# On the Forward Kinematics of Cable-Driven Parallel Robots 

Andreas Pott ${ }^{1}$ and Valentin Schmidt ${ }^{2}$


#### Abstract

Most cable-driven parallel robots are kinematically over-constrained mechanisms. This results in a non-trivial computation of the forward kinematic transformation. It is well known that the forward kinematics of parallel robots may have multiple solutions and in general the convergence of numerical methods is unknown. In recent works, it was proposed to formulate the forward kinematics as optimization problem that models the cables as linear springs in order to compute the platform pose which has minimal potential energy in the cables. In this paper, we analyzed this objective function. Using the Hessian matrix, we show that under certain conditions the problem at hand is convex and we can expect a unique and stable minimum. The computations are exemplified for point-shaped platforms as well as for the planar case. For the spatial case, we present an encouraging numerical study. An ordinary least squares method is then applied to find a position approximation and an improvement to previous methods is demonstrated.


## I. INTRODUCTION

In the last decade, a lot of research has been carried out to study both, theory (see e.g. [1], [2], [3]) and implementation [4] of cable-driven parallel robots.

For a mobile platform with $n$ degrees-of-freedom, in general, at least $m=n+1$ cables are required to fully control the motion [5]. Therefore, many cable robots are under-determined with respect to distribution of forces in the cables and over-determined with respect to forward kinematics (Fig. 1). As a consequence of the latter, it is challenging to calculate the forward kinematics of the cable robot in real-time. Thus, one has to estimate the pose of the mobile platform from given length of the cables.

In the literature, different approaches for that problem were suggested. In general, the forward kinematics of parallel robots, with six legs but almost identical topology as cable robots, can have up to 40 solutions and the algorithm by Husty [6] gives deep insight into the number of solutions and their mathematical structure. Unfortunately, it is currently inadequate for real-time implementation and adding additional cables or lengths does not necessarily reduce the number of solutions in the general case and special geometries maintain this maximum solution set [7]. An incremental forward kinematics to follow a trajectory was presented in [8]. Merlet [9] used interval analysis to calculate the forward kinematics of Stewart-Gough platforms in a robust and guaranteed way. A more specialized method for cable robots with linear drives

[^0]

Fig. 1. CAD draft of the spatial cable-driven parallel robot IPAnema with eight cables and six degrees-of-freedom.
and elastic deformation in the cables was also shown in [10]. Other possible methods include neural networks or combinational approaches [11].

Merlet also presented forward kinematics for underconstrained robots [12]. A closed-form kinematic code for the so-called 3-2-1 configuration is well suitable for realtime application [13][14], but relies on a special non-generic geometry. Bruckmann [15] presented a method to cope with winches using pulley mechanisms to guide the cables. An efficient real-time capable numerical scheme for forward kinematics of over-constrained robots was proposed by [16] and extensions for pulley mechanisms have been implemented [17].

Within this contribution, we follow the line of research of the latter contributions where we provided numerical results for the convergence in order to achieve real-time efficiency. In this paper, we analyze the convergence of the energy minimization method for forward kinematics. Furthermore, using this knowledge of the convergence, we present a new pose estimation method in order to generate the starting position for a numerical optimization technique. This estimation scheme is employed in global navigation satellite system receivers, which constitutes a very similar problem statement of finding position from distances to satellites [18], [19], [20].

Verhoeven created and proved the completeness of the motion pattern for fully-parallel cable robots. According to this list, only the types 1T, 2T, 3T, 1R2T, 2R3T, and 3R3T exist [1]. In this contribution, we restrict ourself to a thorough consideration of convergence behavior for robots of the 2 T , 3 T , and 1R2T class.


Fig. 2. Geometry and kinematics of a general cable robot.

## II. FORWARD KINEMATICS OF CABLE-DRIVEN PARALLEL ROBOTS

For better reference, the kinematic foundations of cable robots are briefly reviewed to introduce our notation. Fig. 2 shows the kinematic structure of a spatial cable robot, where the vectors $\mathbf{a}_{i}$ denote the proximal anchor points on the robot base, the vectors $\mathbf{b}_{i}$ are the relative positions of the distal anchor points on the movable platform, and $\mathbf{l}_{i}$ denote the vector of the cables. The length of the cables is abbreviated by $l_{i}=\left\|\mathbf{l}_{i}\right\|_{2}$. Applying a vector loop, the closure-constraint reads

$$
\begin{equation*}
\mathbf{a}_{i}-\mathbf{r}-\mathbf{R} \mathbf{b}_{i}-\mathbf{l}_{i}=\mathbf{0} \quad \text { for } \quad i=1, \ldots, m \tag{1}
\end{equation*}
$$

where the vector $\mathbf{r}$ is the Cartesian position of the platform and the rotation matrix $\mathbf{R}$ represents the orientation of the platform frame $\mathcal{K}_{p}$ with respect to the world frame $\mathcal{K}_{0}$. From (1) we receive $m$ nonlinear equations $\nu_{i}$ for the forward kinematics

$$
\begin{equation*}
\nu_{i}(\mathbf{l}, \mathbf{r}, \mathbf{R})=\left\|\mathbf{a}_{i}-\mathbf{r}-\mathbf{R} \mathbf{b}_{i}\right\|_{2}^{2}-l_{i}^{2}=0, i=1, \ldots, m \tag{2}
\end{equation*}
$$

that form an over-constrained system for the considered case with $m>n$. In general, we cannot expect to solve the above equation analytically, but we can minimize the error which can be interpreted as minimizing the potential energy in pretensed springs [16] which yields the function for forward kinematics

$$
\begin{equation*}
\boldsymbol{\Phi}(\mathbf{l})=\min _{\mathbf{r}, \mathbf{R}} \sum_{i}^{m} \nu_{i}^{2}(\mathbf{l}, \mathbf{r}, \mathbf{R}), \tag{3}
\end{equation*}
$$

where the given vector $\mathbf{l}=\left[l_{1}, \ldots, l_{m}\right]^{\mathrm{T}}$ is the vector containing the cable lengths. Then, the function $\Phi(1)$ yields the values $\mathbf{r}^{*}, \mathbf{R}^{*}$ that minimize the right hand side of (3). The function $\Phi: \mathbb{R}^{m} \rightarrow \mathbb{R}^{n}$ can only be computed using a numerical procedure.

To further characterize the optimization problem at hand, we consider the objective function $g: \mathbb{R}^{m} \rightarrow \mathbb{R}$ as follows

$$
\begin{equation*}
g(\mathbf{l}, \mathbf{r}, \mathbf{R})=\sum_{i}^{m}\left[\left\|\mathbf{a}_{i}-\mathbf{r}-\mathbf{R} \mathbf{b}_{i}\right\|_{2}^{2}-l_{i}^{2}\right]^{2} \tag{4}
\end{equation*}
$$

In order to compute derivatives, we introduce a parameterization of the rotation matrix $R$ through an angular model with the angles $a, b, c$. This can be chosen to be e.g. Euler angles or Bryant angles. The pose is thus denoted by $\mathbf{y}=$ $(x, y, z, a, b, c)$. Computing the gradient $\mathbf{G}$ of $g$ yields

$$
\mathbf{G}=\nabla g(\mathbf{y})=\left(\begin{array}{c}
\frac{\partial g}{\partial x}  \tag{5}\\
\ldots \\
\frac{\partial g}{\partial c}
\end{array}\right)
$$

containing six partial derivatives of the objective function. Since the objective function $g$ is differentiable, the sought optimum corresponds to the pose where $\nabla g=\mathbf{0}$, given exact cable lengths. Furthermore, we will consider the Hessian matrix $\mathbf{H}$ of the function $g$ in order to characterize the number and type of extremal values of $g$. The Hessian of $g$ is given by

$$
\mathbf{H}=\frac{\partial^{2} g}{\partial \mathbf{y}^{2}}=\left(\begin{array}{ccc}
\frac{\partial^{2} g}{\partial x \partial x} & \cdots & \frac{\partial^{2} g}{\partial x \partial c}  \tag{6}\\
\vdots & \ddots & \\
\frac{\partial^{2} g}{\partial c \partial x} & & \frac{\partial^{2} g}{\partial c \partial c}
\end{array}\right)
$$

where the Hessian is symmetric according to the theorem of Schwarz because the function $g$ is continuous differentiable in $\mathbf{y}$.

## III. ANALYSIS OF THE CONVERGENCE BEHAVIOR

Numerical studies as well as experimental results from several years of operation of the robot controller indicate that the kinematic code built from Levenberg-Marquardt optimization of $g$ shows both stable and reliable performance in practice. However, little analysis have been made yet to elaborate a theoretical foundation. In the following section, we present some case studies for point-shaped platforms as well as for planar robots with one rotational degree-offreedom and two translational degrees-of-freedom (1R2T).

## A. The $2 T$ and $3 T$ case

In this section, we analyze the procedure for the generalized robot geometry. The 2 T and 3 T robot types, can be described as the 2D and 3D motions of a point respectively. Therefore, when considering the objective function $g$, rotation can be ignored. The geometric condition for robots with these two motion patterns is that all cables are connected to the same point on the platform and thus that all vectors $\mathbf{b}_{i}$ are equal. Without loss of generality, we can therefore assume $\mathbf{b}_{i}=\mathbf{0}$. Consequently, the equations of the objective function are greatly simplified. To further characterize the optimization problem at hand, we consider the objective function $g: \mathbb{R}^{m} \rightarrow \mathbb{R}$ in 2 T case as follows

$$
\begin{equation*}
g(\mathbf{l}, \mathbf{r})=\sum_{i}^{m}\left(\left\|\mathbf{r}-\mathbf{a}_{i}\right\|_{2}^{2}-l_{i}^{2}\right)^{2} \tag{7}
\end{equation*}
$$

and substituting the parameters of the position $(x, y)$ for the vectors into the expression gives

$$
\begin{equation*}
g=\sum_{i}^{m}\left(\left(x-a_{i x}\right)^{2}+\left(y-a_{i y}\right)^{2}-l_{i}^{2}\right)^{2} \tag{8}
\end{equation*}
$$

Thus, the gradient $\mathbf{G}$ can be computed as follows

$$
\begin{equation*}
\mathbf{G}=\sum_{i}^{m}\binom{4\left(\left(x-a_{i x}\right)^{2}+\left(y-a_{i y}\right)^{2}-l_{i}^{2}\right)\left(x-a_{i x}\right)}{4\left(\left(x-a_{i x}\right)^{2}+\left(y-a_{i y}\right)^{2}-l_{i}^{2}\right)\left(y-a_{i y}\right)} \tag{9}
\end{equation*}
$$

and the Hessian $\mathbf{H}$ becomes

$$
\begin{align*}
\mathbf{H} & =\sum_{i}^{m}\left(\begin{array}{ll}
H_{x x} & H_{x y} \\
H_{x y} & H_{y y}
\end{array}\right) \text { with, }  \tag{10}\\
H_{x x} & =\sum_{i}^{m} 12\left(x-a_{i x}\right)^{2}+4\left(y-a_{i y}\right)^{2}-4 l_{i}^{2}  \tag{11}\\
H_{x y} & =\sum_{i}^{m} 8\left(x-a_{i x}\right)+8\left(y-a_{i y}\right)  \tag{12}\\
H_{y y} & =\sum_{i}^{m} 4\left(x-a_{i x}\right)^{2}+12\left(y-a_{i y}\right)^{2}-4 l_{i}^{2} \tag{13}
\end{align*}
$$

where for the $3 T$ case the gradient $G$ is extended with the respective terms for the $z$-coordinate and the Hessian consists of some additional trivial derivatives. A sufficient condition for the optimum of the function $g$ to be unique is that the gradient $G=\mathbf{0}$ and the Hessian $\mathbf{H}$ is positive definite. For a symmetric $2 \times 2$ matrix, this check can be done by testing if the determinant is positive. The eigenvalues of a symmetric matrix are real, therefore, both eigenvalues are positive if the determinant is positive. To demonstrate the procedure, we use the geometric parameters for $\mathbf{a}_{i}$ given in Tab. I. With actual numbers for the geometry, the determinant of $\mathbf{H}$ becomes a multivariate polynomial in the position $(x, y)$ and the cable length $\left(l_{1}, \ldots, l_{m}\right)$. This polynomial allows to consider the general relation for arbitrary cable length. To remove the dependency from the cable length, the inverse kinematic equation is to give the cable length is inserted into $\mathbf{H}$. This corresponds to the ideal situation without measurement errors or disturbance in the cable length. Executing the substitution with computer algebra gives a surprisingly simple expression

$$
\begin{equation*}
\operatorname{det} \mathbf{H}_{\text {ideal }}=1024\left(x^{2}+4(y-1)^{2}+4\right) \tag{14}
\end{equation*}
$$

where for the determinant of $\mathbf{H}_{\text {ideal }}$ the geometric parameters $\mathbf{a}_{i}$ listed in Tab. I where used. This expression is obviously always positive. Therefore, we have shown for the sample robot that the solution of the forward kinematics by the energy method is always unique. The result is also illustrated in Fig. 3 that shows the eigenvalues of the matrix $\mathbf{H}_{\text {ideal }}$ over the area covered by the frame and in Fig. 4 we plot the determinant in the same region. From the positive definiteness of the Hessian, we conclude that the objective function is convex which means that we can find a unique solution in our optimization problem to solve the forward kinematics.

## B. The planar case $1 R 2 T$

We apply the same approach to the 1R2T case where the equations are slightly more complex. Again, we express the position of the platform with the coordinates $\mathbf{r}=(x, y)$ and the rotation is given by the rotation matrix $\mathbf{R}$ which is parameterized by the angle $\phi$. Thus, for the 1R2T case

TABLE I
A Sample Planar Robot with 1R2T Motion Pattern: Platform Vectors b and Base Vectors a

| cable $i$ | platform vector $\mathbf{b}_{i}[\mathrm{~m}]$ | base vector $\mathbf{a}_{i}[\mathrm{~m}]$ |
| :--- | :--- | :--- |
| 1 | $[-2.0,2.0]^{\mathrm{T}}$ | $[-0.05,0.1]^{\mathrm{T}}$ |
| 2 | $[2.0,2.0]^{\mathrm{T}}$ | $[0.05,0.1]^{\mathrm{T}}$ |
| 3 | $[2.0,0]^{\mathrm{T}}$ | $[0.05,-0.0]^{\mathrm{T}}$ |
| 4 | $[-2.0,0]^{\mathrm{T}}$ | $[-0.05,-0.0]^{\mathrm{T}}$ |



Fig. 3. Evaluation of the smallest eigenvalues $\lambda_{\min }$ of the Hessian $\mathbf{H}_{\text {ideal }}$ within the frame of the 2 T robot. ( x and y in [m])
the geometry $\mathbf{b}_{i}$ of the platform cannot be removed from the equation and we deal with the general case of having arbitrary vectors $\mathbf{b}_{i}$. Substituting the known quantities into the general over-constrained objective function (4) yields

$$
\begin{align*}
g= & \sum_{i}^{m}\left(\left(x+\cos (\phi) b_{i x}-\sin (\phi) b_{i y}-a_{i x}\right)^{2}\right. \\
& +\left(y+\sin (\phi) b_{i x}+\cos (\phi) b_{i y}-a_{i y}\right)^{2} \\
& \left.-l_{i}^{2}\right)^{2} \tag{15}
\end{align*}
$$

We compute the gradient $\mathbf{G}=\left(G_{x}, G_{y}, G_{\phi}\right)^{\mathrm{T}}$ as follows

$$
\begin{align*}
G_{x}= & \sum_{i}^{m} 4\left(\left(x+\cos (\phi) b_{i x}-\sin (\phi) b_{i y}-a_{i x}\right)^{2}\right. \\
& \left.\left(y+\sin (\phi) b_{i x}+\cos (\phi) b_{i y}-a_{i y}\right)^{2}-l_{i}^{2}\right) \\
& \left(x+\cos (\phi) b_{i x}-\sin (\phi) b_{i y}-a_{i x}\right)  \tag{16}\\
G_{y}= & \sum_{i}^{m} 4\left(\left(x+\cos (\phi) b_{i x}-\sin (\phi) b_{i y}-a_{i x}\right)^{2}\right. \\
& \left.\left(y+\sin (\phi) b_{i x}+\cos (\phi) b_{i y}-a_{i y}\right)^{2}-l_{i}^{2}\right) \\
& \left(y+\sin (\phi) b_{i x}+\cos (\phi) b_{i y}-a_{i y}\right) \tag{17}
\end{align*}
$$



Fig. 4. Evaluation of the determinant of the Hessian $\mathbf{H}_{\text {ideal }}$ within the frame of the 2 T robot. ( x and y in [m])

$$
\begin{align*}
G_{\phi}= & \sum_{i}^{m} 4\left[\left(x+\cos (\phi) b_{i x}-\sin (\phi) b_{i y}-a_{i x}\right)^{2}\right. \\
& \left.+\left(y+\sin (\phi) b_{i x}+\cos (\phi) b_{i y}-a_{i y}\right)^{2}-l_{i}^{2}\right] \\
& {\left[\left(x+\cos (\phi) b_{i x}-\sin (\phi) b_{i y}-a_{i x}\right)\right.} \\
& \left(-\sin (\phi) b_{i x}-\cos (\phi) b_{i y}\right) \\
& +\left(y+\sin (\phi) b_{i x}+\cos (\phi) b_{i y}-a_{i y}\right) \\
& \left.\left(\cos (\phi) b_{i x}-\sin (\phi) b_{i y}\right)\right] \tag{18}
\end{align*}
$$

Evaluating the Hessian is possible repeating the procedure in case study for the 2T type; however, we do not reproduce the coefficients of the matrix here due to space limitation. To study the expected convergence of the optimization problem, we apply the procedure outlined above. Substituting both a geometry given by Tab. I and the ideal cable length into the Hessian provides the desired equations for the determinant of the Hessian. The evaluation with computer algebra provides an expression with around 250 operations to compute the determinant for a pose $\mathbf{y}=(x, y, \phi)$. Results from the computation of the determinant are shown in Fig. 5. As one can see in the figures, the determinant is positive for two coordinate planes. A numerical search also shows no zero crossings within the workspace. Therefore, we expect the solution to be unique inside the robot machine frame.

## IV. ESTIMATION OF POSITION FOR 3T3R

It was shown in the previous section that the 3 T class problem is convex and therefore a much simpler problem than the general case. Simplifying the 3T3R to the 3 T problem in the first step gives the opportunity to obtain a fast and accurate position estimation. An estimate is made using tools for linear regression estimators, which in the simplest


Fig. 5. Evaluation of the determinant of the Hessian $\mathbf{H}_{\text {ideal }}$ within the frame of the 1R2T robot. The upper plot shows the value of the determinant for $\phi=0$ in the $x y$-plane where the lower plot shows the value of the determinant for $y=0.5$ in the $x \phi$-plane. (x and y in [m] and $\phi$ in [rad])
form is the ordinary least squares estimator

$$
\begin{equation*}
\hat{\beta}=\left(\mathbf{X}^{\mathrm{T}} \mathbf{X}\right)^{-1} \mathbf{X}^{\mathrm{T}} y \tag{19}
\end{equation*}
$$

where $\beta$ is the unknown parameter, $\mathbf{y}$ the dependent variable and $\mathbf{X}$ the design matrix containing the regressors.

The common name for the estimation method is the line of position, as the geometrical equivalent is finding a line of position between two spheres (defined by the cable length around center points $\mathbf{a}_{i}$ ), and then the estimate for the position from the line intersections using (19) is made.

We begin with (1) and assume that $\mathbf{R}=\mathbf{I}_{3}$. This enables to combine the parameter vectors $\mathbf{a}_{i}$ and $\mathbf{b}_{i}$ to a single vector $\alpha_{i}$ by

$$
\begin{equation*}
\alpha_{i}=\mathbf{a}_{i}-\mathbf{b}_{i} \tag{20}
\end{equation*}
$$

TABLE II
IPANEMA 1 GEOMETRICAL PARAMETERS: PLATFORM VECTORS b AND
BASE VECTORS a

| cable $i$ | platform vector $\mathbf{b}_{i}[\mathrm{~m}]$ | base vector $\mathbf{a}_{i}[\mathrm{~m}]$ |
| :--- | :--- | :--- |
| 1 | $[-2.0,1.5,2.0]^{\mathrm{T}}$ | $[-0.06,0.06,0.0]^{\mathrm{T}}$ |
| 2 | $[2.0,1.5,2.0]^{\mathrm{T}}$ | $[0.06,0.06,0.0]^{\mathrm{T}}$ |
| 3 | $[2.0,-1.5,2.0]^{\mathrm{T}}$ | $[0.06,-0.06,0.0]^{\mathrm{T}}$ |
| 4 | $[-2.0,-1.5,2.0]^{\mathrm{T}}$ | $[-0.06,-0.06,0.0]^{\mathrm{T}}$ |
| 5 | $[-2.0,1.5,0.0]^{\mathrm{T}}$ | $[-0.06,0.06,0.2]^{\mathrm{T}}$ |
| 6 | $[2.0,1.5,0.0]^{\mathrm{T}}$ | $[0.06,0.06,0.2]^{\mathrm{T}}$ |
| 7 | $[2.0,-1.5,0.0]^{\mathrm{T}}$ | $[0.06,-0.06,0.2]^{\mathrm{T}}$ |
| 8 | $[-2.0,-1.5,0.0]^{\mathrm{T}}$ | $[-0.06,-0.06,0.2]^{\mathrm{T}}$ |

and coincidentally results in the same equation as shown for the 3 T case (7). We expand this to get

$$
\begin{equation*}
g(\mathbf{l}, \mathbf{r}, \mathbf{R})=\sum_{i}^{m}\|\mathbf{r}\|_{2}^{2}+\left\|\alpha_{i}\right\|_{2}^{2}-2 \alpha_{i}^{\mathrm{T}} \mathbf{r}-l_{i}^{2} \tag{21}
\end{equation*}
$$

Now, the non-linear term $\|\mathbf{r}\|_{2}^{2}$ can be eliminated by subtracting the equation of one cable from all others. Assuming perfect cable lengths to solve for $\mathbf{r}$, we generate a set of equations along the lines of

$$
\begin{equation*}
\mathbf{r}=\frac{\left\|\alpha_{i}\right\|_{2}^{2}-\left\|\alpha_{i+1}\right\|_{2}^{2}-l_{i}^{2}+l_{i+1}^{2}}{2\left(\alpha_{i}^{T}-\alpha_{i+1}^{T}\right)} \tag{22}
\end{equation*}
$$

$$
\text { for } i=1, \ldots, N-1 \text {. }
$$

Interestingly enough, only four cables are needed to get the estimate in three dimensional space. However, since we expect some errors, all cables can be used to generate the estimate. If we assume $N$ cables are used for the estimate, $N-1$ circle pairs are generated. A more detailed investigation of the possible pairs is given by [20]. Using ordinary least squares, we can now form an estimate for position $\widehat{\mathbf{r}}$. To form this, we use matrix representation of (22) using

$$
\begin{align*}
\mathbf{A} & =\left[\begin{array}{c}
2 \alpha_{1}^{T} \\
\vdots \\
2 \alpha_{n}^{T}
\end{array}\right], \quad \mathbf{y}=\left[\begin{array}{c}
\left\|\alpha_{1}\right\|_{2}^{2}-l_{1}^{2} \\
\vdots \\
\left\|\alpha_{n}\right\|_{2}^{2}-l_{n}^{2}
\end{array}\right]  \tag{23}\\
\mathbf{D} & =\left[\begin{array}{ll}
-1 & \mathbf{I}_{N-1}
\end{array}\right] \tag{24}
\end{align*}
$$

using (19) results in

$$
\begin{equation*}
\widehat{\mathbf{r}}=\left(\mathbf{A}^{\mathrm{T}} \mathbf{D}^{\mathrm{T}} \mathbf{D} \mathbf{A}\right)^{-1} \mathbf{A}^{\mathrm{T}} \mathbf{D}^{\mathrm{T}} \mathbf{D} \mathbf{y} \tag{25}
\end{equation*}
$$

The choice which cables to use for the estimation is not arbitrary. If the distribution of error is unknown, then all cables can be used. Mathematically, a singular matrix $\left(\mathbf{A}^{\mathrm{T}} \mathbf{D}^{\mathrm{T}} \mathbf{D A}\right)$ should be avoided. This will occur when the vectors $\alpha_{i}$ are linearly dependent or congruent with the axes of the coordinate system (only likely when using only four cables). While the first case is not a likely robot configuration, the second can be avoided through rotating the entire world coordinate system ( $\mathcal{K}_{0}$ in Fig. 2) when necessary.

If errors are not equal for each cable, then the selection of cables can be performed to reflect this. For example if we do expect rotation, we can chose cables which have the smallest $\left\|\mathbf{b}_{i}\right\|_{2}$ and thus are least effected by rotation.

Since we have shown that the $3 T$ case has a single minimum, this method gives the exact position in one step when no rotation is conducted. This is often the case in cable robots as the workspace is limited in rotating degrees-of-freedom. As far as rotation is concerned, the position is only an estimate which can be compared with the one based on maximal cable intervals presented in [16]. This "interval method" provided an estimate for further evaluation using iterative techniques by bisecting the workspace into an area in which the platform must be located due to the cable length limits. An iterative technique needed to follow in order to give a precise estimate of the pose, $\mathbf{r}, \mathbf{R}$, from (1).

Fig. 6 shows two pose estimators for a grid of almost 4000 poses for the IPAnema 1 robot (Tab. II). For each pose the cable lengths were calculated using the inverse kinematics (1) and from these the position estimated using the previous method and the improved method. In a second test, the wire lengths generated have normally distributed error of up to 0.1 m . We plot the estimation error $\varepsilon$ calculated from

$$
\begin{equation*}
\varepsilon=\|\mathbf{r}-\hat{\mathbf{r}}\|_{2} \tag{26}
\end{equation*}
$$

in a histogram for all evaluated poses. Each bar represents an error range of 0.1 m . The generated poses were all inside the frame defined by vectors $\mathbf{a}_{i}$.


Fig. 6. Error $\varepsilon$ for different pose estimation algorithms (left: perfect cable length, right: normalized error of 0.1 m )

As can be seen in Fig. 6, the pose estimation technique proposed here gives the exact minimum for the poses which have perfect cable lengths. This is to be expected as shown in section III-A the rotation-less robot in 3D has one local minimum which is quickly estimated by the ordinary least squares. Even when cable length errors are considered, it will find an acceptable solution within the cable length error magnitude. Whether this is the case for the rotational 3R3T
is not determined, but the estimate is consistently closer than the previous pose estimation method. When repeated for one million poses with rotation, the previous estimate had a mean $\varepsilon$ of 0.395 m and the new estimate of 0.050 m .

As in the previous real-time algorithm from [16], starting with the position estimate, a numerical algorithm can be used to iterate the platform pose through a least square approach of the over-constrained nonlinear equations. This can be a Gauss-Newton Method, or a Levenberg-Marquardt algorithm as used by the IPAnema controller.

## V. CONCLUSIONS

In this paper, we investigated the forward kinematics problem by minimizing the potential energy in the cables. Analysis of the second derivatives showed that the forward kinematic problem is actually conditioned well. For the 2 T and 3 T case, a numerical evaluation of the Hessian determinant can show that a unique solution exists within the bounds of the workspace. This implies good convergence for numerical optimization tools. Rotation greatly increases complexity of this problem, but can be tackled using this approach. Calculating the terms numerically can be used as a check on the geometry of the cable robot, separating feasible from infeasible designs.

We have also shown that an ordinary least squares estimate gives the solution for robots without rotation and thus can be used as a good starting point for iterative solvers. It remains to be seen if iterative techniques can be optimized further to provide more certainty in calculations of the kinematics in real-time. Interval bounds defined through the rotation of the platform and the longest $\mathbf{b}_{i}$ could be a reasonable starting point for further techniques.

## REFERENCES

[1] Richard Verhoeven. Analysis of the Workspace of Tendon-based Stewart Platforms. PhD thesis, University of Duisburg-Essen, Duisburg, 2004.
[2] Manfred Hiller, Shiqing Fang, Sonja Mielczarek, Richard Verhoeven, and Daniel Franitza. Design, analysis and realization of tendon-based parallel manipulators. volume 40, pages 429-445.
[3] Marc Gouttefarde, Jean-Pierre Merlet, and David Daney. Wrenchfeasible workspace of parallel cable-driven mechanisms. In ICRA, pages 1492-1497, Roma and Italy, 2007.
[4] Jean-Pierre Merlet and David Daney. A new design for wire-driven parallel robot. In 2nd Int. Congress, Design and Modelling of mechanical systems, Monastir, 2007.
[5] Aiguo Ming and Toshiro Higuchi. Study on multiple degree-offreedom positioning mechanism using wires (part 1) - concept, design and control. Int. Journal of the Jap. Soc. for Precision Engineering, 28(2):131-138, 1994.
[6] Manfred L. Husty. An algorithm for solving the direct kinematic of stewart-gough-type platforms. Mechanism and Machine Theory, 31(4):365-380, 1996.
[7] Manfred L. Husty, Sonja Mielczarek, and Manfred Hiller. A redundant spatial stewart-gough platform with maximal forward kinematics solution set. In $A R K$, pages 147-154, Caldes de Malavella and Spain, 2002. Kluwer Academic Publishers.
[8] Shiqing Fang. Design, Modeling and Motion Control of Tendon-Based Parallel Manipulators. Fortschritt-Berichte VDI, Reihe 8, Nr. 1076. VDI Verlag, Düsseldorf, 2005.
[9] Jean-Pierre Merlet. Solving the forward kinematics of a gough-type parallel manipulator with interval analysis. International Journal of Robotics Research, 23(3):221-235, 2004.
[10] Jean-Pierre Merlet. Kinematics of the wire-driven parallel robot marionet using linear actuators. In Proceedings of the IEEE International Conference on Robotics and Automation, Pasadena, CA and USA, 2008.
[11] Valentin Schmidt, Bertram Müller, and Andreas Pott. Solving the forward kinematics of cable-driven parallel robots with neural networks and interval arithmetic. In Computational Kinematics, volume 15, pages 103-110. 2014.
[12] J.-P. Merlet. On the redundancy of cable-driven parallel robots. In New Trends in Mechanism and Machine Science, volume 24 of Mechanisms and Machine Science, pages 31-39. Springer International Publishing, Cham, 2015.
[13] Frederico Thomas, Erika Ottaviano, Lluís Ros, and Marco Ceccarelli. Uncertainty model and singularities of 3-2-1 wire-based tracking systems. In ARK, Caldes de Malavella and Spain, 2002. Kluwer Academic Publishers.
[14] Andreas Pott. Forward kinematics and workspace determination of a wire robot for industrial applications. In $A R K$, pages $451-458$, Baz-sur-Mer and France, 2008. Springer-Verlag.
[15] Tobias Bruckmann, Lars Mikelsons, Thorsten Brandt, Manfred Hiller, and Dieter Schramm. Wire robots part i - kinematics, analysis and design. In Parallel Manipulators - New Developments, ARS Robotic Books, Vienna and Austria, 2008. I-Tech Education and Publishing.
[16] Andreas Pott. An algorithm for real-time forward kinematics of cabledriven parallel robots. In 12th International Symposium on Advances in Robot Kinematics, Piran Portoroz and Slovenio, 2010. SpringerVerlag.
[17] Valentin Schmidt and Andreas Pott. Implementing extended kinematics of a cable-driven parallel robot in real-time. In Cable-Driven Parallel Robots, pages 287-298. 2013.
[18] S. Jazaeri, A. R. Amiri-Simkooei, and M. A. Sharifi. Iterative algorithm for weighted total least squares adjustment. Survey Review, 46(334):19-27, 2014.
[19] Niilo Sirola. Closed-form algorithms in mobile positioning: Myths and misconceptions. In 2010 7th Workshop on Positioning, Navigation and Communication (WPNC), pages 38-44.
[20] J. J. Caffery. A new approach to the geometry of toa location. In Vehicular Technology Conference Fall 2000. IEEE VTS Fall VTC2000. 52nd Vehicular Technology Conference, pages 1943-1949, 24-28 Sept. 2000.


[^0]:    ${ }^{1}$ Andreas Pott is with the Department of Robot and Assistive Systems, Fraunhofer IPA, 70569 Stuttgart, Germany andreas.pott@ipa.fraunhofer.de
    ${ }^{2}$ Valentin Schmidt is with the Department of Robot and Assistive Systems, Fraunhofer IPA, 70569 Stuttgart, Germany valentin.schmidt@ipa.fraunhofer.de

