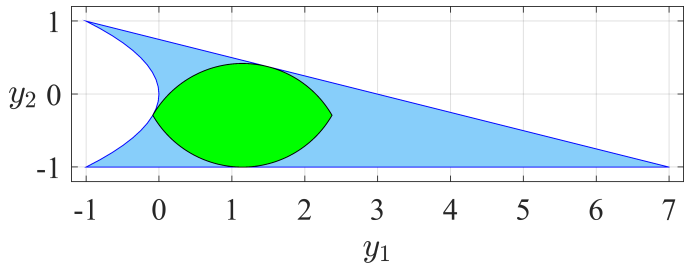


Figure 6.3: Largest boat in concavified triangle [*green* - design, *blue* - container]

In the remainder we investigate the question whether the discretization points are needed for the algorithms with linear information. To do so, we use Algorithm $\mathcal{A}_{\mathsf{GSIP}}$ but ignore all constraints induced by the discretization points in the solution of problem $\widehat{\mathsf{SIP}}$. This means we only respect the constraints induced by the linearizations. We choose a grid on the rectangle $[-2, 8] \times [-2, -2]$:

$$X = \left\{ -2 + \frac{10 \cdot i}{20} \mid 0 \leq i \leq 20 \right\} \times \left\{ -2 + \frac{4 \cdot j}{10} \mid 0 \leq j \leq 10 \right\} .$$

In a first experiment we use for every $(x_3, x_4) \in X$ the following initial point:

$$\boldsymbol{x}^1 = (0, 1, x_3, x_4)^\top .$$

For no such choice of an initial point the modified version of Algorithm $\mathcal{A}_{\mathsf{GSIP}}$ converges, if the discretization points are ignored. Next we modify the starting points and choose for the first two entries the optimal ones given in Equation (6.4). This means, for $(x_3, x_4) \in X$, we use:

$$\boldsymbol{x}^1 = (x_1^*, x_2^*, x_3, x_4)$$

as an initial point. The convergent and non-convergent runs are shown in Figure 6.4. One can see that, if the starting values are chosen close enough to the limit, the algorithm converges often without discretization points. However, if the starting values are not chosen good enough, the discretization points are needed to guarantee convergence. Which reflects the observation we have made in Lemma 4.3.

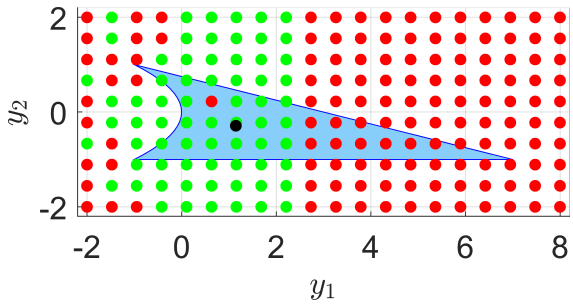


Figure 6.4: Convergent vs non-convergent starting values for modified Algorithm $\mathcal{A}_{\mathsf{GSIP}}$ without discretization points [*blue* - container, *red points* - non-convergent starting points, *green points* - convergent starting points, *black point* - optimal solution]

6.2 Increase of problem dimension

After we have commented on the number of iterations and the time needed for every example, we turn now to the nonlinear problems solved within every iteration. For Algorithm $\mathcal{A}_{B\&F}$ we solve a discretized problem in every iteration. Its complexity depends on the number of discretization points and the semi-infinite constraints $\tilde{\boldsymbol{g}}$. For the Algorithms $\mathcal{A}_{\mathsf{GSIP}}$ and $\mathcal{A}_{\mathsf{SIP}}$ a further constraint, based on a linearization, is added. This additional constraint can affect the time needed for solving the approximate problem.

We consider the second example from the previous Section. In Table 6.3 one can see for $m = 3$ that, although Algorithm $\mathcal{A}_{B\&F}$ needs approximately five times more iterations than Algorithm $\mathcal{A}_{\mathsf{GSIP}}$, the runtime is only larger by a factor of three. This means that the solution of the approximate problems consisting only of the discretization points is faster. Indeed, for Algorithm $\mathcal{A}_{B\&F}$ the time needed per iteration ranges from 0.06 seconds for the first iteration to 0.4 seconds for the last iteration. In comparison, Algorithm $\mathcal{A}_{\mathsf{GSIP}}$ needs already 0.22 seconds for the first iteration which increases up to 0.5 seconds in the last iteration. In Figure 6.5 we show how the step size between two iterates decreases over time. One can see that at the beginning Algorithm $\mathcal{A}_{B\&F}$ reaches smaller step sizes in less time. We can see the effect of the quadratic rate of convergence later. Once smaller step sizes are reached, Algorithms $\mathcal{A}_{\mathsf{SIP}}$ and $\mathcal{A}_{\mathsf{GSIP}}$ become much faster.

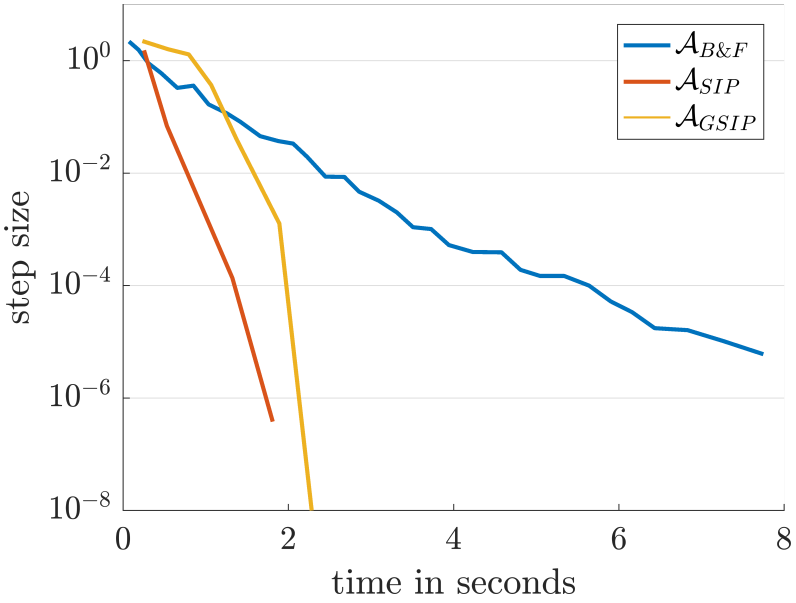


Figure 6.5: Time needed by the algorithms to reach specific step size for design-centering problem three dimensional ellipse into simplex

For the same example one can observe that the time needed per iteration changes for dimension $m = 6$. Here Algorithm $\mathcal{A}_{B\&F}$ still needs approximately 5 times more iteration and now also the runtime is 10 times larger. While the first approximate problems can still be solved faster (between 0.2 to 1 second), the last iterations take up to 20 seconds. The slowest iteration for the Algorithms $\mathcal{A}_{\mathsf{SIP}}$ and $\mathcal{A}_{\mathsf{GSIP}}$ takes 5 seconds.

The larger runtime to solve a single approximate problem for $\mathcal{A}_{B\&F}$ is due to the problem dimension. In iteration k we have for all three algorithms the following number of discretization points:

$$|Z^k| = k \cdot |I| \ .$$

As every discretization point induces $|I|$ constraints in the discretized problem there are $k \cdot |I|^2$ many constraints. In the discussed example one can observe that approximately a fixed number L of points is needed to decrease the current step size of the Blankenship and Falk algorithm $\mathcal{A}_{B\&F}$ by a factor of 10. To reach an accuracy of 10^{-6} we thus need around $6 \cdot L$ discretization points. This means that we have in the last problem

$$6 \cdot L \cdot |I|^2$$

constraints. In the discussed example this means that in the final iteration we consider a problem with 528 constraints for $m = 3$ and 5292 constraints for $m = 6$.

For Algorithm $\mathcal{A}_{\mathsf{GSIP}}$ we only need to reach an accuracy of 10^{-2}. After this accuracy is reached, we need only two or three further iterations because of the quadratic rate of convergence. This means that the maximal number of constraints is approximately given by:

$$(2 \cdot L + 3) \cdot |I|^2 + |I| \ .$$

In the discussed example we consider in the final iteration a problem with 116 constraints for $m = 3$ and 938 constraints for $m = 6$. For Algorithm $\mathcal{A}_{\mathsf{SIP}}$, we only need to consider a problem with at most 67 constraints, for $m = 3$ and 203 constraints for $m = 6$.

In other words: for the first iterations the constraints, based on the linearization, can influence the compute time needed to solve the approximate problems, but for later iterations the number of discretization points increases and we obtain problems with more and more constraints. This number of constraints becomes the dominant factor. As the algorithms with linear information need less iterations, the problem dimension does not grow as much as for the Blankenship and Falk algorithm.

6.3 Summary

After we have presented the solutions of the numerical examples, we now summarize the observations we have made.

In the first example we have seen that the quadratic rate of convergence reduces the number of iterations. Especially for the 'final steps' only a couple of iterations are needed while the algorithm by Blankenship and Falk still needs many iterations. For achieving a result with high precision, i.e. a small distance to the limit, it is important to benefit from the quadratic rate of convergence.

In the second example, having higher problem dimensions, we saw that this benefit becomes even larger. For a six-dimensional example the original algorithm took almost 10 minutes to converge, while the algorithms with linear information only took less then 6 and 50 seconds respectively. Using the examples we also discussed the influence of the transformation on the earlier iterates. These iterates are needed to first come close to a limit point and do not benefit from the quadratic rate of convergence. The earlier iterates calculated by Algorithm $\mathcal{A}_{\mathsf{GSIP}}$ show a behavior similar to the iterates calculated by the Blankenship and Falk algorithm. In this example, the approximation is closer, if a linearization of the solution in the transformed lower-level problem is used. That is why Algorithm $\mathcal{A}_{\mathsf{SIP}}$ reached in this example the quadratic rate of convergence faster.

In the third example we studied the question whether the discretization points are needed for Algorithm 2 and Algorithm 3 or whether one could simply consider an algorithm without these points. We have seen that no convergence can be guaranteed, if the starting point is not close enough to an optimal solution. This reflects the observations made in Lemmas 4.3 and 5.1.

In Section 6.2 we took a closer look at the nonlinear problems solved in each iteration. We have seen that in the earlier iterations the additional constraint, based on the linearization, can increase the compute time needed to solve the new approximate problems. For further iterations the number of discretization points and constraints increases. As a consequence the optimization solver needs more time to solve the discretized problems. As we need fewer iterations in the algorithms with linearization, this effect is larger for the Blankenship and Falk algorithm. This means that not only the number of iterations is reduced by the Algorithms $\mathcal{A}_{\mathsf{GSIP}}$ and $\mathcal{A}_{\mathsf{SIP}}$ but also the time needed for each iteration.

Summary and further work

The aim of this thesis is to solve semi-infinite optimization problems by adaptive discretization methods with a quadratic rate of convergence. To do so, we first investigated the convergence speed of the classical Blankenship and Falk algorithm. In every iteration of the algorithm a solution of an approximate problem based on the current discretization is calculated, the so-called discretized problem. In a second step, the most violated constraint is determined and added to the discretization, the so-called lower-level problem. The algorithm treats the lower-level problem and the discretized problems separately. To improve the convergence, we introduced a new adaptive discretization algorithm. The key idea is to use derivatives to capture the dependence of the solutions of the lower-level problems on the optimization variables of the semi-infinite problem. Using this technique we achieved a quadratic rate of convergence.

In Chapter 3 we started with the analysis of the Blankenship and Falk algorithm. Therefore we assumed that a local minimum of the current discretized problem is calculated in every iteration. We divided the analysis into two steps.

- In a first part we bounded the distance from the current iterate to a limit point in terms of the maximal violation of the semi-infinite constraints. The derived bounds depend on the order of a local minimum. Before we presented a rigorous statement we constructed a worst case example with arbitrarily slow convergence. This example can also be used as a counterexample to bounds presented in the literature. We excluded this example by demanding that the radii of the local solutions do not vanish in the limit.

- The second part consisted of bounding the maximal violation of the semi-infinite constraints for the current iterates. We showed that the violation of the current iterate can be bounded by a constant times the square of the previous step size.

Combining both bounds provided quadratic convergence in the special case of a minimum of order one. However, not every minimum is of order one. We introduced an example with a minimum of order two and showed that in this case only a linear rate of convergence holds.

The Blankenship and Falk algorithm achieves linear convergence in this example mainly because of the separated nature of the Blankenship and Falk algorithm. In the solution of the discretized problems, only the current discretization points are considered. All other points in the index set are ignored. This observation motivated a new adaptive discretization method. Instead of adding only the solutions of the lower-level problems, we also calculated a linearization of these solutions. Using this linearization, we injected an additional constraint which accounts for the points not yet added to the discretization.We solve a new approximate problem on every iteration, having the constraints induced by the discretization points together with the injected constraint. We proved the Quadratic Convergence Theorem in Chapter 4. This states that a sequence of stationary points converges to a strongly stable stationary point in the new algorithm, with a quadratic rate under some mild regularity assumptions.

One of the strengths of the Blankenship and Falk algorithm is its convergence properties. We showed that many of them also hold for our new method in the first part of Chapter 4. We showed that any accumulation point is a feasible point. We showed that iterates that are stationary points of the approximate problems, converge to a stationary point of the semi-infinite problem.To investigate the accumulation points of local solutions, we studied second-order sufficient conditions. We established properties of the iterates that are sufficient to ensure that an accumulation point is a local minimum.

In [Sch13], classical adaptive discretization methods are applied to a variable index-set via a transformation. We used this transformation and presented two techniques for applying the new method to generalized semi-infinite optimization problems. We showed that our Quadratic Convergence Theorem and other convergence statements also hold in this case.

In the last chapter we implemented the Blankenship and Falk algorithm alongside our new method, with linear information. We solved three numerical problems. In each case, our method converged in fewer iterations. As a result, the number of discretization points added was reduced. The approximate problems thus had fewer constraints, and could be solved faster. Our method thus required fewer iterations and less compute time per iteration . This advantage was especially pronounced in the more complex problems we considered.

We focused in this thesis on theoretical statements about the convergence properties of a new minimization method. These include the rate of convergence of the method, and properties of its limit points. Our chosen numerical examples motivate further questions.

- We observed that earlier iterations could be solved faster without the additional constraints we devised via a linearization. These constraints are needed to guarantee the quadratic rate of convergence once the iterates are close to a limit point, but could be omitted early in the calculation. We could, for instance, introduce the additional constraints once the step size falls below a threshold.

- Solving the earlier iterates without the linearization makes it more likely that the method will find a global minimum. As we noted in Chapter 4, our new method is not guaranteed to converge to a global solution. Starting without the linearization, however, makes it more likely that the method will get close to a global solution. Once it is sufficiently close, it is guaranteed to converge quadratically to this solution.

- We generally assumed that we could solve the approximate and lower-level problems in every iteration using standard solvers. To ensure the quadratic convergence of our method, we demanded very high precision of these solvers for every step in the solution process. However, this high precision is not necessary for the earlier iterates. It would be interesting to understand how the required precision depends on the iteration. For instance, lower precision requirements early on might not affect overall convergence, but could save substantial compute time. These issues could be investigated practically via numerical experiment, and also theoretically.

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Semi-infinite programming can be used to model a large variety of complex optimization problems. Interesting applications include cutting and packing, or coverage problems. The simple description of such problems comes at a price: semi-infinite problems are often harder to solve than finite nonlinear problems.

In this thesis we solve semi-infinite optimization problems using adaptive discretization methods having a quadratic rate of convergence. First, we investigate the classical Blankenship and Falk algorithm. We present an example which shows that quadratic convergence is not possible for a minimum of order higher than one.

Motivated by this example we suggest a new adaptive discretization algorithm with guaranteed quadratic convergence. This rate holds even for minima of order higher than one. We prove the Quadratic Convergence Theorem, which rigorously establishes quadratic convergence under mild regularity conditions and investigate further convergence properties.

We compare the Blankenship and Falk algorithm to our new method by considering a series of numerical examples. In these examples, our new method outperforms the Blankenship and Falk algorithm.

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